All (qubit) decoherences: Complete characterization and physical implementation

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We investigate decoherence channels that are modelled as a sequence of collisions of a quantum system (e.g., a qubit) with particles (e.g., qubits) of the environment. We show that collisions induce decoherence when a bi-partite interaction between the system qubit and an environment (reservoir) qubit is described by the controlled-$U$ unitary transformation (gate). We characterize decoherence channels and in the case of a qubit we specify the most general decoherence channel and derive a corresponding master equation. Finally, we analyze entanglement that is generated during the process of decoherence between the system and its environment.

\section{I. INTRODUCTION}

One of the most distinctive features of quantum systems is their ability to "exist" in superpositions of mutually exclusive (orthogonal) states \cite{1}. Providing a quantum system has been prepared in a pure state $|\Psi\rangle$ then we can write $|\Psi\rangle = \sum_k c_k |\psi_k\rangle$, where $|\psi_k\rangle$ are orthonormal vectors that compose a basis ($\langle \psi_l | \psi_k \rangle = \delta_{lk}$). All bases are unitarily equivalent and we can express the same state in different bases. In fact, we can always select a basis such that $|\Psi\rangle$ is a basis vector so in its matrix representation the vector $|\Psi\rangle$ is represented by a single diagonal element. According to quantum postulates for the isolated system any evolution is governed by unitary transformations and the original information about the state preparation of the quantum system is preserved. As soon as an interaction with an environment comes into the play (the quantum system is open) the situation becomes dramatically different and the state is no longer described by the single diagonal element in some basis. Depending on properties of the environment and the character of the interaction our system evolves non-unitarily and its state is, in general, described by a statistical mixture. Among various possible dynamics of an open quantum system interacting with its environment a specific role is played by a process in which the off-diagonal elements of the original state $\rho = |\Psi\rangle \langle \Psi|$ in some basis are continuously suppressed in time, i.e.

$$\rho \rightarrow \rho_{t \rightarrow \infty} = \text{diag}(|\rho|), \quad (1.1)$$

This is a process of decoherence during which some of the information about the initial state of the quantum system might be irreversibly lost \cite{2–4}. The basis in which the decoherence takes place is specified by properties of the environment and the character of the interaction \cite{4}. There are at least two aspects of quantum decoherence that keep it in the center of interests in multiple investigations related to foundations of quantum mechanics and in quantum information processing. The first aspect is, that decoherence is presently viewed as a mechanism via which classicality emerges from the realm of quantum (see e.g. \cite{2–6}). In this context it is of paramount importance to specify the basis (the so called pointer basis \cite{4}) in which the decoherence takes place. In the field of quantum information the decoherence is an evil - it degrades quantum resources (superpositions of states and quantum entanglement) that are needed for quantum information processing \cite{7}. The degradation of resources is caused by random interactions (errors) between a quantum system under consideration (e.g. a qubit or a quantum register) with its environment. If nothing else then these two facets of quantum decoherence are enough to justify an investigation of decoherence channels (transformations).

As mentioned above the decoherence is caused by (unavoidable) interactions between the system and its environment. Consequently, the whole process of decoherence can be completely described within the framework of the quantum theory as a unitary process that governs the joint evolution of the quantum systems and its environment \textsuperscript{1} \cite{2–4}. There are plentiful theoretical models describing the decoherence within the framework of the standard quantum theory that have been in accordance with various experiments \cite{9,10}. These models either use Hamiltonian evolution of the composite system-plus-environment structure (the Hamiltonian itself is time-independent).

Alternatively, the description of decoherence can be based on a simple collision-like models, i.e. a sequence of

\footnote{Another possibility would be to include decoherence into the basic dynamical equation, i.e. to add a non-hamiltonian part into the Schrödinger equation \cite{8}. However, the modifications of the basic quantum dynamical law are out of scope of this paper.}
interactions between the object under consideration and particles from environment leads to decoherence. These models allow us to study microscopic dynamics of open systems, in which the flow of information from the system to the environment and creation of entanglement can be analyzed. In fact, collision models are equivalent to more general models of causal memory channels [11]. In this case, the memory is represented by the system under decoherence, whereas the reservoir (environment) plays the role of input/output systems.

In the present paper we will focus our attention on collision-like models of decoherence of qubits. Our first aim is to completely classify all possible decoherence channels of a qubit. The second task is to show that all decoherence maps of qubits can be modelled as sequences of collisions. The paper is organized as follows: Sections II and III are devoted to a description of general properties of all decoherence channels. In Sec. IV we present a generic collision-like model. In the Sec. V the master equations for collision models are derived and all possible master equations describing decoherence of a qubit are presented. In Sec. VI we analyze how entanglement is created during a sequence of collisions. Finally, in Sec. VII we summarize our results and formulate some open problems.

II. DECOHERENCE CHANNELS

The aim of this section is to classify all possible completely positive trace-preserving maps (quantum channels) that describe quantum decoherence. Let us denote by \( \mathcal{D} \) the set all maps \( \mathcal{E} \) satisfying the decoherence conditions, i.e.

\[
\langle \phi_e | \mathcal{E} | \phi_e \rangle = \langle \phi_e | \phi_e \rangle \quad \text{for all } k \tag{2.1}
\]

\[
| \langle \phi_e | \mathcal{E} | \phi_e \rangle | < | \langle \phi_e | \phi_e \rangle | \quad \text{for all } k \neq l, \tag{2.2}
\]

with \( \mathcal{B} = \{ |\phi_e \rangle \} \) being the decoherence basis. For our purposes it is useful to fix one basis \( \mathcal{B} \) and to analyze all decoherences (forming the set \( \mathcal{D}_B \)) with respect to this basis. The general decoherence maps are then just unitary rotations of elements from \( \mathcal{D}_B \), that correspond to a change of the decoherence basis. In particular, if \( \mathcal{E} \) is a decoherence map, then also \( \mathcal{E}' = U_1 \mathcal{E} U_2 \) is such a map. We used the notation \( U_j | \phi \rangle = U_j \phi U_j^\dagger \) with \( U_j \) unitary operators. From the definition it is clear that decoherence channels are unital (they preserve the total mixture, i.e. \( \mathcal{E}[I] = I \)) and are not strictly contractive (they might have more than a single fixed point).

Denoting by \( \mathcal{D}_B \) the set of all decoherence maps with respect to a fixed basis \( \mathcal{B} \) we can write \( \mathcal{D} = \cup_{B} \mathcal{D}_B \). Each decoherence map \( \mathcal{E} \in \mathcal{D} \) belongs only to one class \( \mathcal{D}_B \). Elements of \( \mathcal{D}_B \) and \( \mathcal{D}_B' \) are unitarily related, i.e.

\[
\mathcal{D}_B' = \{ \mathcal{E}' | \mathcal{E}'[\phi] := \mathcal{E}(U \phi U^\dagger), \mathcal{E} \in \mathcal{D}_B, B' = U B \} = \mathcal{D}_{UB}.
\]

This defines a new decoherence class only if \( B' \neq B \). That is, the unitary operation \( U \) does not commute with all projectors \( |e_k \rangle \langle e_k| \), or equivalently the basis \( B \) is not an eigenbasis of the transformation \( U \). If \([U, |e_k \rangle \langle e_k|]\) = 0 for all \( k \) then from a given \( \mathcal{E} \in \mathcal{D}_B \) we obtain different decoherence maps within the fixed set \( \mathcal{D}_B \).

A. Qubit decoherences

In what follows we will analyze the case of qubit decoherence channels. In this case the set \( \mathcal{D} \) has surprisingly simple form. We will use the so-called left-right notation, in which the evolution map is represented by a 4x4 matrix [12]. Let us choose the following operator basis

\[
S_0 = I; \quad S_1 = |\psi \rangle \langle \psi | + |\psi^\perp \rangle \langle \psi^\perp |;
S_2 = i|\psi \rangle \langle \psi^\perp | - i |\psi^\perp \rangle \langle \psi |;
S_3 = |\psi \rangle \langle \psi | - |\psi^\perp \rangle \langle \psi^\perp |, \tag{2.3}
\]

where \( \mathcal{B} = \{ |\psi \rangle, |\psi^\perp \rangle \} \) is the decoherence basis. The elements of \( \mathcal{B} \)-basis satisfy the same properties as the Pauli operators, because \( S_j = W \sigma_j W^\dagger \) with \( W \) being a unitary operation. In this basis the operators (states) take the form of four-dimensional vectors \( \rho = \frac{1}{2}(I + r \cdot \vec{S}) \leftrightarrow \vec{r} = (1, r_3) \), where \( r = \text{Tr}[\rho S_j] \). The evolution \( \mathcal{E} \) is described by 4x4 matrix with elements given by the equation \( \mathcal{E}_{ik} = \frac{1}{2} \text{Tr}(S_i \mathcal{E}[S_j]) \). Because of the trace-preservation we have \( \mathcal{E}_{00} = 1 \) and \( \mathcal{E}_{01} = \mathcal{E}_{02} = \mathcal{E}_{03} = 0 \). Consequently, we obtain the Bloch sphere representation [7] of the state space, in which the states are illustrated as points (three-dimensional real vectors \( \vec{r} \) lying inside a sphere with a unit radius. The action of \( \mathcal{E} \) corresponds to an affine transformation of the Bloch vector \( \vec{r} \), i.e.

\[
\vec{r} \rightarrow \vec{r}' = T \vec{r} + \vec{t}, \quad T_{jk} = \mathcal{E}_{jk} \quad \text{(for } j, k = 1, 2, 3 \text{)} \quad \text{and} \quad t_j = \mathcal{E}_{j0}.
\]

The translation vector \( \vec{t} \) describing the shift of the Bloch sphere (including its center, i.e. the total mixture) is related to the unitality of the channel. For unital maps \( \vec{t} = 0 \).

Diagonal elements of the state \( \rho \) are in this case associated with the mean value \( z = \text{Tr}[\rho S_3] \). The conservation of the diagonal elements implies that the corresponding components of \( \rho \) are preserved. Combining the unitality with this property we find the following form for decoherence maps

\[
\mathcal{E} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a & b & 0 \\
0 & c & d & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \tag{2.4}
\]

from where it follows that the set of all possible qubit decoherence maps is at most four-parametric.

Each unital map can be written as [12]

\[
\mathcal{E} \rho = R U_1 \Phi \rho R U_2 = U_1 \Phi \rho U_2^\dagger U_1^\dagger, \tag{2.5}
\]

where the role of input/output systems.
where $R_{U_1}, R_{U_2}$ are orthogonal rotations corresponding to unitary transformations $U_1, U_2$; $\Phi_{\mathcal{E}} = \text{diag}\{1, \lambda_1, \lambda_2, \lambda_3\}$ and $\lambda_j$ are the singular values of the matrix $\mathcal{E}$. In fact, the above relation is the singular-value decomposition of the matrix $\mathcal{E}$. The conditions of the complete positivity restricts the possible values of $\lambda_j$. In particular, the allowed points $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ must lie inside a tetrahedron with vertices that have coordinates $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, and $(-1, -1, 1)$, respectively.

Applying these facts to the decoherence map under consideration ($\mathcal{E}$ from Eq.(2.4)) we obtain that $\Phi_{\mathcal{E}} = \text{diag}\{1, \lambda_1, \lambda_2, 1\}$, i.e. $\lambda_3 = 1$. Let us note that in this case we use unitaries that do not change the decoherence basis, so we are still dealing with all decoherences that belong to a fixed basis $B$. The condition of complete positivity restricts the values to the points $\vec{\lambda} = (\lambda, \lambda, 1)$ with $-1 \leq \lambda \leq 1$, i.e. to a line connecting the two vertices of the tetrahedron representing the identity ($\lambda = 1$) and the unitary rotation $S_3$ ($\lambda = -1$). Consequently, the general decoherence channel $\mathcal{E} \in \mathcal{D}_B$ reads

$$\mathcal{E} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & c_1 & s_1 & 0 \\
0 & -s_1 & c_1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & c_2 & s_2 & 0 \\
0 & -s_2 & c_2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

(2.6)

where $s_j = \sin \varphi_j$ and $c_j = \cos \varphi_j$ represent rotations $R_{U_j}$ around the $z$-axis by an angle $\varphi_j$. From here it follows that a general decoherence map $\mathcal{E}$ takes the form

$$\mathcal{E} = \begin{pmatrix}
1 & 0 & \lambda \cos(\varphi_1 + \varphi_2) & \lambda \sin(\varphi_1 + \varphi_2) \\
0 & \lambda \cos(\varphi_1 + \varphi_2) & \lambda \sin(\varphi_1 + \varphi_2) & 0 \\
0 & -\lambda \sin(\varphi_1 + \varphi_2) & \lambda \cos(\varphi_1 + \varphi_2) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

(2.7)

and, consequently, it is specified only by two real parameters $a = \lambda \cos(\varphi_1 + \varphi_2)$ and $b = \lambda \sin(\varphi_1 + \varphi_2)$, i.e.

$$\mathcal{E} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a & b & 0 \\
0 & -b & a & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

(2.8)

As a result we obtain that any map $\mathcal{E}$ of the above form with the numbers $a, b$ satisfying the condition $a^2 + b^2 \leq 1$ is completely positive. Therefore we can conclude that the set of all decoherence maps of a qubit is characterized just by two parameters. Moreover, to obtain the decoherence (to secure the suppression of off-diagonal terms) the inequality must be strict, i.e. $a^2 + b^2 < 1$. Otherwise the map $\mathcal{E}$ describes a unitary rotation around the $z$ axis. Defining the rotation map

$$R_{\varphi} = \begin{pmatrix}
\cos \varphi & \sin \varphi \\
-\sin \varphi & \cos \varphi
\end{pmatrix},
$$

(2.9)

and using the relation $\varphi = \varphi_1 + \varphi_2$, we can write the most general decoherence channel ($\mathcal{E} \in \mathcal{D}_B$) in a very compact form

$$\mathcal{E} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda R_{\varphi} & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

(2.10)

This form is suitable for our purposes, because the powers of the map $\mathcal{E}$ read

$$\mathcal{E}^n = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda^n R_{n\varphi} & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

(2.11)

### III. STRUCTURAL PROPERTIES OF DECOHERENCE CHANNELS

In this section we will briefly review structural properties of the set of all possible decoherence completely positive maps $\mathcal{E}$. Let us denote this set by $\mathcal{D}$.

- **Convex structure**
  The set of all decoherence maps $\mathcal{D}$ is not convex, i.e. a convex combination of two decoherence channels $\mathcal{E}_\mu = \mu \mathcal{E}_1 + (1-\mu) \mathcal{E}_2$ is not again a decoherence channel. This is true except the case when the decoherence bases of $\mathcal{E}_1, \mathcal{E}_2$ coincide, i.e. the set $\mathcal{D}_B$ is convex. The extremal points of $\mathcal{D}_B$ correspond to unitary transformations. However, these are not elements of $\mathcal{D}_B$, because they do not not fulfill the second decoherence condition (2.2).

![FIG. 1. (Color online) The cube corresponds to all positive unital trace-preserving maps. The condition of complete positivity confines quantum channels into the tetrahedron with (generalized) Pauli matrices as vertices. In this picture the set of decoherence channels $\mathcal{D}_B$ forms a line connecting the points $I$ and $S_z$.](image)
We have already mentioned that for qubits the set of all possible $\Phi_E$ channels form a tetrahedron and up to unitary transformations each channel belongs to this tetrahedron. Those channels that correspond to decoherence maps form a line connecting the points $(1,1,1)$ and $(-1,-1,1)$. From this picture (see Fig. 1) the convexity of $D_B$ is transparent and also the extremal points can be easily identified as unitary channels. It follows that each decoherence map can be written as a convex sum of only two unitary channels. In fact all maps $\Phi_E$ for which one of the $\lambda$'s equals to unity and all the others are the same define a decoherence with respect to some basis. This means that all edges of the tetrahedron correspond to decoherence channels. It illustrates that the set $D$ as a whole is not convex, but is composed of a continuous number of "convex" subsets $D_B$ corresponding to each orthonormal basis $B$.

- Composition
A composition of two decoherence channels $E = E_1 \circ E_2$ is not, in general, a decoherence channel. So the set $D$ is not closed under the operation of multiplication. The channel $E$ belongs to $D$ only if the decoherence bases of $E_1$ and $E_2$ coincide, i.e. again only the sets $D_B$ are closed under the composition.

- Classical capacity
The decoherence basis is preserved by the decoherence map. Therefore it is possible to exploit these bases states to transmit the maximally possible amount of information, i.e. the capacity achieves its maximum $C = \log_2 d$ with $d = \dim H$.

- Tensor product
The tensor product of two decoherence maps describes a decoherence. However, $D_{12} \neq D_1 \otimes D_2$, because the decoherence basis of $E_1 \otimes E_2$ is always separable. The open problem is whether the whole set $D_{12}$ can be obtained from the sets $D_1, D_2$ by global unitary rotations. Properties of decoherence channels under tensor products is an interesting topic, which is related to our ability of controlling the decoherence. For example, how the decoherence of a sub-system affects characteristics of the whole system?

IV. COLLISION MODEL

In what follows we will study whether an arbitrary decoherence channel can be implemented via a sequence of bi-partite collisions. Each of the collisions is described by a unitary transformation $U$. Our task will be to derive all possible unitary transformations that force the system to decohere. Our analysis will be performed only for qubits, but up to technical details all results hold for qudits.

Let us consider that initially the system qubit is decoupled from an environment (reservoir) that is modelled as a set of qubits, i.e. $\Omega_{in} = \varrho \otimes \Xi_{res}$. Moreover, we will simplify the model by assuming that initially the reservoir qubits are in a factorized state $\Xi_{res} = \xi^\otimes N$ and each reservoir qubit interacts with the system qubit just once. In addition we assume that reservoir qubits do not interact between themselves. Under such conditions the evolution of the system qubit is induced by the sequence of maps $E_1 = \ldots = E_N \equiv E$. In particular, the state of the system after the $n$-th interaction equals to

$$\varrho^{(n)} = E_n \ldots E_1[\varrho] = E^n[\varrho], \quad (4.1)$$

where $E[\varrho] = \mbox{Tr}_{\mbox{res}}[U(\varrho \otimes \xi)U^\dagger]$. We will refer to this picture as to a collision model. The system qubit collides with reservoir qubits.

In order to obtain the decoherence channel, i.e.

$$\varrho \rightarrow \varrho^{(n)} = \begin{pmatrix} \varrho_{00} & \varrho_{12}^{(n)} \\ \varrho_{21}^{(n)} & \varrho_{11} \end{pmatrix}$$

with $\varrho_{12}^{(n)} = [\varrho_{21}^{(n)}]^* \rightarrow 0$ for $n$ goes to infinity, we have to ensure that the map $E$ preserves diagonal elements of each state $\varrho$ in a given (decoherence) basis.

In order to preserve the diagonal elements of pure states $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ (decoherence basis) the bi-partite unitary transformation $U$ must necessarily satisfy the relations

$$|00\rangle \rightarrow |0\psi\rangle;$$

$$|01\rangle \rightarrow |0\psi^+\rangle;$$

$$|10\rangle \rightarrow |1\phi^+\rangle;$$

$$|11\rangle \rightarrow |1\phi\rangle. \quad (4.2)$$

In what follows we will prove our main result that the class of possible bi-partite interactions that induce decoherence in collision models coincides with the set of all controlled-U transformations (the so-called U-processors as introduced in Ref. [13]), where the system under consideration plays the role of the control and the reservoir particle is a target. Certainly, we have to identify those transformations for which the off-diagonal elements of the system density operator do vanish in the limit of infinitely many collisions with reservoir particles.

The unitary bi-partite transformation (the controlled-U operation) defined by the relations (4.2) can be rewritten into the following operator form

$$U = |0\rangle\langle 0| \otimes V_0 + |1\rangle\langle 1| \otimes V_1, \quad (4.3)$$

where $V_0, V_1$ are unitary rotations of a reservoir qubit. In particular, $V_0 = |\psi\rangle\langle 0| + |\psi^+\rangle\langle 1|$ and $V_1 = |\phi^+\rangle\langle 0| + |\phi\rangle\langle 1|$. Thus, the initial state $\Omega = \varrho \otimes \xi$ of a bi-partite system evolves according to a transformation

$$\Omega \rightarrow \Omega' = U\Omega U^\dagger = \sum_{j,k=0} \varrho_{jk} |j\rangle \langle k| \otimes V_j \xi V_k^\dagger \quad (4.4)$$
and by performing the partial trace over the reservoir qubit we obtain the induced map

$$\rho \rightarrow \rho' = \mathcal{E}[\rho] = \text{Tr}_p \Omega' = \sum_{j,k=0}^1 \varrho_{jk} \text{Tr}[V_j \xi V_k^\dagger] |j\rangle\langle k|$$

$$= \text{diag}[\varrho] + \varrho_{01} (\langle X \rangle\xi |0\rangle\langle 1| + \varrho_{10} (\langle \mathbf{X}^\dagger \rangle\xi |1\rangle\langle 0|)$$,

where $X = V_1^\dagger V_0$ and $\langle X \rangle\xi = \text{Tr}[X \xi]$ stands for the mean value of the operator $X$ in the state $\xi$.

Applying the transformation $\mathcal{E}$ in a sequence of $n$ collisions the state of the system qubit is described by the density operator

$$\rho^{(n)} = \mathcal{E}^n[\rho] = \text{diag}[\varrho] + \varrho_{01} (\langle X \rangle\xi |0\rangle\langle 1| + \varrho_{10} (\langle \mathbf{X}^\dagger \rangle\xi |1\rangle\langle 0|)$$,

from where we can conclude, that providing $\langle |X|\rangle\xi < 1$ and $\langle |\mathbf{X}^\dagger|\rangle\xi < 1$ the off-diagonal terms vanish. However, because $XX^\dagger = \mathbf{X}^\dagger X = I$, i.e. $X$ is unitary, its eigenvalues are just complex square roots of the unity. Therefore, for the eigenvectors of $X$ the off-diagonal terms do not tend to zero.

The fact that for convex combinations of the eigenvectors the off-diagonal elements still vanish might sound counterintuitive. But it can be seen from the following consideration: Let us denote by $e^{i\varphi}$ and $e^{i\eta}$ the eigenvalues of $X$ associated with the eigenvectors $|f_1\rangle$ and $|f_2\rangle$, respectively. Then the mean value $\langle X \rangle\xi$ for the convex combination $\xi = a |f_1\rangle\langle f_1| + (1-a) |f_2\rangle\langle f_2|$ equals to $\langle X \rangle\xi = e^{i\varphi} a + e^{i\eta} (1-a)$. The condition $\langle X \rangle\xi < 1$ can be rewritten as the inequality $2a(1-a)[1-\cos(\varphi - \eta)] < 0$, which is satisfied only if $\cos(\varphi - \eta) \neq 1$, or $a \neq 0$ and $a \neq 1$. The latter property means that $\xi$ is the eigenstate.

The first property requires $\varphi = \eta$, i.e. the operator $X$ is proportional to the identity, $X = e^{i\varphi} I$. However, under this assumption $V_1 = e^{i\varphi} V_0$, i.e. we have no interaction and $U = (e^{i\varphi}|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes V_1$. Hence, we can conclude that whenever the reservoir state is not an eigenstate of $X$ and the interaction is not trivial, the described collision model with controlled-$U$ interaction forces the system to decohere.

It is straightforward to show that unitary interactions $U = |0\rangle\langle 0| \otimes V_0 + |1\rangle\langle 1| \otimes V_1$ induce maps of the left-right form (see Eq.(2.8)) with the parameters

$$a = \frac{1}{2} (\langle X \rangle\xi + \langle \mathbf{X}^\dagger \rangle\xi)$$

$$b = \frac{i}{2} \left( \langle X \rangle\xi - \langle \mathbf{X}^\dagger \rangle\xi \right)$$

or, equivalently, $\langle X \rangle\xi = \lambda e^{i\varphi}$. So given a decoherence map $\mathcal{E}$ one can, in principle, find an interaction $U$ and an initial state of the reservoir qubits $\xi$, such that the desired decoherence process is implemented via a sequence of collisions.

V. MASTER EQUATION

In this section we will derive a master equation that describe the decoherence process induced by collisions of the system qubit with reservoir particles. Although the studied decoherence model is intrinsically discrete, we will show that we can perform a continuous-time approximation that enable us to write down the master equation (see, e.g. [14]).

As shown in the previous section the collision model is described by a set of maps $\mathcal{E}_n = \mathcal{E}^n$ that form a discrete semigroup, i.e. $\mathcal{E}_n \mathcal{E}_m = \mathcal{E}_{n+m}$ for all integer $m$, $n$ and $\mathcal{E}_0 = I$. The question is whether we can introduce a continuous one-parametric set of transformations $\mathcal{E}_t$ such that $\mathcal{E}_{nt} = \mathcal{E}_n$ for $n = t \tau$ (as is a time scale roughly corresponding to the time interval between two interactions). It turns out that a simple relation $n \rightarrow t/\tau$ can be used to accomplish the task. The obtained continuous set of transformations $\mathcal{E}_t$ will be used to derive the generator $\mathcal{G}$ of the dynamics by using a simple formula $\mathcal{G}_t = \mathcal{E}_t \mathcal{E}_t^{-1}$.

With the help of results from Sec. III (namely, Eq.(2.11)) we can directly write

$$\mathcal{E}_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda R \varphi & 0 & 0 \\ 0 & 0 & \lambda^2 R \varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ,$$

(5.1)

where for simplicity we set the time scale $\tau = 1$. It is easy to see that the one-parametric set of transformations $\mathcal{E}_t$ possesses the semigroup property, i.e. $\mathcal{E}_t \mathcal{E}_s = \mathcal{E}_{t+s}$, for all real $t$, $s$. It means that the generator and the associated master equation will be of the Lindblad form [15], i.e. the process under consideration is Markovian.

The corresponding generator reads

$$\mathcal{G} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \ln \lambda & -\varphi & 0 \\ 0 & \varphi & \ln \lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

(5.2)

where we used the identity $R_t \varphi = \varphi R_t \varphi + \pi / 2$ and $\frac{\pi}{2!} [\lambda^t R_{\varphi} \lambda^{-t} R_{-\varphi}] = \ln \lambda R_t + \varphi R_t / 2$. This step can be performed only if $\lambda$ is non-negative (i.e., when the logarithm is defined), which, in general, is not the case. The parameter $\lambda$ belongs to the open interval $(-1, 1)$. Consequently, it seems that the generator cannot be derived in all cases. However, using the equality $-\lambda R_{\varphi} = \lambda R_{\varphi + \pi}$ for $\lambda$ nonnegative, one can write $[\lambda^t R_{\varphi + \pi}]$ instead of $\lambda^t R_{\varphi}$ in the expression for $\mathcal{E}_t$ with $\lambda < 0$. Then the generator is slightly different and contains the term $\varphi + \pi$ instead of $\varphi$, and $\ln |\lambda|$ instead of $\ln \lambda$. Thought this is not a problem, because in terms of parameters of the collision model $\langle X \rangle\xi = \lambda e^{i\varphi}$, i.e. the parameter $\lambda = |\langle X \rangle|\xi$ is always positive. Therefore we can consider the generator $\mathcal{G}$ as the most general one.

The general master equation in Lindblad form reads

$$\dot{\rho}_i = \mathcal{G}[\rho_i] = -i[H, \rho_i] + \frac{1}{2} \sum_{a,b} c_{ab}([S_a, \rho_i S_b] + [S_a \rho_i, S_b]) .$$
If the numbers $c_{ab}$ are time-independent and form a positive matrix, then the generated evolution is Markovian and satisfies the semigroup property. To find the values of the coefficients $c_{ab}$ we will use the following relations (see Ref. [14])

$$h_1 = \frac{[\mathcal{G}]_{32} - [\mathcal{G}]_{23}}{4} ; \quad h_2 = \frac{[\mathcal{G}]_{13} - [\mathcal{G}]_{31}}{4} ; \quad h_3 = \frac{[\mathcal{G}]_{23} - [\mathcal{G}]_{12}}{4} ;$$
$$e_{23} = \frac{[\mathcal{G}]_{10}}{4} ; \quad e_{31} = \frac{[\mathcal{G}]_{20}}{4} ; \quad e_{12} = \frac{[\mathcal{G}]_{30}}{4} ;$$
and

$$d_{11} = \frac{[\mathcal{G}]_{11} - [\mathcal{G}]_{22} - [\mathcal{G}]_{33}}{4} ; \quad d_{12} = \frac{[\mathcal{G}]_{12} + [\mathcal{G}]_{21}}{4} ;$$
$$d_{22} = \frac{[\mathcal{G}]_{22} - [\mathcal{G}]_{11} - [\mathcal{G}]_{33}}{4} ; \quad d_{23} = \frac{[\mathcal{G}]_{23} + [\mathcal{G}]_{32}}{4} ;$$
$$d_{33} = \frac{[\mathcal{G}]_{33} - [\mathcal{G}]_{11} - [\mathcal{G}]_{22}}{4} ; \quad d_{13} = \frac{[\mathcal{G}]_{13} + [\mathcal{G}]_{31}}{4} ,$$

(5.4)

where $[\mathcal{G}]_{kl}$ correspond to matrix elements of the generator $\mathcal{G}$, $c_{ab} = d_{ab} - i e_{ab}$ and $H = \sum_a h_a S_a$. Note that $d_{ab}$ form a symmetric matrix and $e_{ab}$ is an antisymmetric matrix.

Using these expressions one finds that the non-vanishing parameters are

$$h_3 = \frac{1}{2} \varphi ; \quad d_{33} = - \frac{1}{2} \ln \lambda$$

(5.5)

and the corresponding master equation reads

$$\dot{\rho}_t = -i \frac{\varphi}{2} [S_3, \rho_t] - \frac{\ln \lambda}{2} (S_3 \rho_t S_3 - \rho_t) .$$

(5.6)

A typical evolution driven by this equation is depicted in Fig. 2.

![Bloch sphere](image)

**FIG. 2.** (Color online) The decoherence of a qubit governed by Eq.(5.6). The Bloch sphere that represents the initial state space of a qubit is mapped into the line connecting the decoherence basis states. On the right the evolution of Bloch-vector components for two different initial states is depicted.

Let us now address the following question: Is there any other master equation describing a decoherence of a qubit? The preservation of the $S_z$ component (determined by the decoherence basis) together with the unitarity of the transformation implies that

$$\mathcal{G} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

(5.7)

The corresponding matrix $C = \frac{1}{2} [c_{ab}]$ then reads

$$C = \frac{1}{4} \begin{pmatrix} a - d & c + b & 0 \\ c + b & d - a & 0 \\ 0 & 0 & -a - d \end{pmatrix} .$$

(5.8)

This matrix is positive only when $a = d$ and $b = -c$. Moreover, $a$ must be negative. These restrictions leave only a single element that does not vanish, namely, $c_{33} = -a/2$. Consequently, the Hamiltonian part takes non-vanishing value for $h_3 = b/2$. Therefore the family of all master equations describing the decoherence is only two-parametric

$$\dot{\rho}_t = -i \frac{b}{2} [S_3, \rho_t] - \frac{a}{2} (S_3 \rho_t S_3 - \rho_t) .$$

(5.9)

This general master equation is of the same form as the one derived for the collision model (5.6). The parameters $\lambda, \varphi$ are related to the parameters of the underlying unitary interaction via the formula $\langle X \rangle_{\xi} = \lambda e^{i \varphi}$. Let us note the constraint $\lambda = |\langle X \rangle_{\xi}| \in [0, 1]$, since $X$ is unitary. Therefore $\ln \lambda \leq 0$ as it is required by the condition on possible values of $a$.

**VI. ENTANGLEMENT IN DECOHERENCE VIA COLLISIONS**

We start with definitions of entanglement quantities that we will evaluate. Let us denote the joint state of the system of $N+1$ qubits (the system qubit and $N$ reservoir qubits) by $\Omega$. The bipartite entanglement shared between a pair of qubits $j$ and $k$ can be quantified in terms of the concurrence [16]

$$C_{jk} = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} ,$$

(6.1)

where $\lambda_i$ are decreasingly ordered square roots of the eigenvalues of the matrix $R_{jk} = \rho_{jk} \sigma_y \otimes \sigma_y \rho_{jk} \sigma_y \otimes \sigma_y$ and $\rho_{jk} = \text{Tr}_{\bar{k}} \Omega$ is the state of two qubits under consideration.

The case of multi-partite entanglement is a more complex phenomenon and there is no unique way of its quantification. Fortunately, for pure multi-qubit systems there is an accepted method of characterization (identification) of intrinsic multi-partite entanglement. Specifically, let us consider how strongly the $j$-th qubit is correlated with the rest of qubits in the multi-partite system.
This degree of entanglement can be quantified via the so-called tangle (see Ref. [17])
\[
\tau_{j} = 4 \det g_{j} = 2(1 - \text{Tr} \Omega^{2}) , \tag{6.2}
\]
where \(g_{j} = \text{Tr}_{2} \Omega\) is the state of the \(j\)-th qubit. Then we evaluate bi-partite concurrences between the given \(j\)-th qubit and any other qubit in the system, i.e. we evaluate \(N\) quantities \(C_{jk}\).

Wooters and his coworkers have found (see Ref. [17]) that for pure three-qubit states the inequalities
\[
\sum_{j \neq k} |C_{jk}|^{2} \leq \tau_{k} \quad \forall k = 1, 2, 3 , \tag{6.3}
\]
hold. In addition they have conjectured that such inequalities also hold for any number of qubits. This conjecture (to so-called Coffman-Kundu-Wooters (CKW) inequality) has been recently proved by Osborne [18]

These inequalities quantify the property which is known as the monogamy of entanglement (the entanglement cannot be shared freely in multipartite systems).

As a consequence of the CKW inequality one can define a measure of intrinsic multipartite entanglement \(\Delta_{j}\) as
\[
\Delta_{j} = \tau_{j} - \sum_{k \neq j} \tau_{jk} , \tag{6.4}
\]
where we have used the notation \(\tau_{jk} = |C_{jk}|^{2}\). It is important to note that in the multi-partite case (in particular for more than three qubits) the differences \(\Delta_{j} := \tau_{k} - \sum_{j \neq k} \tau_{jk}\) take different values for different \(j\). Therefore, a weighted sum \(\Delta = \frac{1}{n} \sum_{j} \Delta_{j}\) is an appropriate measure of an intrinsic multipartite entanglement. Based on this quantity we can argue that there are multi-partite entangled states for which the entanglement has purely bi-partite origin, as for example the family of \(W\) states [20] that saturate the CKW inequalities, i.e. \(\Delta = 0\).

Let us assume that the system qubit is initially prepared in the state \(|\psi\rangle = a|0\rangle + b|1\rangle\) and each qubit of the reservoir is in a pure state \(|\psi\rangle\), i.e. the joint initial state is \(|\Omega_{0}\rangle = |\psi\rangle \otimes |\psi\rangle^{\otimes N}\). After \(n\) collisions governed by bi-partite controlled unitary operations (4.3) the whole system evolves into the state
\[
|\Omega_{n}\rangle = |a|0\rangle \otimes |V_{0}\psi\rangle^{\otimes n} + b|1\rangle \otimes |V_{1}\psi\rangle^{\otimes n} \otimes |\psi\rangle^{(N-n)} . \tag{6.5}
\]

In order to be able to evaluate the entanglement quantities we have to specify all two-qubits and single-qubit density operators. In particular, for \(k \leq n, j \leq k\) the bi-partite states are given by expressions
\[
\varrho_{0k}(n) = |a|^{2}(|0\rangle\langle 0| + |b|^{2}|1\rangle\langle 1| + ab^{*}|0\rangle\langle 1| + c.c. ; \tag{6.6}
\]
\[
\varrho_{jk}(n) = |a|^{2}|0\rangle\langle 0| + |b|^{2}|1\rangle\langle 1| + c.c. ; \tag{6.7}
\]
where we used the notation \(|\psi_{0}\rangle = V_{0}|\psi\rangle\) and \(|\psi_{1}\rangle = V_{1}|\psi\rangle\). The single qubit states are as follows:
\[
\varrho_{0}(n) = |a|^{2}|0\rangle\langle 0| + |b|^{2}|1\rangle\langle 1| + ab^{*}|0\rangle\langle 1| + c.c. \quad \text{describes the system qubit after } n\text{-th collision, and}
\]
\[
\varrho_{k}(n) = |a|^{2}|\psi_{0}\rangle\langle \psi_{0}| + |b|^{2}|\psi_{1}\rangle\langle \psi_{1}| \quad \text{describes the } k\text{-th qubit of the reservoir after the collision with the system qubit. Evaluation of the tangles is straightforward and results in expressions}
\]
\[
\tau_{0}(n) = 4|a|^{2}|b|^{2}(1 - |\langle \psi_{0}|\psi_{1}\rangle|^{2}) ; \tag{6.9}
\]
\[
\tau_{k}(n) = 4|a|^{2}|b|^{2}|\langle \psi_{0}|\psi_{1}\rangle|^{2} ; \tag{6.10}
\]
\[
\tau_{0k}(n) = 4|a|^{2}|b|^{2}|\langle \psi_{0}|\psi_{1}\rangle|^{2(n-1)}|\langle \psi_{0}|\psi_{1}\rangle|^{2} ; \tag{6.11}
\]
\[
\tau_{jk}(n) = 0 . \tag{6.12}
\]
One can directly verify the validity of the CKW inequalities
\[
\sum_{k=0, k \neq j}^{N} \tau_{jk}(n) = \tau_{0}(n) = 4|ab|^{2}|\langle \psi_{0}|\psi_{1}\rangle|^{2(n-1)}|\langle \psi_{0}|\psi_{1}\rangle|^{2} \leq 4|ab|^{2}|\langle \psi_{0}|\psi_{1}\rangle|^{2} = \tau_{j}(n) ; \tag{6.13}
\]
\[
\sum_{k=1}^{N} \tau_{0k}(n) = n \times 4|ab|^{2}|\langle \psi_{0}|\psi_{1}\rangle|^{2(n-1)}|\langle \psi_{0}|\psi_{1}\rangle|^{2} \leq 4|ab|^{2}(1 - |\langle \psi_{0}|\psi_{1}\rangle|^{2n}) = \tau_{0}(n) , \tag{6.14}
\]
where we have used the relations \(|\langle \psi_{0}|\psi_{1}\rangle|^{2} = 1 - |\langle \psi_{0}|\psi_{1}\rangle|^{2} = 0.25\).

\[\text{FIG. 3. (Color online) The behavior of entanglement as a function of number } n \text{ of collisions between the system qubit and reservoir qubits. The degree of entanglement between the system qubit and } n \text{ reservoir qubits after } n \text{ collisions is given by } \tau_{0} - \text{it increases with the number of collisions (time) to a steady-state value. On the contrary, all reservoir qubits after their interaction with the system qubit are entangled with the constant degree of entanglement (see the tangle } \tau_{k}. \text{The bi-partite entanglement } \tau_{0k} \text{ (the square of the concurrence } C_{0k}) \text{ is zero until the } k\text{-th reservoir qubit collides with the system qubit. After the collision the entanglement takes a non-zero value, though it decreases due to subsequent collisions of the system qubit with other reservoir qubits. It is interesting to note that all } \tau_{0k}(n) \text{ for } n \geq k \text{ are described by the same function. We assume the following initial state of the system qubit } |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \text{ and } |\langle \psi_{0}|\psi_{1}\rangle|^{2} = 0.75.\]
In the limit of large number of interactions \((n \to \infty)\) all two-qubit correlations vanish (i.e. finally there is no bi-partite entanglement between qubits in reservoir), but the entanglement between the system qubit and the whole reservoir converges to a finite value

\[
\tau_0 \to 4|ab|^2 ; \\
\tau_{0k} \to 0 .
\] (6.15) (6.16)

It means that after the process of decoherence the system qubit is not entangled with the reservoir via bipartite entanglements, but is entangled to the reservoir via multi-partite correlations. The final state belongs to the family of Greenberger-Horn-Zeilinger states that exhibit purely multi-partite correlations.

From the above one can see how the entanglement is related to the decoherence. Given the relation \(|\psi_0\rangle|\psi_1\rangle = |\langle X|\psi\rangle| = \lambda\) we conclude that the decoherence rate restricts the maximum amount of created entanglement and simultaneously it determines the decrease of entanglement with the number of collisions.

\[\xi = \frac{\ln \lambda}{\gamma} = \frac{\ln \lambda}{\lambda_{eff}}\]

\(\lambda_{eff}\) is a decoherence rate that is adjusted by a suitable choice of the interaction \(U\).

\[|\psi_{env}\rangle = \left[|a|^2|\psi_0\rangle|\psi_1\rangle\right]^{\otimes n} + \left[|b|^2|\psi_0\rangle|\psi_1\rangle\right]^{\otimes n} \otimes |\psi\rangle|\psi\rangle^{\otimes (N-n)} ,
\]

where \(\lambda_{eff}\) is the maximum amount of created entanglement in any model with the controlled-\(U\) bi-partite collisions (the system particle plays the role of the control while particles from the reservoir are targets) a decoherence of qudits can be described as well.

\[\omega_{env}(n) = \text{Tr} \Omega_n \langle \Omega_n | = \left[|a|^2|\psi_0\rangle|\psi_0\rangle\right]^{\otimes n} + \left[|b|^2|\psi_0\rangle|\psi_1\rangle\right]^{\otimes n} \otimes |\psi\rangle|\psi\rangle^{\otimes (N-n)} ,
\]

\[\omega_{env}(n) = \frac{1}{2}\gamma [H, [H, \rho]] ,
\]

where we use the notation \(H = \frac{1}{2}S_3 = \frac{\omega}{2}S_3\). We have shown that the double commutator term is well known and usually appears in decoherence models even for higher-dimensional systems. For example, Milburn in his work on intrinsic decoherence and the Schrödinger equation exactly in the form (7.1).

\[\omega_{env}(n) = \frac{1}{2}\gamma [H, [H, \rho]] ,
\]

We have shown that the decoherence in the collision model is accompanied (caused) by a creation of entanglement between the system and the reservoir. Unlike in the process of homogenization described in [19–21], in which the created entanglement saturates the CKW inequalities, in the case of decoherence the entanglement results in the Greenberger-Horn-Zeilinger type of correlations [22]. This means that decoherence process (as described by our collision model) does not create an entanglement between the environment particles. Specifically, if we trace over the system qubit (which decoheres) in the nth step of the evolution (see Eq.(6.5)), we find that the environment is in a separable state.

\[\omega_{env}(n) = \frac{1}{2}\gamma [H, [H, \rho]] ,
\]

where all the parameters are specified in Sec. VI. The decoherence rate \(\lambda\) and the rotation parameter \(\varphi\) can be adjusted by a suitable choice of the interaction \(U\) and the state of the reservoir \(\xi\). The collision model reflects microscopic origins of both these parameters that enter the decoherence master equations. The eigenvalues of
the Hamiltonian $H$ are given by the value of $\varphi$ and the parameter $\gamma$ is specified by both these parameters. The eigenvectors of $H$ form the decoherence basis.

We have shown explicitly that an arbitrary decoherence channel for a qubit can be represented via the collision model with a particularly chosen controlled-$U$ interaction. However, this result holds for arbitrary dimension (i.e., for qudits) as well. Let remind us that an arbitrary decoherence channel (represented by a unitary transformation) must be of the form of the controlled-$U$ operation. An open question is whether each decoherence master equation (even for $\dim H = \infty$) can be derived from the collision model. Knowing a decoherence master equation (i.e., knowing a generator $G$) it is easy to “fix” a time step $t = \tau$ and define $E_t = \mathcal{E}$. This map is for sure a decoherence channel and can be realized by a collision $U$. By applying this “elementary” map many times (a sequence of collisions) we obtain a discrete semigroup of the powers of $\mathcal{E}$.

The inverse task is trickier, that is, how do we interpolate between these discrete sequence of transformations (parameterized by number of collisions) to obtain a continuously parameterized channel. From a construction of the problem we know that the solution exists (we have started our analysis from the master equation). The question is whether this interpolation for qudit channels can be performed as easily as for qubits, i.e., by replacing the discrete powers of $n$ with continuous parameter $t$. Nevertheless, given the fact that we have started with a continuous set of channels $\mathcal{E}_t$ and by replacing $t \rightarrow \tau$ we obtained $\mathcal{E}_t = \mathcal{E}$. Consequently, it is possible to replace $n \rightarrow t/\tau$ to obtain the original continuous semigroup of decoherence channels $\mathcal{E}_t$. As a result we have found that a collision model can be used not only to describe any decoherence master equation, but can also be used to describe any quantum evolution governed by the Lindblad equation. On the other hand, it has to be stressed that collision models describe evolutions that might not be “interpolated” by continuous semigroup of quantum channels.

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Mathematically, this is related to the property of infinite divisibility of the matrix $\mathcal{E}$, i.e. to the possibility to calculate all real powers.

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6. M. Schlosshauer, *Decoherence, the measurement problem, and interpretations of quantum mechanics*, Rev. Mod. Phys. 76, 1267 (2004); see also quant-ph/0312059

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