Stability of Pairwise Entanglement in a Decoherent Environment

Jian-Ming Cai, Zheng-Wei Zhou and Guang-Can Guo

Key Laboratory of Quantum Information,
University of Science and Technology of China (CAS),
Hefei 230026, People’s Republic of China.

Abstract

Consider the dynamics of a two-qubit entangled system in the decoherence environment, we investigate the stability of pairwise entanglement under decoherence. We find that for different decoherence models, there exist some special class of entangled states of which the pairwise entanglement is the most stable. The lifetime of the entanglement in these states is larger than other states with the same initial entanglement. In addition, we also investigate the dynamics of pairwise entanglement in the ground state of spin models such as Heisenberg and XXY models.

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I. INTRODUCTION

Quantum entanglement is nonlocal and excess-classical correlation between separate parties, which is a most important character of quantum mechanics \[1\]. Lots of interest has focused on the nature of entanglement and the structure of entangled states \[2\]. Besides, entanglement is also the most central and indispensable resource in quantum information processing such as quantum computation \[2\], quantum teleportation \[3\], quantum dense coding \[4\], and quantum key distribution \[5\]. In a word, quantum entanglement is not only importance in theory but also in practical applications.

One the other hand, it is well known that decoherence \[6\] is a vital factor that should not be neglected in quantum information processing. The coupling between any quantum system and its environment is inevitable. Thus the entanglement will evidently be reduced and even disappear because of this system-environment coupling, e.g. in a large scale quantum computer or during the course of entanglement distribution via noisy channels. The stability of entanglement depends on the initial entangled system – its entanglement structure and its size \[7, 8, 9, 10\]. We may look on entanglement as a bond between different qubits, just like the chemical bond between different atoms. People studied the behavior of chemical bonds in different environment to understand how chemical bonds are formed. Thus investigating the entanglement dynamics of different types of entangled states in the decoherence environment may help to gain some insight into the properties of the decoherence and the entanglement, which will provide useful hints for maintaining entanglement. And what kind of entanglement bond is the most stable under different decoherence models is an interesting problem.

In this paper, we investigate the evolution of pairwise entanglement for two-qubit entangled states in the decoherence model which is described by general Pauli channels. We use the concurrence of Wootters \[11\], which is related straightforwardly to the entanglement of formation (EOF), as the measure of entanglement for two-qubit entangled states. The most interesting problem is that given some general decoherence model, what kind of entangled states can maintain entanglement best. We find that with the same initial entanglement, the lifetime of entanglement in some specific class of entangled states is the longest. For a special decoherence model, that is depolarizing channels, all pure states together with some mixed states, which we call as Decoherence Path States (DPS) is the most entanglement-stable.
We present the analytic dynamics of two-qubit entanglement for these special entangled states. Furthermore, we also study the stability of the nearest neighbor entanglement in the ground state of some spin models such as Heisenberg and XXY model. Coincidentally, the conclusion is that in some noise models, the nearest neighbor entanglement in the ground state is also the most stable, though it is not maximized [12].

The paper is organized as follows. In Sec. II we introduce the entanglement measure of two-qubit entanglement and the decoherence model, which can be viewed as a completely positive map. In Sec. III we investigate the dynamics of two-qubit entanglement under the influence of decoherence and try to find the special entangled states of which the pairwise entanglement is the most stable. In Sec. VI we examined the evolution of entanglement for some specific and maybe important mixed states, e.g. the ground states of spin models and the maximally entangled mixed states etc. In Sec. IV conclusions and discussions, together with some interesting open questions are presented.

II. ENTANGLEMENT MEASURE AND DECOHERENCE MODEL

There have been a number of measures for two-qubit entanglement, such as the entanglement of formation [13, 14], negativity [15] and relative entropy of entanglement [16] etc. In this paper, we adopt the well-established measure of entanglement concurrence as the measure of two-qubit entanglement. Consider a general two-qubit state, the density matrix is \( \rho \). Then its time-reversed matrix is defined as

\[
\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)
\]

The concurrence of \( \rho \) is given by [11]

\[
C = \max \left\{ 0, \sqrt{\lambda_1 - \lambda_2} - \sqrt{\lambda_3 - \lambda_4} \right\}
\]

where \( \lambda_i \)s are the eigenvalues of \( \rho \tilde{\rho} \) in decreasing order. The corresponding entanglement of formation can be evaluated as

\[
\xi (C) = h \left( \frac{1 + \sqrt{1 - C^2}}{2} \right)
\]

where \( h (x) = -x \log_2 x - (1 - x) \log_2 (1 - x) \) is the Shannon’s entropy function.

However, it is not a very simple task to calculate the concurrence of a two-qubit system in an analytic way. Here we adopt a new method of calculating the entanglement of formation
and thus the concurrence, which is based on Lorentz singular-value decomposition \[17\]. For an arbitrary \(2 \times 2\) state \(\rho\), there exists a \(4 \times 4\) matrix with elements \(R_{ij} = \text{Tr}(\rho \sigma_i \otimes \sigma_j)\). In the real \(R\)-picture, the density matrix \(\rho\) can be written as

\[
\rho = \frac{1}{4} \sum_{ij=0}^3 R_{ij} \sigma_i \otimes \sigma_j
\]  

where \(\{\sigma_i\}\) are the Pauli matrices.

**Lemma 1:** The \(4 \times 4\) matrix \(R\) can be decomposed as \(R = L_1 \Sigma L_2^T\), with \(L_1, L_2\) finite proper orthochronous Lorentz transformations given by \(L_1 = T(A \otimes A^*)T^\dagger\), \(L_2 = T(B \otimes B^*)T^\dagger\), where \(T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & 1 & 1 & \cdot \\ \cdot & i & -i & \cdot \\ 1 & \cdot & \cdot & -1 \end{pmatrix}\). The normal form \(\Sigma\) is either of real diagonal form \(\Sigma = \text{diag}[s_0, s_1, s_2, s_3]\) with \(s_0 \geq s_1 \geq s_2 \geq |s_3|\), or of the form

\[
\begin{pmatrix}
  a & \cdot & \cdot & b \\
  \cdot & d & \cdot & \cdot \\
  \cdot & d & \cdot & \cdot \\
  c & \cdot & b + c - a & \cdot
\end{pmatrix}
\]

with \(a, b, c, d\) real. And the Lorentz singular values of the second normal form are given by \([s_0, s_1, s_2, s_3] = [\sqrt{(a - b)(a - c)}, \sqrt{(a - b)(a - c)}, d, -d]\).

**Lemma 2:** Given a state \(\rho\) and the corresponding matrix \(R\), the concurrence of \(\rho\) is \(C = \max \{0, (-s_0 + s_1 + s_2 - s_3)/2\}\) depending on the Lorentz singular values of \(R\). And \(s_0 - s_1 - s_2 + s_3 = \min_{L_1, L_2} \text{Tr}(L_1 RL_2^T)\), where \(L_1, L_2\) are proper orthochronous Lorentz transformations.

Based on the above two useful lemmas, we can see that the concurrence of a given density matrix \(\rho\) is only determined by the Lorentz singular values of the corresponding \(R\)-matrix, which are the only invariants of a state under determinant 1 SLOCC operations \[17\]. In the following, we adopt this method for calculating the residual entanglement of an initial entangled state in the decoherence environment. It is shown that the influence of the environment on the pairwise entanglement is reflected by changing the Lorentz singular values. Before proceeding to the details, we first introduce the decoherence model generated by Pauli operators.
For a general decoherence model, it can be denoted as a completely positive map with an operator-sum representation. The effect of the general Pauli channels on a qubit $\rho$ is described as follows

$$\varepsilon(\rho_i) = p_0 \rho + \sum_{i=1}^{3} p_i \sigma_i \rho \sigma_i$$

where $p_i \geq 0$, $\sum_{i=0}^{3} p_i = 1$, and $\sigma_i$ are Pauli operators. This decoherence model includes some representative noise channels. When $p_1 = p_2 = p_3$ it is just the depolarizing channel, which describes the decoherence process related to the couplings of quantum system to the thermal reservoir in the large temperature limit. And when $p_1 = p_2 = 0$ the noise model is the dephasing channel, without energy exchange between the system and the environment, and only lose phase information. These kinds of decoherence models are common in several physical systems. In the rest of this paper, we will investigate how the decoherence will influence the pairwise entanglement in details.

III. DYNAMICS OF PAIRWISE ENTANGLEMENT UNDER DECOHERENCE

A. General Pauli Channels

We first assume that each qubit is independently coupled to the environment. The environment is characterized by the noisy channels generated by Pauli operators as shown in Eq (6). The initial state $\rho$, associated with this state $R = L_1 \Sigma L_2^T$, is an entangled two-qubit state. Then after some time, $\rho$ will be transformed to another state $\rho'$ with much less entanglement, due to the action of the noisy channels, that is $\rho' = \varepsilon_1 \varepsilon_2(\rho)$. According to Eq (6), it can be obtained easily that

$$\rho' = \sum_{i,j=0}^{3} (M_i \otimes N_j) \rho (M_i \otimes N_j)^\dagger$$

where $M_i = \sqrt{p_i} \sigma_i$ and $N_j = \sqrt{p_j} \sigma_j$. Now we can transformed this map into the $R$-picture. Denote the $R$-matrix associated with the state $\rho'$ as $R'$, then

$$R' = \sum_{i,j=0}^{3} L_{M_i} R L_{N_j}^T = (\sum_{i=0}^{3} L_{M_i}) R (\sum_{j=0}^{3} L_{N_j}^T)$$

where $L_{M_i}$ and $L_{N_j}$ are Lorentz transformations given by $L_{M_i} = T(M_i \otimes M_i^\dagger)T^\dagger$ and $L_{N_j} = T(N_j \otimes N_j^\dagger)T^\dagger$. 
For simplification, we can introduce $L_1 = \sum_{i=0}^{3} L_{M_i}$ and $L_2 = \sum_{j=0}^{3} L_{N_j}$. Therefore, the state evolution under decoherence is simply characterized by $R' = L_1 R L_2^T$ in the real $R$-picture. For the decoherence model we discussed here, $L_1 = \sum_{i=0}^{3} T(M_i \otimes M_i^*)T^\dagger$, $L_2 = \sum_{j=0}^{3} T(N_j \otimes N_j^*)T^\dagger$. After simple calculation, it can be seen that $L_1 = L_2 = diag[1, Q_1, Q_2, Q_3]$, where $Q_1 = p_0 + p_1 - p_2 - p_3$, $Q_2 = p_0 - p_1 + p_2 - p_3$ and $Q_3 = p_0 - p_1 - p_2 + p_3$. It is obvious that the action of the noisy channels on the entanglement can be viewed as shrinking the Lorentz singular values by the above three coefficients.

If the initial entangled states are set as pure states, then according to the Schmidt decomposition theorem [2], an arbitrary two-qubit pure state $|\Omega\rangle$ can be expressed as $|\Omega\rangle = \lambda_1 |01\rangle + \lambda_2 |10\rangle$, where here $\lambda_1$ and $\lambda_2$ are non-negative real numbers satisfying $\lambda_1^2 + \lambda_2^2 = 1$. That is there always exist local unitary operations $U$ and $V$, which satisfy $|\Omega\rangle = (U \otimes V)|\Omega_0\rangle$, where $|\Omega_0\rangle = \lambda_1 |01\rangle + \lambda_2 |10\rangle$. Here $|0\rangle$, $|1\rangle$ are the +1, −1 eigenstates of the Pauli $\sigma_z$ matrix. In the $R$-picture, $|\Omega_0\rangle\langle\Omega_0|$ corresponds to the matrix:

$$R_0 = \begin{pmatrix}
1 & \cdot & \cdot & \lambda_2^2 - \lambda_1^2 \\
\cdot & 2\lambda_1\lambda_2 & \cdot & \cdot \\
\cdot & \cdot & 2\lambda_1\lambda_2 & \cdot \\
\lambda_1^2 - \lambda_2^2 & \cdot & \cdot & -1
\end{pmatrix}$$

And the local unitary operations on $|\Omega_0\rangle$ correspond to left and right multiplication of $R_0$ with orthogonal matrices, therefore in the $R$-picture, an arbitrary pure state $|\Omega\rangle\langle\Omega|$ corresponds to the matrix:

$$R = L_U R_0 L_V^T$$

$$\begin{pmatrix}
1 & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & 1
\end{pmatrix}$$

where $L_U = (\cdot O_1 \cdot) = T(U \otimes U^*)T^\dagger$ and $L_V = (\cdot O_2 \cdot) = T(V \otimes V^*)T^\dagger$, with $O_1$ and $O_2$ are real $3 \times 3$ orthogonal matrices with determinant 1.

In the following we will investigate how the pairwise entanglement changes under decoherence in the real $R$-picture and try to find what kind of entangled states, with the same initial entanglement, can maintain entanglement best.

We starting by considering the initial entangled state is in the Schmidt decomposition form $|\Omega_0\rangle$. As discussed above, due to the coupling between the system and environment,
$|\Omega_0\rangle$ is transformed into another mixed states $\rho'_0 = \varepsilon_1\varepsilon_2(|\Omega_0\rangle \langle \Omega_0|)$. In the $R$-picture, this action can be expressed as $R'_0 = L_1R_0L_2^T$, that is:

$$
R'_0 = \begin{pmatrix}
1 & \cdot & \cdot & (\lambda_2^2 - \lambda_1^2)Q_3 \\
\cdot & 2\lambda_1\lambda_2Q_1^2 & \cdot & \cdot \\
\cdot & \cdot & 2\lambda_1\lambda_2Q_2^2 & \cdot \\
(\lambda_1^2 - \lambda_2^2)Q_3 & \cdot & \cdot & -Q_3^2
\end{pmatrix}
$$

(11)

Therefore the concurrence of $\rho'_0$ can be obtained easily according to lemma 2.

$$
C' = \max \{0, C'_0(Q_1^2 + Q_2^2) + Q_3^2 - 1\}
$$

(12)

where $C_0 = 2\lambda_1\lambda_2$ is the initial entanglement. Actually, this result can also be obtained by the conventional way of calculating the concurrence. Note that

$$
\rho'_0 = \begin{pmatrix}
A & \cdot & \cdot & B \\
\cdot & D & C & \cdot \\
\cdot & C & E & \cdot \\
B & \cdot & \cdot & A
\end{pmatrix}
$$

(13)

where $A = (1 - Q_3^2)/4$, $B = (Q_1^2 - Q_2^2)\lambda_1\lambda_2/2$, $C = (Q_1^2 + Q_2^2)\lambda_1\lambda_2/2$, $D = (1 + Q_3^2)/4 + Q_3(\lambda_1^2 - \lambda_2^2)/2$ and $E = (1 + Q_3^2)/4 - Q_3(\lambda_1^2 - \lambda_2^2)/2$. For a state having a density matrix of the above form, the concurrence is given by $C' = \max \{0, C_1, C_2\}$, where $C_1 = 2|B| - \sqrt{DE}$ and $C_2 = 2(C - A)$. We note $C_2$ is always larger than $C_1$, thus $C' = \max \{0, 2(C - A)\}$ which agrees with the above result.

For an arbitrary pure state $|\Omega\rangle$, the corresponding $R$ matrix is shown in Eq. (10). In the decoherence environment, $|\Omega\rangle$ is transformed to $\rho'$. In the real $R$-picture, the evolution of the state is described by

$$
R' = L_1RL_2^T = L_1L_UR_0L_1^T L_2^T
$$

(14)

We then multiply $L_2^T$ and $L_V$ to $R$ from left and right respectively, and get another $R$-matrix $R'' = L'_1R_0L'_2$, where $L'_1 = L_1^TL_1L_1U$ and $L'_2 = L_1^TL_2L_1L_V$. This corresponds to local unitary operations on the state $\rho'$ in the $\rho$-picture, thus the concurrence of $R''$ is identical to the one of $R'$. As we have discussed above, the concurrence of a given $R$ matrix is only determined by the Lorentz singular values, and the action of the decoherence on the entanglement can be viewed as shrinking the Lorentz singular values. If we look
on \((s_0, s_1, s_2, s_3)\) as the components of a vector, then the influence of decoherence is just shrinking this vector according to the coefficients \(Q_1, Q_2\) and \(Q_3\). This can be reflected by the residual entanglement shown in Eq. (12). The action of the noisy channels is characterized by the three shrinking coefficients. In addition, we note that \(L_U, L_V \in SO(3)\). Therefore the effects of \(L_U\) and \(L_V\) on \(L_1\) and \(L_2\) respectively is changing the shrinking directions. Then if we order the coefficients \(\{n_1, n_2, n_3\} = \{Q_1, Q_2, Q_3\}\) such that \(n_1^2 \leq n_2^2 \leq n_3^2\). Based on the above analysis, it can easily be seen that the maximum residual entanglement is

\[
C'_{\text{max}} = \max \left\{ 0, C_0 \left( n_1^2 + n_2^2 \right) \right\} - 1
\]

where \(C_0\) is the initial entanglement. The above maximum residual entanglement can be achieved by appropriate local unitary operations. The corresponding initial pure states are

\[
\varphi = \{ |\Omega \rangle = (U \otimes U)|\Omega_0 \rangle \mid L_U^T L_1 L_U = \text{diag}[1, \pm n_1^2, \pm n_2^2, \pm n_3^2] \}\]

This set of pure states present those special states that are the most entanglement-stable. The minimum residual entanglement and the corresponding initial pure states can be also derived in a similar way. Given a specific decoherence model, the relation between \(Q_1, Q_2,\) and \(Q_3\) is known, then the pure states in \(\varphi\) can be written explicitly. For example, in the dephasing channels model, the coefficients are \(Q_1 = Q_2 = p_0 - p_3\) and \(Q_3 = 1\). Therefore both \(|\Omega_0 \rangle\) and \((\sigma_x \otimes \sigma_x)|\Omega_0 \rangle\) belong to the set \(\varphi\). The other states contained in \(\varphi\) can also be obtained easily according Eq. (16).

Although all the pure states with the same entanglement is equivalent under local unitary operations, it can be seen from the above results that under decoherence the behavior of different pure states are not all the same. This reflect the properties of the decoherence model and its influence on the entanglement. Our results suggest that if the decoherence model is fixed, there exists a special class of pure states, with the same initial entanglement, which are more favorable for maintaining entanglement. This gives some useful hints for maintaining and distributing entanglement. For example, if we want to distribute an entangled pure state between two separate parties through noisy channels, it will be helpful to apply some local unitary operations beforehand to transform the entangled pure state into the form of the states in the above set \(\varphi\).

For an initial general mixed entangled state \(\rho\) with the initial entanglement \(C_0\), to drive an analytic evolution equation for its entanglement maybe intractable. However, we derive
a upper bound of the residual entanglement in the following. This upper bound is the corresponding residual entanglement $C'_\varphi$ for the states in the set $\varphi$ with the same initial entanglement $C_0$. It has be shown in [11], there exists an optimal decompositi

$$\rho = \sum_i p_i \varphi_i \langle \varphi_i |,$$

with $\sum_i p_i = 1$. Then under decoherence $\rho$ is transformed into another state $\rho' = \sum_i p_i \varepsilon_1 \varepsilon_2 (|\varphi_i \rangle \langle \varphi_i |)$. According to the convexity of the concurrence, we know that $C' \leq \sum_i p_i C'_i$. Since $C'_i \leq C'_\varphi$, it is obvious that $C' \leq C'_\varphi$. This suggests that the state in the set $\varphi$ is the most entanglement-stable of all the states with the same initial entanglement, no matter pure or general mixed states. It is well known that, an arbitrary pure state can be transformed into the states, with the same entanglement, of the set $\varphi$ by local unitary operations. Thus in the sense discussed in this paper, the pairwise entanglement in pure states are more favorable for maintaining entanglement compared to the generic mixed states. This conclusion is valid for any decoherence model which can be verified from the above discussions.

Up till now, we have investigate the entanglement dynamics of two-qubit states assuming that each qubit is independently coupled to the environment. In the following, we consider the situation that only one qubit is under decoherence and the initial state is a pure entangled state. We are interested in whether the lifetime of entanglement also depends on the initial entanglement in this situation. As discussed above, the entanglement dynamics for pure states with the same initial entanglement are also dependent on their forms. To be comparable, we set the initial pure state in the Schmidt decomposition form $|\Omega_0 \rangle = \lambda_1 |01 \rangle + \lambda_2 |10 \rangle$. If only the first qubit is under decoherence, then the two-qubit system becomes $\rho' = \varepsilon_1 (|\Omega_0 \rangle \langle \Omega_0 |)$. In the $R$-picture, the matrix corresponding to $\rho'$ is $R' = L_1 R_0$, that is

$$R' = \left( \begin{array}{cccc} 1 & \cdot & \cdot & \lambda_1^2 - \lambda_2^2 \\ \cdot & 2\lambda_1 \lambda_2 Q_1 & \cdot & \cdot \\ \cdot & \cdot & 2\lambda_1 \lambda_2 Q_2 & \cdot \\ (\lambda_1^2 - \lambda_2^2) Q_3 & \cdot & \cdot & -Q_3 \end{array} \right).$$

(17)

Therefore the concurrence of $\rho'$ is

$$C' = 2 \max \{0, \frac{C_0}{2} (|Q_1 - Q_2| - Q_3 - 1), \frac{C_0}{2} (|Q_1 + Q_2| + Q_3 - 1)\}$$

(18)

This result shows that the lifetime of the entanglement is independent on the initial entanglement. No matter how much the initial two-qubit state is entangled, it becomes separable
in a constant time. This somewhat interesting phenomena reflect that entanglement is some kind of nonlocal property.

B. Depolarizing Channels

In the above, we discuss the dynamics of entanglement under the noisy channels generated by Pauli matrices. When the parameters satisfy $p_1 = p_2 = p_3 = p/4$ and $p_0 = 1 - 3p/4$, the decoherence model in Eq. (6) is the depolarizing channels. The depolarizing channels describe the system-environment coupling in the large temperature limit $T \to \infty$. It can be realized by random von Neumann measurements. Again we assume that each qubit is independently coupled to the environment. The shrinking coefficients are $Q_1 = Q_2 = Q_3 = 1 - p$. Taking into account of the strength of the system-environment coupling and the interaction time, we can write $1 - p(t) = e^{-\kappa t}$. Thus the corresponding matrices in the $R$-picture are $L_1 = L_2 = \text{diag}[1, e^{-\kappa t}, e^{-\kappa t}, e^{-\kappa t}]$. Given an arbitrary initial entangled pure state $|\Omega\rangle$, the corresponding Schmidt decomposition normal form is $|\Omega_0\rangle$ associated with the $R$-matrix $R_0$. Then after time $t$, the residual entanglement is dependent on the $R'' = L_1 R_0 L_2^T$. Note that $L_U^T L_1 L_U = L_1$ and $L_V^T L_2 L_V = L_2$ here. Therefore the concurrence at time $t$ is

$$C'(t) = \max\{0, (C_0 + \frac{1}{2})e^{-2\kappa t} - \frac{1}{2}\} \quad (19)$$

From the above evolution function of the pairwise entanglement, we can find that the residual entanglement at time $t$, only depends on its initial entanglement $C_0$. Thus for all two-qubit pure states coupled with the same depolarizing environment, the stability of the entanglement is only governed by their initial entanglement, although these pure states could be in different forms. In other words, all pure states are the most entanglement-stable, need not to apply local unitary operations beforehand. Recalling the above analysis in the $R$-picture, the reason for this interesting result is that the shrinking of the Lorentz singular values of the associated $R$-matrix, which is introduced by the depolarizing channels, is isotropic. Besides, it is also obviously that even for two generic mixed state, if there are LU equivalent then the dynamics of two-qubit entanglement are also equal. Furthermore, there are some special mixed entangled states which have the same entanglement dynamics as pure states, that is also the most entanglement-stable. We will discuss the situation of mixed states in the next section. In Fig. 1 we present a visual example by plotting the
dynamics of residual entanglement for pure states and some other generic mixed states with the same initial entanglement.

FIG. 1: (Color online) The dynamics of residual entanglement in a two-qubit system. The initial states are chosen as pure states and some other generic mixed states with the same initial entanglement. We set $C_0 = 2/3$. Residual entanglement $C'$ as function of time $t$. Pure state (Solid Curve); some other generic mixed states (Dashed and Dotted Curve).

IV. EXAMPLES: MIXED STATES

A. Decoherence Path States

Definition: Given a decoherence model characterized by a completely positive map $\Lambda$, Decoherence Path States (DPS) are those transient states $\rho$ obtained from the pure states, that is $\exists |\psi\rangle$ which satisfies $\rho = \Lambda(|\psi\rangle\langle\psi|)$.

We consider the depolarizing channels, and the initial entangled state $\rho$ is a decoherence path state with the initial entanglement $C_0 \geq 0$. Thus there exists $|\psi\rangle$ with the entanglement $C$ and some time $t_0$ that satisfy $\rho = \Lambda(t_0)(|\psi\rangle\langle\psi|)$ and $C_0 = (C + \frac{1}{2})e^{-2\kappa t_0} - \frac{1}{2}$. Then after some time $t$, the decoherence path state $\rho$ evolves to another state $\rho'(t)$ with the entanglement $C'(t) = (C + \frac{1}{2})e^{-2\kappa(t_0+t)} - \frac{1}{2}$. This can be simplified to $C'(t) = \max\{0, (C_0 + \frac{1}{2})e^{-2\kappa t} - \frac{1}{2}\}$. It is obvious that this is the same with the dynamics of entanglement for pure states with the same initial entanglement $C_0$, as shown in Eq. (19). In fact, the familiar Werner states belong to the decoherence path states.
Therefore in the depolarizing channels not only all pure states but also some special mixed states have the same entanglement dynamics. In other words, the pairwise entanglement of the decoherence path states are also the most stable. We plot the dynamics of entanglement in decoherence path states with different initial entanglement, as depicted in Fig. 2.

From Fig. 2 (a) it can be seen that the entanglement in the two-qubit system decreases with time due to its interaction with the decoherence environment. There exists some time $T_c$ when its entanglement $C(t) = 0$ for $t \geq T_c$. Thus $T_c$ is the critical time when the system entanglement disappears, i.e. the two-qubit system becomes separable. We can easily calculate the critical time:

$$T_c = \frac{\ln[2C_0 + 1]}{2\kappa}$$  \hspace{1cm} (20)

The relation between $T_c$ and the initial entanglement $C_0$ is depicted in Fig. 2 (b). Certainly, the lifetime of entanglement is longer if the initial entanglement is larger. For the singlet state, the lifetime of entanglement is $\kappa T_c = 0.549$.

![FIG. 2: (Color online) The dynamics of residual entanglement in a two-qubit system with initial decoherence path states. (a) Residual entanglement $C'$ as function of the initial entanglement $C_0$ and time $t$. (b) Critical time $\kappa T_c$ as function of the initial entanglement $C_0$.](image)

However, we can not exclude the possibility that there exist non-DPS states with the same entanglement dynamics as DPS states. In the following, we can see a concrete example. But as a special class of mixed states, the decoherence path states of a given decoherence model are expected to exhibit some other interesting properties.

### B. Ground State of Spin Models

In this section we will investigate the stability of the nearest neighbor entanglement of the ground states of spin models. The Hamiltonian of the translationally invariant XXZ
spin chain with periodic boundary condition is

\[ H = \sum_{i=1}^{N} [\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \gamma \sigma_i^z \sigma_{i+1}^z] \]  

(21)

when \( \gamma = 1 \) the above Hamiltonian represents the Heisenberg antiferromagnetic model. The ground state \( \rho_g \) [12, 18] is translationally invariant and the total z component of spin is zero. Therefore in the depolarizing channels, the nearest neighbor entanglement of the ground state of the Heisenberg antiferromagnetic model is the most stable, though it is not maximized [12].

Thus the reduced density matrix of the \( i \) and \( i + 1 \) site is:

\[ \rho_{i,i+1} = \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & x & z & 0 \\ 0 & z^{*} & y & 0 \\ 0 & 0 & 0 & v \end{pmatrix} \]  

(22)

When \( u = v \) and \( x = y \) it corresponds to the ground state of the Heisenberg antiferromagnetic model. The density matrices of the maximally entangled mixed states [19, 20, 21] are also in the above form. In the general Pauli channels of (6), the state \( \rho_{i,i+1} \) evolve to

\[ \rho'_{i,i+1} = \begin{pmatrix} A & 0 & 0 & E \\ 0 & B & F & 0 \\ 0 & F^{*} & C & 0 \\ E^{*} & 0 & 0 & D \end{pmatrix} \]  

(23)

where \( A = w \eta_{1}^{2} + v \eta_{2}^{2} + (x + y) \eta_{1} \eta_{2}, B = x \eta_{1}^{2} + y \eta_{2}^{2} + (u + v) \eta_{1} \eta_{2}, C = y \eta_{1}^{2} + x \eta_{2}^{2} + (u + v) \eta_{1} \eta_{2}, D = v \eta_{1}^{2} + u \eta_{2}^{2} + (x + y) \eta_{1} \eta_{2}, E = (z + z^{*}) \eta_{3} \eta_{4} \) and \( F = z \eta_{3}^{2} + z^{*} \eta_{4}^{2} \) with \( \eta_{1} = p_{0} + p_{3}, \eta_{2} = p_{1} + p_{2}, \eta_{3} = p_{0} - p_{3} \) and \( \eta_{4} = p_{1} - p_{2} \). We note that \( \eta_{1} \geq |\eta_{3}| \) and \( \eta_{2} \geq |\eta_{4}| \). In addition, \( xy \geq |z|^{2} \) because \( \rho_{i,i+1} \) is positive. Thus \( |E| \leq (BC)^{1/2} \). Therefore the concurrence of \( \rho'_{i,i+1} \) is given by \( C' = 2 \max\{0, |F| - (AD)^{1/2}\} \).

For the special Heisenberg antiferromagnetic model and the depolarizing channels, the residual entanglement is \( C' = \max\{0, \eta_{3}^{2}C_{0} - 2\eta_{1} \eta_{2}\} \), where \( C_{0} = 2(|z| - u) \). Note that \( \eta_{1} = (1 + e^{-\kappa t})/2, \eta_{2} = (1 - e^{-\kappa t})/2 \) and \( \eta_{3} = e^{-\kappa t} \). Thus \( C' = \max\{0, (C_{0} + 1/2)e^{-2\kappa t} - 1/2\} \), which is the same as the pure states. Therefore in the depolarizing channels, the nearest neighbor entanglement of the ground state of the Heisenberg antiferromagnetic model is the most stable, though it is not maximized [12]. This result study the entanglement of the ground states of spin models from a new point of view. In addition, we can verify that \( \rho_{i,i+1} \) does not belong to the decoherence path states. This suggest that in this certain decoherence model, several mixed states other than decoherence path states are also the most entanglement-stable.
V. CONCLUSIONS AND DISCUSSIONS

In conclusion, we have investigated the entanglement dynamics of a two-qubit system under a general decoherence model, that is Pauli channels. Given a decoherence model, we find the special class of pure states that are the most entanglement-stable and present the analytic entanglement dynamics of these states. Since any pure states with the same entanglement are $LU$ equivalent, we show that pure states are more favorable for maintaining entanglement than general mixed states. Therefore in the situation of maintaining or distributing entanglement, it is helpful to use pure entangled states and to apply appropriate local unitary operations beforehand to transform the entangled states to the most entanglement-stable form. Particularly, we investigate a certain decoherence model i.e. the depolarizing channels. In this case, a special class of mixed states that is decoherence path states are as entanglement-stable as pure states. The familiar Werner states are indeed DPS states. In addition, we investigate the entanglement dynamics of some specific class of mixed states, such as the ground states of XXZ, Heisenberg antiferromagnetic spin model and the maximally entangled mixed states. It is shown that in the depolarizing channels, the nearest neighbor entanglement of the ground state of Heisenberg antiferromagnetic spin model is coincidentally the most entanglement-stable. Another interesting result is that if only one qubit is coupled with the decoherence environment then the lifetime of entanglement is independent on the initial entanglement. This just reflects that entanglement is some kind of nonlocal property.

For the decoherence model considered in this paper, we find the most entanglement-stable form of entangled states. The extension of this study to more general decoherence models is very meaningful, which is also related to the important work in [22]. Furthermore, we also introduce a special class mixed states i.e. decoherence path states and find that they are also the most entanglement-stable states in the depolarizing channels. However, they are expected to exhibit more interesting properties in general decoherence models.

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