Deriving relativistic momentum and energy. 
II. Three-dimensional case

Sebastiano Sonego* and Massimo Pin†

*Università di Udine, Via delle Scienze 208, 33100 Udine, Italy

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Abstract

We generalise a recent derivation of the relativistic expressions for momentum and kinetic energy from the one-dimensional to the three-dimensional case.

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1 Introduction

We have recently shown [1] how to construct, given a velocity composition law, the expressions of kinetic energy, momentum, the Lagrangian and the Hamiltonian for a particle, in a general mechanical theory satisfying the principle of relativity, and in which elastic collisions between asymptotically free particles exist. For reasons explained in that paper, the discussion in reference [1] was restricted to the case of one spatial dimension only. In the present article we extend the treatment to three dimensions.

Let us briefly review the key elements upon which the derivation in reference [1] is based. The starting point is the definition of kinetic energy for a particle, as a scalar quantity whose change equals the work done on the particle. Mathematically, this amounts to requiring the validity of the fundamental relation

\[ \frac{dT(u)}{du} = u \cdot \frac{dp(u)}{du}, \tag{1.1} \]

where \( T(u) \) and \( p(u) \) are, respectively, the kinetic energy and the momentum of a particle with velocity \( u \). For a system of non-interacting particles, kinetic energy is then necessarily additive, since work is, so one can easily write down a formula that expresses energy conservation during an elastic collision. On requiring that this holds in an arbitrary inertial frame, it follows that another quantity is conserved, in addition to energy — a vector one, that we identify with momentum. As we shall see, this quantity is linked to kinetic energy through a simple equation containing a matrix of functions \( \varphi_{ij} \) (the indices \( i \) and \( j \)).
Suppose that a particle moves with velocity \( u \) with respect to a reference frame \( K \). If \( K \) moves with velocity \( v \) with respect to another reference frame \( \bar{K} \), the particle velocity \( \bar{u} \) with respect to \( \bar{K} \) is given by some composition law
\[
\bar{u} = \Phi(u, v).
\]
(2.1)

(It is important to appreciate that this relation contains vectors belonging to two different spaces. Not only are the basis used to write the vectors \( v \) and \( \bar{u} \), and the vector \( u \), different; they even span different rest spaces, namely those of \( \bar{K} \) and \( K \), respectively.) The relativity principle requires that (2.1) give the composition law of a group, i.e., that:
\[
\Phi(u, 0) = \Phi(0, u) = u, \quad \forall u ;
\]
(2.2)
\[
\forall u, \exists u' \text{ such that } \Phi(u, u') = \Phi(u', u) = u ;
\]
(2.3)
\[
\Phi(\Phi(u, v), w) = \Phi(\Phi(u, v), \Phi(w, u)) , \quad \forall u, v, w .
\]
(2.4)

Note that, although in the Galilean case,
\[
\Phi(u, v) = u + v ,
\]
(2.5)
the composition law is commutative, this is not the case in general, unless velocities are collinear. For example, the relativistic law \([2]\)
\[
\Phi(u, v) = \frac{1}{1 + \frac{u \cdot v}{c^2}} \left[ u \gamma(v) + v \left( \frac{u \cdot v}{v^2} \left( 1 - \frac{1}{\gamma(v)} \right) + 1 \right) \right] ,
\]
(2.6)
where, as usual,
\[
\gamma(v) := \frac{1}{\sqrt{1 - v^2/c^2}}
\]
(2.7)
denotes the Lorentz factor, is not commutative.

From the composition law \((2.1)\) one can define a matrix whose components are, in Cartesian bases,
\[
\varphi_{ij}(u) := \frac{\partial \Phi_j(u, v)}{\partial v_i} \bigg|_{v = 0} .
\]
(2.8)
Equation (2.3) then imposes that \( \varphi_{ij}(0) = \delta_{ij} \). In fact, for the Galilean composition law (2.5), we have \( \varphi_{ij}(u) = \delta_{ij} \) for all \( u \). On the other hand, in the relativistic case (2.6),
\[
\varphi_{ij}(u) = \delta_{ij} - \frac{u_i u_j}{c^2}.
\]

(2.9)

3 General analysis

Let \( T(u) \) be the kinetic energy of a particle with velocity \( u \) in an inertial frame \( K \). During an elastic collision between two particles, energy conservation requires that
\[
T_1(u_1) + T_2(u_2) = T_1(u'_1) + T_2(u'_2).
\]
(3.1)
(Of course, the kinetic energy will also depend on the particle mass; we keep track of this dependence with the indices 1 and 2 on \( T \).)

With respect to another inertial frame \( K \), in which \( K \) moves with velocity \( v \), the particle velocities are \( \bar{u}_1 = \Phi(u_1, v) \), \( \bar{u}_2 = \Phi(u_2, v) \), \( \bar{u}'_1 = \Phi(u'_1, v) \), and \( \bar{u}'_2 = \Phi(u'_2, v) \). Conservation of energy in \( K \) then implies
\[
T_1(\bar{u}_1) + T_2(\bar{u}_2) = T_1(\bar{u}'_1) + T_2(\bar{u}'_2),
\]
(3.2)
where we have used the same functions \( T_1 \) and \( T_2 \) as in (3.1), because of the relativity principle.

The expansion of \( T(\bar{u}) \) around \( v = 0 \) is
\[
T(\bar{u}) = T(u) + v_i \varphi_{ij}(u) \frac{\partial T(u)}{\partial u_j} + O(v^2).
\]
(3.3)

Doing this for each term in equation (3.2), then using equation (3.1) and considering the terms of first order in \( v \), we find the following additional conservation law:
\[
\varphi_{ij}(u_1) \frac{\partial T_1(u_1)}{\partial u_j} + \varphi_{ij}(u_2) \frac{\partial T_2(u_2)}{\partial u_j} = \varphi_{ij}(u'_1) \frac{\partial T_1(u'_1)}{\partial u_j} + \varphi_{ij}(u'_2) \frac{\partial T_2(u'_2)}{\partial u_j}.
\]
(3.4)

Thus, one arrives at the conclusion that the vector \( p(u) \), whose components are
\[
p_i(u) = \varphi_{ij}(u) \frac{\partial T(u)}{\partial u_j},
\]
(3.5)
is conserved during a collision, in addition to energy. In the one-dimensional case, this quantity can be identified with linear momentum \( \mathbf{p} \), and we suggest doing the same in three dimensions.\(^2\) Note that, with this definition, linear momentum turns out to be a one-form rather than a vector, which is very satisfactory from a formal point of view.

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\(^1\)We adopt the convention of summing over repeated indices.

\(^2\)Actually, the most general conserved quantity has the form
\[
\lambda_{ij} \varphi_{jk}(u) \frac{\partial T(u)}{\partial u_k} + \mu_i T(u) + \nu_i,
\]
where \( \lambda_{ij}, \mu_i \) and \( \nu_i \) are a tensor and two vectors that do not depend on \( u \). The simplest choice \( \lambda_{ij} = \delta_{ij}, \mu_i = \nu_i = 0 \) corresponds to the requirement that space be isotropic, and is the one which leads to the standard expressions for \( T(u) \) and \( \mathbf{p}(u) \).
If we know the function $T(u)$, we can find $p$. If we do not already know $T(u)$, we can define it by requiring that it satisfies the fundamental relation (1.1), which expresses the equality between the work done on the particle and the variation of its kinetic energy [1]. On rewriting the differentials $dT$ and $dp$ in (1.1) in terms of $du_i$, one gets

$$\frac{\partial T(u)}{\partial u_i} = u_j \frac{\partial p_j(u)}{\partial u_i}. \quad (3.6)$$

Taken together, equations (3.5) and (3.6) allow one to determine $T(u)$. Using again (3.5), one can then find $p(u)$.

The free particle Lagrangian must satisfy the relation

$$p_i(u) = \frac{\partial L(u)}{\partial u_i}. \quad (3.7)$$

Using equation (3.5), we obtain

$$dL(u) = \varphi_{ij}(u) \frac{\partial T(u)}{\partial u_j} du_i. \quad (3.8)$$

Obviously, it is only for $\varphi_{ij} = \delta_{ij}$ (i.e., in Newtonian dynamics) that $L = T + \text{const}$ — a feature already emphasised in reference [1].

Turning now to the Hamiltonian, we need only notice that (3.6) gives, basically, half of Hamilton’s equations of motion for a system with Hamiltonian $H(p) = T(u(p)) + \text{const}$. Indeed,

$$u_i = \frac{\partial u_j(p)}{\partial p_i} \frac{\partial T(u)}{\partial u_j} = \frac{\partial H(p)}{\partial p_i} \quad (3.9)$$

or, symbolically, $u = \nabla_p H$. This allows us to identify $H(p)$ with $T(u(p))$, up to an additive $p$-independent constant.

### 4 Isotropy

The previous discussion was general, in the sense that it was based only on the principle of relativity and on the hypothesis of space homogeneity (implicit in our use of inertial systems). With the further requirement that space be isotropic, one can restrict $\varphi_{ij}(u)$ to having the functional form

$$\varphi_{ij}(u) = \delta_{ij} + f(u) u_i u_j, \quad (4.1)$$

where $f$ is an arbitrary function of the magnitude $u$ of $u$. This follows immediately by considering that no other vector except $u$ can be used in writing $\varphi_{ij}$. In fact, even the class (4.1) is too wide, because relativity, homogeneity, and isotropy together, force $f$ to be a constant.\(^3\) Simple physical considerations then require that such a constant be non-positive, so we shall write from now on

$$\varphi_{ij}(u) = \delta_{ij} - K u_i u_j, \quad (4.2)$$

\(^3\)Since $f$ depends only on the magnitude $u$ of the velocity, this result can be established simply by comparison with the one-dimensional case [1, 3].
where $K \geq 0$. The cases $K = 0$ and $K = 1/c^2$ correspond to the Galilei and Einstein composition law.

From equation (3.6) we find

$$u_i \frac{\partial T}{\partial u_i} = u_i \frac{\partial (p_j u_j)}{\partial u_i} - p_i u_i . \quad (4.3)$$

Inserting (3.5) with the form (4.2) for $\phi_{ij}$ into (4.3), we obtain

$$2 u_i \frac{\partial T}{\partial u_i} = u_i \frac{\partial}{\partial u_i} \left( u_j \frac{\partial T}{\partial u_j} - K u^2 u_j \frac{\partial T}{\partial u_j} \right) + K u^2 u_i \frac{\partial T}{\partial u_i} . \quad (4.4)$$

Using the mathematical identity

$$u_i \frac{\partial}{\partial u_i} = 2 u^2 \frac{d}{du^2} = 2 \frac{d}{d\xi} , \quad (4.5)$$

where $\xi := \ln u^2$, equation (4.4) can be rewritten as

$$(2 + K e^\xi) \frac{dT}{d\xi} = 2 \left( 1 - K e^\xi \right) \frac{d^2T}{d\xi^2} . \quad (4.6)$$

This is a simple differential equation for $T$, that can be solved by first finding $dT/d\xi$:

$$\frac{dT}{d\xi} = A \exp \left( \frac{1}{2} \int d\xi \frac{2 + K e^\xi}{1 - K e^\xi} \right) = \frac{A e^\xi}{(1 - K e^\xi)^{3/2}} , \quad (4.7)$$

where $A$ is an arbitrary constant. By a further integration we find, for $K \neq 0$,

$$T(u) = \frac{A/K}{(1 - K u^2)^{1/2}} + B ; \quad (4.8)$$

while, for $K = 0$,

$$T(u) = A u^2 + B . \quad (4.9)$$

In both cases, $B$ is a further integration constant. On requiring that $T(0) = 0$, we can write $B$ in terms of $A$, which can then be expressed using the more familiar parameter $m$ — the particle mass. We thus obtain, for $K \neq 0$,

$$T(u) = \frac{m/K}{(1 - K u^2)^{1/2}} - m/K ; \quad (4.10)$$

and, for $K = 0$,

$$T(u) = \frac{1}{2} m u^2 . \quad (4.11)$$

Note that (4.11) coincides with the limit of (4.10) for $K \to 0$. For $K = 1/c^2$ one recovers the standard expression for kinetic energy in Einstein’s dynamics,

$$T(u) = \frac{m c^2}{(1 - u^2/c^2)^{1/2}} - m c^2 . \quad (4.12)$$
It is then a straightforward exercise to find the usual expressions for momentum, the Lagrangian, and the Hamiltonian, using equations (3.5), (3.8), and the remarks at the end of section 3. The results are the same as in the one-dimensional case, so we do not list them here.

In closing this section, we note that the calculations can be simplified by arguing that $T(u)$ can only be a function of $u$, and that $p(u)$ must be of the form $p(u) = \alpha(u)u$, with $\alpha$ an unspecified function of $u$. However, we have preferred not to rely on these results, because they need not be derived separately, but are actually consequences of equations (3.5), (3.6), and (4.2).

5 Conclusions

In this paper we have extended the derivation of relativistic energy and momentum given in [1], from one to three dimensions. Although nothing changes in the results, it is obvious that the discussion is not as elementary as for the one-dimensional case, since it requires some familiarity with multivariable calculus. Hence, this material is not suitable for an introductory course, contrary to what happens for the treatment in reference [1]. It can, however, be presented in a standard course on analytical mechanics, at the level, e.g., of reference [5]. Indeed, we believe that students would benefit from being exposed to this approach, which relies on a very small number of physical hypotheses, and has therefore the advantage of being logically very simple.

Appendix: Differential constraints

We present two mathematical relations — equations (A.1) and (A.4) below — that have not been used in the body of the article, but that nevertheless are somewhat interesting by their own, since they represent differential constraints on momentum and on the Hamiltonian.

Replacing (3.6) into (3.5), one arrives at a single equation for $p(u)$:

$$p_i(u) = \varphi_{ij}(u) \frac{\partial p_j(u)}{\partial u_j} u_k . \tag{A.1}$$

On defining $\psi_{ij}(u)$ as the components of the inverse matrix of the $\varphi_{ij}(u)$, that is $\psi_{ij} \varphi_{jk} = \delta_{ik}$, equation (A.1) can be rewritten as

$$\frac{\partial p_j(u)}{\partial u_i} u_j = \psi_{ij}(u) p_j . \tag{A.2}$$

Taking now the second derivative of Hamilton’s equations (3.9) we get

$$\frac{\partial u_i}{\partial p_j} = \frac{\partial^2 H}{\partial p_i \partial p_j} . \tag{A.3}$$

Replacing (A.3) into equation (A.2), we arrive at the following differential constraint on $H(p)$:

$$\frac{\partial H}{\partial p_i} = \frac{\partial^2 H}{\partial p_i \partial p_j} \psi_{jk}(\nabla_p H) p_k . \tag{A.4}$$
Deriving relativistic momentum and energy. II

References


