Transport Reversal in a Thermal Ratchet

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Abstract

Transport of a Brownian particle moving in a periodic potential is investigated in the presence of symmetric unbiased external force. The viscous medium is alternately in contact with the two different heat reservoirs. We present the approximatively analytical expression of the net current at quasi-steady state limit. It is found that the interplay of the symmetric parameter of potential and the temperature difference generates a rich variety of cooperation effects such as current reversal. The mutual interplay between the competitive driving factors is a necessary but not sufficient condition for current reversals.

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1. INTRODUCTION

Transport phenomena play a crucial role in many processes from physical, biological to social systems. There has been an increasing interest in transport properties of nonlinear systems which can extract usable work from unbiased nonequilibrium fluctuations [1, 2, 3, 4]. This comes from the desire of understanding molecular motors [5], nanoscale friction [6], surface smoothening [7], coupled Josephson junctions [8], optical ratchets and directed motion of laser cooled atoms [9], and mass separation and trapping schemes at microscale [10].

The focus of research has been on the noise-induced unidirectional motion over the last decade. A ratchet system is generally defined as a system that is able to transport particles in a periodic structure with nonzero macroscopic velocity in the absence of macroscopic force on average. In these systems, directed Brownian motion of particles is generated by nonequilibrium noise in the absence of any net macroscopic forces and potential gradients. Typical examples are rocking ratchets [4, 11], flashing ratchets [12], diffusion ratchets [13], correlation ratchets [4, 14]. In all these studies, the potential is taken to be asymmetric in space. It has also been shown that a unidirectional current can also appear for spatially symmetric potentials if there exists an external random force either asymmetric or spatially-dependent.

The current reversal is very important in new particle separation devices such as electrophoretic separation of micro-particles [15]. The phenomena of current reversal is of also interest in biology [32]. When considering the motion of macromolecules where the probable mechanism of vesicle transport inside eukaryotic cells was found to be the motion of proteins along a microtubule modelled as a ratchet [29]. Myosin moves along actin filaments towards their plus extremity, and kinesins and dyneins move along tubulin filaments towards their plus and minus extremities respectively. It is well known that the current reversal effect allows one pair of motor proteins to move simultaneously in opposite directions along the microtubule inside the eukaryotic cells. Several biological molecular motors, for instance kinesin and non-claret disjunctional, belonging to the same superfamily of motor proteins move towards opposite ends of the microtubules. The ratchet mechanism was used for obtaining efficient separation methods of nanoscale objects, e. g., DNA molecules, proteins, viruses, cells, etc. To date, the feasibility of particle transport by man-made devices has
been experimentally demonstrated for several ratchet types. The current reversal in ratchet systems can be engendered by varying system parameters. The current can be reversed, for example, by a noise of Gaussian force with non-white power spectrum in present of stationary periodic potential. The current reversal can also be obtained in two-state ratchets if the long arm is kinked. Bier and Astumian have also found the current reversal in a fluctuating three-state ratchet. In the presence of a kangaroo process as the driving force, the current reversal can be triggered by varying the noise flatness, the ratio of the fourth moment to the square of the second moment. The current reversal can be induced by both an additive Gaussian white and an additive Ornstein-Uhlenbeck noise in a correlation ratchet. The current reversal also appear in forced inhomogeneous ratchets.

The previous works regarding the current reversal are limited to case of one heat reservoir. The present study extends the study of current reversal to the case of two heat reservoirs. When a positive driving factor meets a negative one, the current may reverse its direction. The mutual interplay between the competitive driving factors is necessary but not sufficient for the current reversal. Our emphasis is on finding conditions of generating current reversal. This is achieved by using a quasi-steady state limit to solve the Fokker-Planck equation.

2. NET CURRENT OF THE THERMAL RATCHET

Consider a Brownian particle moving in a sawtooth potential with an unbiased external force where the medium is alternately in contact with the two heat reservoirs. This model was first proposed to describe molecular motor in biological system. The particle motion satisfies the dimensionless Langevin equation of motion

\[ m \frac{d^2x}{dt^2} = -\beta \frac{dx}{dt} - \frac{dU(x)}{dx} + F(t) + \sqrt{2k_BT(x)}\beta\xi(t), \]  

where \( x \) stands for the position of Brownian particle, \( m \) the mass of the particle, \( \beta \) the viscous friction drag coefficient, \( k_B \) Boltzmann constant, \( T(x) \) absolute temperature. \( \xi(t) \) is a randomly-fluctuating Gaussian white noise of zero mean and the autocorrelation function \( < \xi(t)\xi(s) > = \delta(t-s) \). Here \( < ... > \) denotes an ensemble average over the distribution of the fluctuating forces \( \xi(t) \). \( F(t) \) is an external periodic force (Fig. 1b), satisfy

\[ F(t+\tau) = F(t), \int_0^T F(t)dt = 0. \]
The geometry of symmetric potential $U(x) = U(x + L)$ is displayed in Fig. 1a and $U(x)$ within the interval $0 \leq x \leq L$ is described by

$$U(x) = \begin{cases} \frac{U}{L_1}x, & 0 \leq x < L_1; \\ \frac{U}{L_2}(L-x), & L_1 \leq x \leq L, \end{cases}$$ \hspace{1cm} (3)$$

where $L = L_1 + L_2$, is the period of the potential. The temperature $T(x)$ has the same period as the potential $U(x)$. Therefore, $T(x) = T(x + L)$,

$$T(x) = \begin{cases} T + \delta, & 0 \leq x < L_1; \\ T, & L_1 \leq x \leq L. \end{cases} \hspace{1cm} (4)$$

FIG. 1: Schematic illustration of the potential and the driving force. (a) Potential, $U(x) = U(x+L)$, $U(x)$ is a piecewise linear and periodic potential. The period of the potential is $L = L_1 + L_2$, and $\Delta = L_1 - L_2$. The temperature profiles is also shown. (b) Driving force $F(t)$ which preserved the zero mean $< F(t) > = 0$, and $F(t + \tau) = F(t)$, $F_0$ is amplitude of $F(t)$.

Because the motion of the ratchet is highly overdamped in general \[30\], the inertia term can be neglected. Hence, Eq. (1) reduces to, when $\beta = 1$ and $k_B = 1$,

$$\frac{dx}{dt} = -\frac{dU(x)}{dx} + F(t) + \sqrt{2T(x)}\xi(t).$$ \hspace{1cm} (5)$$

The probability density for $x$ satisfies the associated Fokker-Planck equation \[30\],

$$\frac{\partial P(x, t)}{\partial t} = \frac{\partial}{\partial x}[(U'(x) - F(t))P(x, t) + \frac{\partial}{\partial x}(T(x)P(x, t))] = -\frac{\partial j(x, t)}{\partial x}. \hspace{1cm} (6)$$
\[ j(x, t) = -[U'(x) - F(t)]P(x, t) - \frac{d}{dx}[T(x)P(x, t)], \]  

(7)

Here \( j \) is the probability current density. The prime stands for the derivative with respect to the space variable \( x \). \( P(x, t) \) is the probability density for the particle at position \( x \) at time \( t \) and satisfy the normalization condition and the periodicity condition,

\[ P(x, t) = P(x + L, t), \]  

(8)

\[ \int_0^L P(x, t)dx = 1. \]  

(9)

If \( F(t) \) changes very slowly with respect to \( t \), namely, its period is longer than any other time scale of the system, there exists a quasi-steady state. In this case, follow the previous method \[30, 31\], we can obtain the current \( j(F(t)) \) from Eq. (7)-Eq. (9),

\[ j(F(t)) = \frac{-Q}{G_1G_2 + HQ}, \]  

(10)

where \( Q, G_1, G_2 \) and \( H \) are

\[ Q = e^{a-b} - 1, \]

\[ G_1 = \frac{L + \Delta}{2a(T + \delta)}(1 - e^{-a}) + \frac{L - \Delta}{2b}e^{-a}(e^b - 1), \]  

(11)

\[ G_2 = \frac{L + \Delta}{2a}(e^a - 1) + \frac{L - \Delta}{2b}e^a(1 - e^{-b}), \]

\[ H = A + B + C, \]

\[ A = \frac{1}{T + \delta}(\frac{L + \Delta}{2a})^2(a + e^{-a} - 1), \]

\[ B = \frac{L^2 - \Delta^2}{4abT}(1 - e^{-a})(e^b - 1), \]

\[ C = \frac{1}{T}(\frac{L - \Delta}{2b})^2(e^b - 1 - b), \]  

(12)

\[ a = \frac{2U_0 - F(t)(L + \Delta)}{2(T + \delta)}, \]

\[ b = \frac{2U_0 + F(t)(L - \Delta)}{2T}. \]  

(13)
The average current is
\[ J = \frac{1}{\tau} \int_{0}^{\tau} j(F(t))dt, \tag{14} \]
where \( \tau \) is the period of the driving force \( F(t) \). \( \tau \) is assumed to be longer than any other time scale of the system at the adiabatic limit. For the external force \( F(t) \) shown in Fig. 1b,
\[ J = \frac{1}{2} [j(F_0) + j(-F_0)]. \tag{15} \]

3. RESULTS AND DISCUSSION

3.1 Current

![Graph of J versus Delta with axes labeled J on the y-axis and Delta on the x-axis. The graph shows a positive linear relationship with J increasing as Delta increases.](image)

FIG. 2: Current \( J \) versus asymmetric parameter \( \Delta \) of the potential at \( U_0 = 5, F_0 = 3.0, L = 1.0, T = 1.0 \) and \( \delta = 0 \).

Figure 2 shows the current \( J \) as a function of the asymmetric parameter \( \Delta \) of the potential at \( \delta = 0 \). The current is negative for \( \Delta < 0 \), zero at \( \Delta = 0 \) and positive for \( \Delta > 0 \). Therefore, we can have the current reversal by changing the sign of \( \Delta \), the asymmetry of the potential.

Figure 3 shows the current \( J \) versus temperature difference \( \delta \) in a symmetric potential \( \Delta = 0 \). The temperature difference \( \delta \) not only controls the magnitude but also the direction...
FIG. 3: Current $J$ versus temperature difference $\delta$ at $U_0 = 5$, $F_0 = 3.0$, $L = 1.0$, $T = 10$ and $\Delta = 0$.

of the current. When $\delta = 0$ and $\Delta = 0$, there is no current. For the asymmetric potentials, varying temperature difference is another way of inducing a net current.

3.2 Current Reversal

The current $J$ as the function of $T$ is shown in Fig. 4 for different combinations of $\Delta$ and $\delta$. The curve is observed to be bell-shaped, which shows the feature of resonance. When $T \to 0$, $J$ tends to zero for all values of $\delta$ and $\Delta$. Therefore, the particle can not pass the barrier and there is no currents. When $T \to \infty$ so that the thermal noise is very large, the ratchet effect disappear and $J \to 0$, also. There is an optimized value of $T$ at which the current $J$ takes its maximum value. There is no current reversal at $\delta = 0.1, \Delta = 0.9$; $\delta = 0.1, \Delta = 0.1$; $\delta = -0.1, \Delta = -0.1$; and $\delta = -0.1, \Delta = -0.9$. In fact, we can not have the current reversal via temperatrue if $\Delta \delta > 0$ (Fig. 2-Fig. 4).

In Fig. 5a, we plot the current $J$ as a function of temperature $T$ for different combinations of $\Delta$ and $\delta$. When temperature difference ($\delta = 0.1$) is positive, the current may reverse its direction as increasing temperature for negative $\Delta$ ($\Delta = -0.8$). It is observed that the current reversal may occur for negative $\delta$ and positive $\Delta$ ($\delta = -0.1, \Delta = 0.8$). We can also
FIG. 4: Current $J$ versus temperature $T$ for different asymmetric parameters $\delta$ and $\Delta$ at $U_0 = 5$, $F_0 = 3.0$ and $L = 1.0$.

FIG. 5: (a) Current $J$ versus temperature $T$ for different values of the asymmetric parameters $\Delta$ at $U_0 = 5$, $F_0 = 3.0$, $L = 1.0$. (b) Current $J$ versus temperature $T$ at $U_0 = 5$, $F_0 = 3.0$, $L = 1.0$, $\delta = 0.1$ and $\Delta = -0.4$. 
have the current reversal twice at $\delta = 0.1, \Delta = -0.4$ (see Fig. 5b). Therefore, there may exit current reversal for $\Delta \delta < 0$. However, $\Delta \delta < 0$ is not sufficient condition for current reversal, for example, the current is always positive for $\delta = 0.1, \Delta = -0.3$ and negative for $\delta = -0.1, \Delta = 0.2$ (see Fig. 5a).

FIG. 6: Current counter $\Delta$ vs $T$ at $U_0 = 5, F_0 = 3.0, L = 1.0$ and $\delta = 0.1$. Here $\Delta_c (-0.3987)$ is maximum $\Delta$ for zero current, and $\Delta_b (-0.4913)$ is asymmetric parameter of potential for zero current when $T \rightarrow 0$, $T_a$ is temperature for zero current when $\Delta = -1.0$.

In order to present current reversal in detail, the current counter are represented in Fig. 6 and Fig. 7. In Fig. 6, the current values are negative under the current counter curve valued $J = 0$ (dash line), while they are positive above it. When $T > T_a$ or $\Delta > \Delta_c$, no current reversal can be obtained (see the case $\delta = 0.1, \Delta = -0.3$ in Fig. 5a). When one changes $T$ or $\Delta$, current reversal may occur for the case of $T < T_a$ or $\Delta < \Delta_c$ (see the case $\delta = 0.1, \Delta = -0.8$ in Fig. 5a). Especially, it is observed that the current reverses its direction twice for the case of $\Delta_b < \Delta < \Delta_c$ (see the case $\delta = 0.1, \Delta = -0.4$ in Fig. 5b).

From Fig. 7, we can see that the current is always negative and there is no current reversal for $\delta \leq 0$, $\Delta = -0.6$. When $\delta_a < \delta < \delta_c$, the current may reverse its direction as increasing temperature. Especially, when $\delta_b < \delta < \delta_c$, the current may changes its direction twice.
FIG. 7: Current counter $\delta$ vs $T$ at $U_0 = 5$, $F = 3.0$, $L = 1.0$ and $\Delta = -0.6$. Here $\delta_a = 0$, $\delta_b (0.138)$ is temperature difference for zero current when $T \to 0$, and $\delta_c (0.151)$ is maximum temperature difference for zero current.

Therefore, we can not have the current reversal when $\delta \Delta \geq 0$. When $\delta \Delta < 0$, the current may reverse its direction. It is worth to note that $\delta \Delta < 0$ is not, however, the sufficient condition for current reversal. In our systems, $\delta \Delta < 0$ is necessary but not sufficient condition for current reversal, which shows that the mutual interplay between the competitive driving factors may induce current reversal.

4. CONCLUDING REMARKS

The transport of a Brownian particle moving in a periodic potential is studied in the presence of an unbiased fluctuation and two heat reservoirs. In a quasi-steady state limit, we obtain the current analytically. It is found that the asymmetric parameter $\Delta$ of the potential and the temperature difference $\delta$ are the two pivotal factor for obtaining a net current. When the two positive or the two negative driving factors meet ($\Delta \delta > 0$), no current reversal occur. The current reversal can not occur for one driving factor ($\Delta \delta = 0$). When a positive driving factor meets a negative driving factor ($\Delta \delta < 0$), current reversal may occur. Especially, current reversal may occur twice for some cases. However, the
condition $\Delta \delta < 0$ is not a sufficient but a necessary condition for current reversal.

To summarize, it is remarkable that the interplay of the asymmetric parameter $\Delta$ and the temperature difference $\delta$ with an unbiased external force generates a rich variety of cooperation effects such as current reversal with the temperature. We expect that our analysis should be applicable for particle separation devices, control of molecular motors and other microscale phenomena.