Emergence and resilience of social networks: a general theoretical framework

George C.M.A. Ehrhardt and Matteo Marsili
The Abdus Salam ICTP, Strada Costiera 11, I-34014, Trieste. Italy.

Fernando Vega-Redondo
Universidad de Alicante, Facultad de Economicas, Universidad de Alicante, 03071, Alicante. Spain

We introduce and study a general model of social network formation and evolution based on the concept of preferential link formation between similar nodes and increased similarity between connected nodes. The model is studied numerically and analytically for three definitions of similarity. In common with real-world social networks, we find coexistence of high and low connectivity phases and history dependence. We suggest that the positive feedback between linking and similarity which is responsible for the model’s behaviour is also an important mechanism in real social networks.

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I. INTRODUCTION

There is a growing consensus among social scientists that many social phenomena display an inherent network dimension. Not only are they “embedded” in the underlying social network but, reciprocally, the social network itself is largely shaped by the evolution of those phenomena. The range of social problems subject to these considerations is wide and important. It includes, for example, the spread of crime [2,3] and other social problems (e.g. teenage pregnancy [4,5]), the rise of industrial districts [6,7,8], and the establishment of research collaborations, both scientific [9,10] and industrial [11,12]. Throughout these cases, there are a number of interesting observations worth highlighting:

(a) **Sharp transitions:** The shift from a sparse to a highly connected network often unfolds rather "abruptly," i.e. in a short timespan. For example, concerning the escalation of social problems in some neighborhoods of large cities, Crane [4] writes that “if the incidence [of the problem] reaches a critical point, the process of spread will explode.” Also, considering the growth of research collaboration networks, Goyal et al. [10] report a steep increase in the per capita number of collaborations among academic economists in the last three decades, while Hagerdoorn [11] reports an even sharper (ten-fold) increase for R&D partnerships among firms during the decade 1975-1985.

(b) **Resilience:** Once the transition to a highly connected network has taken place, the network is robust, surviving even a reversion to “unfavorable” conditions. The case of California’s Silicon Valley, discussed in a classic account by Saxenian [7], illustrates this point well. Its thriving performance, even in the face of the general crisis undergone by the computer industry in the 80’s, has been largely attributed to the dense and flexible networks of collaboration across individual actors that characterized it. Another intrinsically network-based example is the rapid recent development of Open-Source software (e.g. Linux), a phenomenon sustained against large odds by a dense web of collaboration and trust [13]. Finally, as an example where “robustness” has negative rather than positive implications, Crane [4] describes the difficulty, even with vigorous social measures, of improving a local neighborhood once crime and other social pathologies have taken hold.

(c) **Equilibrium co-existence:** Under apparently similar environmental conditions, social networks may be found both in a dense or sparse state. Again, a good illustration is provided by the dual experience of poor neighborhoods in large cities [4], where neither poverty nor other socio-economic conditions (e.g. ethnic composition) can alone explain whether or not there is degradation into a ghetto with rampant social problems. Returning to R&D partnerships, empirical evidence [11] shows a very polarized situation, almost all R&D partnerships taking place in a few (high-technology) industries. Even within those industries, partnerships are almost exclusively between a small subset of firms in (highly advanced) countries [31].

From a theoretical viewpoint, the above discussion raises the question of whether there is some common mechanism at work in the dynamics of social networks that, in a wide variety of different scenarios, produces the three features explained above: (a) discontinuous phase transitions, (b) resilience, and (c) equilibrium coexistence. Our aim in this paper is to shed light on this question within a general framework that is flexible enough to accommodate, under alternative concrete specifications, a rich range of social-network dynamics.

The recent literature on complex networks has largely focused on understanding what are the generic properties arising in networks under different link formation mechanisms. Those properties are important to gain a proper theoretical grasp of many network phenomena and also provide useful guiding principles for empirical research. The analysis, however, has been mostly static, largely concerned with features such as small-world [16] or scale-free [17] networks. In contrast, our approach in this paper to the issue of network formation is intrinsically dynamic, the steady state being a balance of link formation and removal.

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*Electronic address: gehrhard@ictp.trieste.it
†also University of Essex, Wivenhoe Park, Colchester, CO4 3SQ, UK.
We consider a set of agents – be they individuals or organizations – who establish bilateral interactions (links) when profitable. The network evolves under changing conditions. That is, the favorable circumstances that led at some point to the formation of a particular link may later on deteriorate, causing that link’s removal. Hence volatility (exogenous or endogenous) is a key disruptive element in the dynamics. Concurrently, new opportunities arise that favor the formation of new links. Whether linking occurs depends on factors related to the similarity or proximity of the two parties. For example, in cases where trust is essential in the establishment of new relationships (e.g. in crime or trade networks), linking may be facilitated by common acquaintances or by the existence of a chain of acquaintances joining the two parties. In other cases (e.g. in R&D or scientific networks), a common language, methodology, or comparable level of technical competence may be required for the link to be feasible or fruitful to both parties.

In a nutshell, our model conceives the dynamics of the network as a struggle between volatility (that causes link decay) on the one hand, and the creation of new links (that is dependent on similarity) on the other. The model must also specify the dynamics governing inter-node similarity. A reasonable assumption in this respect is that such similarity is enhanced by close interaction, as reflected by the social network. For example, a firm (or researcher) benefits from collaborating with a similarly advanced partner, or individuals who interact regularly tend to converge on their social norms and other standards of behavior.

We study different specifications of the general framework, each one embodying alternative forms of the intuitive idea that “interaction promotes similarity.” Our main finding is that in all of these different cases the network dynamics exhibits, over a wide range of parameters, the type of phenomenology discussed above. The essential mechanism at work is a positive feedback between link creation and internode similarity. These two factors each exerting a positive effect on the other. Feedback forces of this kind appear to operate in the dynamics of many social networks. We show that they are sufficient to produce the sharp transitions, resilience, and equilibrium co-existence that, as explained, are salient features of many social phenomena.

II. THE MODEL

Consider a set \( \mathcal{N} = \{1, \ldots, n\} \) of agents, whose interactions evolve in continuous time \( t \). Their network of interaction at some \( t \) is described by a non-directed graph \( g(t) \subset \{ij : i \in \mathcal{N}, j \in \mathcal{N}\} \), where \( ij \equiv ji \in g(t) \) iff a link exists between agents \( i \) and \( j \). The network evolves in the following manner. Firstly, each node \( i \) receives an opportunity to form a link with a node \( j \), randomly drawn from \( \mathcal{N} \) (\( i \neq j \)), at rate 1 (i.e. with a probability \( dt \) in a time interval \( [t, t + dt) \)). If this link \( ij \) is not already in place, it forms with probability

\[
P\{ij \rightarrow g(t)\} = \begin{cases} 1 & \text{ if } d_{ij}(t) \leq d \\ \epsilon & \text{ if } d_{ij}(t) > d \end{cases}
\]

where \( d_{ij}(t) \) is the “distance” (to be specified later) between \( i \) and \( j \) prevailing at \( t \). Thus if \( i \) and \( j \) are close, in the sense that their distance is no higher than some given threshold \( d \), the link forms at rate 1; otherwise, it forms at a much smaller rate \( \epsilon \ll 1 \). Secondly, each existing link \( ij \in g(t) \) decays at rate \( \lambda \). That is, each link in the network disappears with probability \( \lambda dt \) in a time interval \( [t, t + dt) \).

We shall discuss three different specifications of the distance \( d_{ij} \), each capturing different aspects that may be relevant for socio-economic interactions. Consider first the simplest possible such specification where \( d_{ij}(t) \) is the (geodesic) distance between \( i \) and \( j \) on the graph \( g(t) \), neighbors \( j \) of \( i \) having \( d_{ij}(t) = 1 \), neighbors of the neighbors of \( i \) (which are not neighbors of \( i \)) having \( d_{ij}(t) = 2 \), and so on. If no path joins \( i \) and \( j \) we set \( d_{ij}(t) = \infty \).

This specific model describes a situation where the formation of new links is strongly influenced by proximity on the graph. It is a simple manifestation of our general idea that close interaction brings about similarity – here the two metrics coincide. When \( d > n - 1 \), the link formation process discriminates between agents belonging to the same network component (which are joined by at least one path of links in \( g \)) and agents in different components. Distinct components of the graph may, for example, represent different social groups. Then Eq. (1) captures the fact that belonging to the same social group is important in the creation of new links (say, because it facilitates control or reciprocity).

Consider first what happens when \( \lambda \) is large. Let \( c \) be the average connectivity (number of links per node) in the network. The average rate \( n\lambda c/2 \) of link removal is very high when \( c \) is significant. Consequently, we expect to have a very low \( c \), which in turn implies that the population should be fragmented into many small groups. Under these circumstances, the likelihood that an agent \( i \) “meets” an agent \( j \) in the same component is negligible for large populations, and therefore new links are created at a rate almost equal to \( nc \). Invoking a simple balance between link creation and link destruction, the average number of neighbors of an agent is expected to be \( c \approx 2c/\lambda \), as is indeed found in our simulations (Fig. 1).

As \( \lambda \) decreases, the network density \( c \) increases gradually, but then, at a critical value \( \lambda_1 \), it makes a discontinuous jump (Fig. 1) to a state containing a large and densely interconnected community covering a finite fraction of the population (the giant component). Naturally, if volatility \( \lambda \) decreases further, the network becomes even more densely connected. But, remarkably, if volatility increases back again beyond the transition point \( \lambda_1 \), the dense network remains stable. The dense network dissolves back into a sparsely connected one only at a second point \( \lambda_2 \). This phenomenology characterizes a wide region of parameter space (see inset of Fig. 1) and is qualitatively well reproduced by a simple mean field approach (see appendix).

A similar phenomenology occurs when \( d = 2 \), i.e. when links are preferentially formed with “friends of friends”, in an appropriate parameter range. This is reminiscent of a model that was recently proposed to describe a situation where (as e.g. in job search) agents find new linking opportunities through current partners. In agents use their
FIG. 1: Mean degree $c$ as a function of $\lambda$ for $\epsilon = 0.2$ when $d_{ij}$ is the distance on the graph and $\bar{d} > n - 1$. The results of a mean field theory for $n = \infty$ (solid line) is compared to numerical simulations ($\times$) starting from both low and high connected states with $n = 20000$. The dashed line corresponds to an unstable solution of the mean field equations, which separates the basins of stability of the two solutions. Indeed the low density state, for finite $n$ simulations ($\epsilon = 20000$), “flips” to the high density state when a random fluctuation in $c$ brings the system across the stability boundary (i.e. when a sizable giant component forms). These fluctuations become more and more rare as $n$ increases. Inset: Phase diagram in mean field theory. Coexistence occurs in the shaded region whereas below (above) only the dense (sparse) network phase is stable. Numerical simulations (symbols) agree qualitatively with the mean field prediction. The high (low) density state is stable up (down) to the points marked with $\times$ ($\circ$) and is unstable at points marked with $\circ$ ($+$). The behavior of $c$ along the dashed line is reported in the main figure.

links to search for new connections, whereas here existing links favor new link formation. In spite of this conceptual difference, the model in Ref. [18] also features the phenomenology (a)-(c) above, i.e. sharp transitions, resilience, and phase coexistence.

We now consider an alternative specialization of the general framework where link formation requires some form of coordination, synchronization, or compatibility. For example, a profitable interaction may fail to occur if the two parties do not agree on where and when to meet, or if they do not speak the same languages, and/or adopt compatible technologies and standards. In addition, it may well be that shared social norms and codes enhance trust and thus are largely needed for fruitful interaction.

To account for these considerations, we endow each agent with an attribute $x_i$, which may take a finite number $q$ of different values, $x_i \in \{1, 2, \ldots, q\}$. $x_i$ describes the internal state of the agent, specifying e.g., its technological standard, language, or the social norms she adopts. The formation of a new link $ij$ requires that $i$ and $j$ display the same attribute, i.e., $x_i = x_j$. This is a particularization of the general Eq. (1) with $d_{ij} = \delta_{x_i, x_j}$ and $\bar{d} = 0$. For simplicity we set $\epsilon = 0$ since in the present formulation there is always a finite probability that two nodes display the same attribute and hence can link.

We assume each agent revises its attribute at rate $\nu$, choosing $x_i$ dependent on its neighbours’ $x_j$s according to:

$$P\{x_i(t) = x\} = \frac{1}{Z} \exp \left[ \beta \sum_{j \in g(t)} \delta_{x, x_j(t)} \right]$$

(2)

where $\beta$ tunes the tendency of agents to conform with their neighbors and $Z$ provides the normalization. This adjustment rule has a long tradition in physics [19] and also occurs in the socio-economic literature as a model of coordination (or social conformity) under local interaction [20, 21, 22]. This is another manifestation of our general idea that network-mediated contact favors internode similarity. We focus on the case where such a similarity-enhancing dynamics proceeds at a much faster rate than the network dynamics. That is, $\nu \gg 1$ so that, at any given $t$ where the network $g(t)$ is about to change, the attribute dynamics on the $x_i$ have relaxed to a stationary state. The statistics of this state is provided by the Potts model in physics, which has been recently discussed for random graphs [23, 24]. We refer to the appendix for details and move directly to discussing the results.

For a given $\beta$, under strong volatility ($\lambda \gg 1$) the link density is very low, there is no giant component and agents $i, j$ chosen at random (for $n$ large) are not coordinated ($P(x_i = x_j) = 1/q$). Hence links form at a node at rate $2/q$. A simple balance of link formation and decay rates implies that $c = 2/(q \lambda)$ in this case. When $\lambda$ decreases, network density $c$ increases. First, it does so gradually but at a critical point $\lambda_1(\beta) c$ becomes sufficiently large that the $x_i$s within the giant component (whose existence is necessary for coordination) become coordinated. Link formation increases since now $P(x_i = x_j) > 1/q$ and this in turn increases the coordination. This positive feedback causes a sharp transition to a coordinated, more highly connected state. Once this sharp transition has taken place, further decreases in $\lambda$ are simply reflected in gradual increases in network density. On the other hand, subsequent changes of $\lambda$ in the opposite direction are met by hysteresis. That is, if $\lambda$ now grows starting at values below $\lambda_1$, the network does not revert to the sparse network at the latter threshold. Rather, it remains in a dense state up to a larger value $\lambda_2 > \lambda_1$, sustained by the same positive feedback discussed above.

This phenomenology, though induced by a different mechanism, is quite similar in spirit to that reported in Fig. 1 for the previous model. In the limit $\beta \to \infty$, the second model becomes equivalent to the first one since with $\beta \to \infty$, all nodes in the same component share the same value of $x_i(t)$, whilst the probability to link two disconnected nodes is $\epsilon = 1/q$. In fact, the roles of $1/\beta$ and $\lambda$ in the model are analogous. If we fix $\lambda$ and parametrize the behavior of the model through $1/\beta$, the same phenomena of discontinuous transitions, hysteresis, and equilibrium co-existence occurs for corresponding threshold values $1/\beta_1$ and $1/\beta_2$, analogous to $\lambda_1$ and $\lambda_2$ in the former discussion.

Finally, we consider a setup where $d_{ij}$ reflects proximity of nodes $i, j$ in terms of some continuous (non-negative) real attributes, $W_i(t), W_j(t)$. These attributes could represent the level of technical expertise of two firms involved in an R&D
partnership, or the competence of two researchers involved in a joint project. It could also be a measure of income or wealth that bears on the quality and prospects of a bilateral relationship. Whatever the interpretation, it may be natural in certain applications to posit that some process of diffusion tends to equalize the levels displayed by neighboring agents. This idea is captured by the following stochastic differential equation:

\[
\frac{dW_i}{dt} = \nu \sum_{j : ij \in g} [W_j(t) - W_i(t)] + W_i(t)\eta_i(t)
\]  

(3)

where \(\eta_i(t)\) is uncorrelated white noise, i.e. \(\langle \eta_i(t_0)\eta_j(t') \rangle = D \delta_{ij} \delta(t - t')\). The first term of Eq. 3 describes the diffusion component of the process, which draws the levels of neighboring agents closer. This homogenizing force competes with the random idiosyncratic growth term \(W_i(t)\eta_i(t)\). Random growth processes subject to diffusion such as that of Eq. 3 are well known in physics. In particular it is known \(^{26}\) that the fluctuation properties of Eq. 5 when \(D\) is larger than a critical value \(D_c\) are qualitatively different to those when \(D < D_c\).

Choosing \(d_{ij} = |\log W_i - \log W_j|\) and updating both the links and WSs at comparable timescales, we have performed extensive numerical simulations of the induced network dynamics. Fig. 2 reports typical results for a simple discretized version of Eq. 3 with \(D > D_c\) (see caption of Fig. 2). As in the two previous models, we find a discontinuous transition between a sparse and a dense network state, characterized by hysteresis effects. When the network is sparse, diffusion is ineffective in homogenizing growth. Hence the distance \(d_{ij}\) is typically beyond the threshold \(d\), thus slowing down the link formation process. On the other hand, with a dense network, diffusion rapidly succeeds in narrowing the gaps between the WSs of different nodes, which in turn has a positive effect on network formation. As before, the phase transition and hysteresis is a result of the positive feedback that exists between the dynamics of the \(W_i\) and the adjustment of the network. In the stationary state we find that \(W(t) \equiv \langle W_i(t) \rangle\) grows exponentially in time, i.e. \(\log W_i(t) \simeq vt\). Notably, the growth process is much faster (i.e. \(v\) is much higher) in the dense network equilibrium than in the sparse one, as shown in the upper panel of Fig. 2.

Finally, we note that when diffusion is very strong compared to the idiosyncratic shocks in Eq. 3 i.e. \(\nu \gg \sqrt{D}\) we expect a much smaller distance \(d_{ij}\) between agents in the same component compared to agents in different components. Thus the model becomes similar to the first one in this limit, in the same way the second model did for \(\beta \rightarrow \infty\).

III. CONCLUSION

In this paper we have proposed a general theoretical setup to study the dynamics of a social network that is flexible enough to admit a wide variety of particular specifications. We have studied three such specifications, each illustrating a distinct way in which the network dynamics may interplay with the adjustment of node attributes. In all these cases, network evolution displays the three features (sharp transitions, resilience, and equilibrium co-existence) that empirical research has found to be common to many social-network phenomena. Our analysis indicates that these features arise as a consequence of the cumulative self-reinforcing effects induced by the interplay of two complementary considerations. On the one hand, there is the subprocess by which agent similarity is enhanced across linked (or close-by) agents. On the other hand, there is the fact that the formation of new links is much easier between similar agents. When such a feedback process is triggered, it provides a powerful mechanism that effectively offsets the link decay induced by volatility.

The similarity-based forces driving the dynamics of the model are at work in many socio-economic environments. Thus, even though fruitful economic interaction often requires that the agents involved display some “complementary diversity” in certain dimensions (e.g. buyers and sellers), a key prerequisite is also that agents can coordinate in a number of other dimensions (e.g. technological standards or trading conventions). Analogous considerations arise as well in the evolution of many other social phenomena (e.g. the burst of social pathologies discussed above) that, unlike what is claimed e.g. by Crane, can hardly be understood as a process of epidemic contagion on a given network. It is by now well understood \(^{23,24}\) that such epidemic processes do not match the phenomenology (a)-(c) reported in empirical research. Our model suggests that a satisfactory account of these phenomena must aim at integrating both the dynamics on the network itself as part of a genuinely
co-evolutionary process.

IV. APPENDIX

We characterize the long run behavior of the network in terms of the stationary degree distribution $P(k)$, which is the fraction of agents with $k$ neighbors. This corresponds to approximating the network with a random graph (see [23]), an approximation which is rather accurate in the cases we discuss here. We focus on the limit $n \to \infty$, for which the analysis is simpler, but finite size corrections can be studied within this same approach. The degree distribution satisfies a master equation [27], which is specified in terms of the transition rates $w(k \to k \pm 1)$ for the addition or removal of a link, for an agent linked with $k$ neighbors. While $w(k \to k - 1) = \lambda k$ always takes the same form, the transition rate $w(k \to k + 1)$ for the addition of a new link depends on the particular specification of the distance $d_{ij}$. For the first model $w(k \to k + 1) = \epsilon$ if the two agents are in different components and $w(k \to k + 1) = 1$ if they are in the same. In the large $n$ limit the latter case only occurs with some probability if the graph has a giant component $G$ which contains a finite fraction $\gamma$ of nodes. For random graphs (see Ref. [25] for details) the fraction of nodes in $G$ is given by $\gamma = 1 - \phi(u)$ where $\phi(s) = \sum_k P(k)s^k$ is the generating function and $u$ is the probability that a link, followed in one direction, does not lead to the giant component. The latter satisfies the equation $u = \phi'(u)/\phi'(1)$. Hence $u^k$ is the probability an agent with $k$ neighbours has no links connecting him to the giant component, and hence is itself not part of the giant component. Then the rate of addition of links, in the first model, takes the form

$$w(k \to k + 1) = 2[\epsilon + (1 - \epsilon)\gamma(1 - u^k)],$$

where the factor 2 comes because each node can either initiate or receive a new link. The stationary state condition of the master equation leads to the following equation for $\phi(s)$

$$\lambda \phi'(s) = 2[\epsilon + (1 - \epsilon)\gamma]\phi(s) - 2(1 - \epsilon)\gamma \phi(us)$$

\hspace{1cm} (4)

which can be solved numerically to the desired accuracy. Notice that Eq. (4) is a self-consistent problem, because the parameters $\gamma$ and $u$ depend on the solution $\phi(s)$. The solution of this equation is summarized in Fig. I. Either one or three solutions are found, depending on the parameters. In the latter case, the intermediate solution is unstable (dashed line in Fig. I), and it separates the basins of attraction of the two stable solutions within the present mean field theory. Numerical simulations reveal that the mean field approach is very accurate away from the phase transition although it overestimates the size of the coexistence region.

Now we turn to the second model, where each node displays one out of a finite set of attributes. In order to simplify the analysis, we approximate the prevailing network $g$ with a random graph with Poisson degree distribution and average degree $\langle k \rangle$, i.e., a graph where any given link $ij$ is present with probability $c/(n - 1)$. Though not exact, this approximation is rather accurate as confirmed by numerical simulations, and it allows us to clarify the behavior of the model in a simple and intuitive way. (A more precise solution, which relies on a more accurate description of the network topology can also be derived, yielding no essential differences.) The solution of the Potts model on random graphs of Ref. [23, 24] (with temperature $T = 1/(2k_B\beta)$) allows us to compute the probability that two randomly chosen nodes $i$ and $j$ have $x_i = x_j$. Given the Poisson approximation, such a probability is given by a function $\pi(c, \beta) = \langle \delta_{x_i, x_j} \rangle$ of the average degree $c$ and $\beta$, as plotted in Fig. 4. Equalizing the link destruction and formation rate $\lambda c/2 = \pi(c, \beta)$ yields an equation for the equilibrium values of $c$, for any given $\beta$. A graphical approach shows that when $\lambda > \lambda_2$ there is a single solution, representing a sparse network. At $\lambda_2$ at least two solutions arise, one of which is unstable as above. At a further point $\lambda_1$ the sparse-network solution merges with the unstable one and both disappear for $\lambda < \lambda_1$, leaving only a solution with a stable and dense network. This reproduces the same phenomenology observed in the numerical simulations of the second model, which is also qualitatively similar to that presented in Fig. I for the first model.

[32] Both $\epsilon$ and $\lambda$ must be comparable to the probability that two arbitrary nodes $i$ and $j$ have $d_{ij} = 2$, which is of order $1/n$ in a network with finite degree.