Virialization in Dark Energy Cosmology

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ABSTRACT

We discuss the issue of energy nonconservation in the virialization process of spherical collapse model with homogeneous dark energy. We propose an approximation scheme to find the virialization radius. By comparing various schemes and estimating the parameter characterizing the ratio of dark energy to dark matter at the turn-around time, we conclude that the problem of energy nonconservation may have sizable effects in fitting models to observations.

Subject headings: cosmology:theory-galaxies:clusters:general-large-scale structure of universe-galaxies:formation

I. INTRODUCTION

Analyzing the effects of dark energy on the nonlinear structure formation process may provide us new ways of constraining the properties of dark energy. Especially, a lot of recent works focused on analyzing the effects of dark energy in the framework of the spherical collapse model (Lahav et al. 1991; Wang & Steinhardt 1998; Maor & Lahav 2005; Mota & de Bruck 2004; Horellou & Berge 2005; Battye & Weller 2003; Iliev & Shapiro 2001; Weinberg & Kamionkowski 2003; Nunes & Mota 2004). The spherical collapse model is a simple but powerful framework to understand the growth of bound systems in the universe (Gunn & Gott 1972). It is also incorporated in the famous Press-Schechter formalism (Press & Schechter 1974).

In spherical collapse model, we consider a top-hat spherical overdensity with mass $M$ and radius $R$. At early times, it expands along with the Hubble flow and density perturbations grow proportionally to the scale factor. After the perturbation exceeds a critical value, the spherical overdensity region will decouple from the Hubble flow and go through three phases: (1) expansion to a maximum radius, $R_{ta}$, after which the overdensity will turn-around to collapse; (2) collapse; (3) virialization at the virialization radius $R_{vir}$. 
A key parameter of spherical collapse model is the ratio of the virialized radius and turn-around radius $x = R_{\text{vir}}/R_{\text{ta}}$. Let’s first review briefly the derivation of the standard result $x = 0.5$ in Einstein-de Sitter cosmology (Peacock 1999a).

As is well-known, the self-energy of a sphere of nonrelativistic particles with mass $M$ and radius $R$ is

$$V_{\text{mm}} = -\frac{3}{5} \frac{GM^2}{R}$$  \hspace{1cm} (1)

After the system virializes, the virial theorem $T_{\text{vir}} = (R/2)dU/dR$ and Eq. (1) tells us that $T_{\text{vir}} = -1/2U_{\text{vir}}$. Substituting this to the energy conservation equation $T_{\text{vir}} + U_{\text{vir}} = U_{\text{ta}}$, we get the standard result $x = 0.5$.

When considering the evolution of spherical overdensities in the presence of homogeneous dark energy (since Dave et al. (2002) have shown that generally dark energy does not cluster on scales less than 100 Mpc), the gravitational potential energy of the spherical dark matter overdensity will be modified by a new term due to the gravitational effects of dark energy on dark matter (Maor & Lahav 2005; Mota & de Bruck 2004):

$$U_{mQ} = \frac{1}{2} \int \rho_m \Phi_Q dV$$  \hspace{1cm} (2)

where $\Phi_Q$ is the potential induced by dark energy

$$\Phi_Q = -2\pi G(1 + 3\omega_Q)\rho_Q \left(R^2 - \frac{r^2}{3}\right).$$  \hspace{1cm} (3)

Note that if we do not consider the pressure of the dark energy, as Wang & Steinhardt (1998) did, then potential is proportional to $\rho$, i.e. the factor $1 + 3\omega_Q$ should be unity in front of the expression for $\Phi_Q$. In a fully relativistic treatment, pressure will also contribute to gravitation, so the potential is proportional to $\rho + 3p$ (Maor & Lahav 2005; Mota & de Bruck 2004).

Thus the total potential energy of spherical overdensity is

$$U = U_{\text{mm}} + U_{mQ} = -\frac{3}{5} \frac{GM^2}{R} - (1 + 3\omega_Q)\frac{4\pi G}{5} M \rho_Q R^2,$$  \hspace{1cm} (4)

where we have substituted Eq. (3) into Eq. (2).

In most of the current literature (Horellou & Berge 2005; Battye & Weller 2003; Iliev & Shapiro 2001), in the presence of smooth dark energy, $x$ is still found by using the energy conservation equation

$$U_{\text{vir}} + T_{\text{vir}} = U_{\text{ta}}.$$  \hspace{1cm} (5)
Then from the virialization theorem \( T_{\text{vir}} = \frac{R_{\text{vir}}}{2} \frac{dU(R)}{dR} |_{R=R_{\text{vir}}} \) and Eq. (4), we can find

\[ T_{\text{vir}} = -\frac{1}{2} U_{\text{mm,vir}} + U_{\text{mQ,vir}}. \] (6)

Substituting Eq. (6) into the energy conservation equation, one can find:

\[ \frac{1}{2} U_{\text{mm,vir}} + 2U_{\text{mQ,vir}} = U_{\text{mm,ta}} + U_{\text{mQ,ta}}, \] (7)

from which we can find the equation determining \( x \)

\[ -4q(1 + 3\omega_Q)y^{-3(1+\omega_Q)}x^3 + 2[1 + (1 + 3\omega_Q)q]x - 1 = 0 \] (8)

where we have defined \( q = \rho_{Q,ta}/\rho_{mc,ta} \) and \( y = a_{\text{vir}}/a_{\text{ta}} \). If we set the \( (1 + 3\omega_Q) \) factor to unity in the first two terms of Eq. (8), we can get the equation found by Lahav et al. (1991); Wang & Steinhardt (1998).

II. THE ENERGY NON-CONSERVATION PROBLEM

The procedure described in Sec. I is problematic when dark energy is dynamical, i.e. \( \omega_Q \neq -1 \). Indeed, when \( \omega_Q > -1 \), \( \rho_Q \) is decreasing with time. When considering collapse of dark matter halo of the cluster scale, since dark energy does not cluster below 100 Mpc (Dave et al. 2002), \( \rho_Q \) in \( U_{mQ} \) should take its background value, i.e. evolving with time. In other words, dark energy does not virialize with dark matter, otherwise it cannot be smooth. Thus, \( U_{mQ} \) will contribute a non-conservative force to the dark matter particle. So the clustering dark matter with potential (4) is a non-conservative system:

\[ U_{\text{vir}} + T_{\text{vir}} < U_{\text{ta}}. \] (9)

So actually, in the presence of dark energy, dark matter cannot reach virialization in the strict sense. But for dark matter halo of the cluster scale, it clusters at the era when the effect of dark energy is still small. So it is reasonable to assume that dark matter particles can reach a quasi-equilibrium state in which virial theorem holds instantaneously. This is supported by the observations of relaxed cluster in our Universe (see e.g. Fabian & Allen (2003)). In the following discussion, we still call this quasi-equilibrium state as virialization. Thus assuming dark matter has reached this quasi-equilibrium state, its total energy can be computed by the virial theorem,

\[ U = U_{\text{vir}} + \frac{R}{2} \frac{dU}{dR} |_{R=R_{\text{vir}}} = -\frac{3}{10} GM^2 R_{\text{vir}}^2 (1 + 3\omega_Q) 4\pi G \rho_{Q,ta} \left( \frac{a}{a_{\text{ta}}} \right)^{3(1+\omega_Q)}, \] (10)
which is decreasing with time. Although this non-conservation effect is small in our discussion, it is worth commenting that in dark energy dominated era, this effect may be large. For example, in the extreme case of phantom dark energy models, the effect of dark energy may be so large that cluster, galaxy and even our solar system will de-virialize in the future (Caldwell et al. 2003).

So using Eq. (8) to find \( x \) will generally overestimate its actual value. In fact, from the dark matter potential (4), we can see that \( U_{\text{total,vir}} = U_{\text{mm,vir}}/2 + 2U_{\text{mQ,vir}} \) is a monotonically increasing function of \( x \). Thus, while in fact we have \( U_{\text{total,vir}} < U_{\text{total,ta}} \), if we still use \( U_{\text{total,vir}} = U_{\text{total,ta}} \) to determine \( x \), we will get a \( x \) larger than its actual value.

If the dark energy density is time-independent, then the system with the potential (4) is conservative and thus we can use energy conservation legitimately. Thus to estimate the virialization radius when dark energy density is changing with time, we can take \( \rho_{\text{Q,vir}} \) to be the same as \( \rho_{\text{Q,ta}} \). With this approximation, the equation determining \( x \) is,

\[
-4q(1 + 3\omega_Q)x^3 + 2[1 + (1 + 3\omega_Q)q]x - 1 = 0.
\]  

(11)

It is worth commenting that taking \( \rho_{\text{Q,vir}} \) to be \( \rho_{\text{Q,ta}} \) does not mean ignoring the background evolution of dark energy, i.e. make it degenerate with a true cosmological constant. First, there is a factor \( 1+3\omega_Q \) in Eq. (11) which is different from the case of a cosmological constant. Second, the value of \( \rho_{\text{Q,ta}} \) is different from \( \rho_{\text{Q,0}} \); while for a true cosmological constant, \( \rho_{\Lambda} \) is constant all the times. Thus using Eq. (11) to estimate \( x \) can give us a more realistic value than Eq. (8) and at the same is able to discriminate among different values of \( \omega \).

Figure 1 shows the virialization radius to the turn-around radius \( x = R_{\text{vir}}/R_{\text{ta}} \) as a function of \( q \). From the figure, it can be seen that when \( q \) is large, i.e. the effects of dark energy is large, Eq. (8) (dotted line) will always predict a larger virialization radius than Eq. (11) (solid line). This showed explicitly the comment following Eq. (10). Thus by assuming the dark energy density to be constant during the virialization process, we can use energy conservation and we can find a lower \( x \) which is closer to the actual one.

It is also interesting to note that all the curve in Fig. 1 is above the standard value \( x = 0.5 \) in an Einstein-de Sitter universe. This is easy to understand. Since dark energy will cause an effective repulsive force on the dark matter, the dark matter particles can reach equilibrium with a larger radius. Thus, \( x > 0.5 \) is a smoking gun of dark energy (see also Maor & Lahav (2005), which reaches similar conclusion).

Recently, Maor & Lahav (2005) considered the possibility that even if dark energy does not fully cluster, it still fully virialize. Then there will also be a energy non-conservation
Fig. 1.— Ratio of the virialization radius to the turn-around radius $x = R_{\text{vir}}/R_{\text{ta}}$ as a function of $q$, which characterizes the strength of dark energy at turn-around, for $\omega_Q = -0.8$. The dotted line is computed by Eq. (8), the solid one is computed by Eq. (11), the dashed line is computed by Eq. (28) of Maor & Lahav (2005) and the dashed-dotted line is computed by Eq. (16)
problem because the virialized system (now containing both dark matter and dark energy) does not cluster in the same rate. Note that this energy non-conservation problem is different from the energy non-conservation problem discussed above. In Maor & Lahav (2005)’s analysis, since both dark matter and dark energy virialize, they included the dark energy self-energy when using the virial theorem. In our case, where dark energy is smooth and do not virialize, the physical picture of what’s going on is just a spherical overdensity of dark matter particles collapsing in the background of smooth dark energy. From the derivation of the virial theorem (Marion 1970), we know that the potential energy appearing in the virial theorem is the one that will give rise to forces on the virialized particles. So if dark energy does not virialize, we should consider only the potential energies giving rise to dark matter self-gravitation and its gravitational interaction with dark energy. This is why our energy corrected equation (11) is very different from that of Maor & Lahav (2005) (Eq. (28) in it).

In Fig. 1, we showed the prediction of the virialization equation found by Maor & Lahav (2005) (dashed line). It can seen that for $q$ not too large, the prediction of Maor & Lahav (2005) is much smaller than our result Eq. (11). This is conceivable. Since in Maor & Lahav (2005)’s analysis, the positive self-energy of dark energy is also included in the total potential energy of virialized particles, $U_{\text{total}}$ will be larger than its actual value. Since $U_{\text{total}}$ is a monotonically increasing function of $x$, with the same initial energy, the equation of Maor & Lahav (2005) will thus predict a smaller $x$.

Mota & de Bruck (2004) have considered the case of fully clustered and virialized dark energy. We think it is interesting to consider the case that only a portion of dark energy cluster and virialize. In this case, we should include the dark energy self-energy that will cluster since this part of the self-energy will contribute to the force felt by the virialized dark energy particles.

In this case, the dark energy evolution equation is

$$\dot{\rho}_Q + (1 - F)3\frac{\dot{R}}{R}(1 + \omega_Q)\rho_Q + F 3\frac{\dot{a}}{a}(1 + \omega_Q)\rho_Q = 0,$$

where $F$ characterizes the fraction of dark energy that cluster. This equation can be integrated to find

$$\rho_Q = \rho_{Q,ta}x^{-3(1+\omega)(1-F)}y^{-3(1+\omega)F}$$

(13)

First, we use our approximation method: we neglect the background evolution of the dark energy, i.e., we take

$$\rho_Q = \rho_{Q,ta}x^{-3(1+\omega)(1-F)}y^{-3(1+\omega)F} \rightarrow \rho_{Q,ta}x^{-3(1+\omega)(1-F)},$$

(14)
in the virialization process. Taking into account the observation that we should only include a fraction $1 - F$ of the dark energy potential energy, we can get the equation determining $x$

$$
-(1 - F)(1 + 3\omega_Q)[7 - 6(1 + \omega_Q)(1 - F)]q^2 x^{-6\omega_Q + 6F(1 + \omega_Q)}
-(2 + 3\omega_Q - F)[4 - 3(1 + \omega_Q)(1 - F)]qx^{-3\omega_Q + 3F(1 + \omega_Q)}
+2[1 + (1 + 3\omega_Q)q + (1 - F)q + (1 - F)(1 + 3\omega_Q)q^2]x - 1 = 0 \tag{15}
$$

For $F = 1$, Eq. (15) will reduce to Eq. (11), while for $F = 0$, it will reduce to the equation found by Mota & de Bruck (2004). Thus, our result Eq. (11) can be continuously connected to the case that dark energy will also collapse with dark matter. This is physically satisfying.

Second, if we adopt the proposal of restoring energy conservation by Maor & Lahav (2005) then when $F = 1$, the virialization equation is

$$
-2(1 + 3\omega_Q)qx^{-3\omega_Q} - 2(1 + 3\omega_Q)qy^{-3(1 + \omega_Q)}x^3 + [1 + (1 + 3\omega_Q)q]x - 1 = 0 \tag{16}
$$

We showed the dependence of $x$ on $q$ from Eq. (16) as the dashed-dotted line in Fig. 1. It can be seen that although the approach of restoring energy conservation are different in Eqs. (16) and (11), their predictions are rather close. This illustrates that although the underlying ideas are different, in practice, our approximation scheme is quantitatively close to the scheme of Maor & Lahav (2005). In both cases, the difference from the old result (8) is large when $q$ is large.

From Fig. 1 we can also see that for $q \sim 10^{-2}$ or smaller, we get $x = 0.5$ in all the four approaches. This is reasonable. In the virialization process, it is the self-energy of matter that plays the dominant role. In fact, $U_{QQ,vir}/U_{mQ,vir}$ and $T_{QQ,vir}/T_{mQ,vir}$ are both of the order

$$
(1 + 3\omega_Q)q \left(\frac{a_{vir}}{a_{ta}}\right)^{-3(1 + \omega_Q)} \left(\frac{R_{vir}}{R_{ta}}\right)^3 \tag{17}
$$

since $\left(\frac{a_{vir}}{a_{ta}}\right)^{-3(1 + \omega_Q)} \approx 1.6^{-3(1 + \omega_Q)}$ and $\left(\frac{R_{vir}}{R_{ta}}\right)^3 \sim 0.1$, for $q < 0.01$ the above ratio is much smaller than 1, and thus we can expect that for small $q$, the problem of energy conservation will not influence the virialization process greatly.

Thus to estimate the effects of dark energy on virialization, and especially the ambiguity of energy-nonconservation, it is necessary to estimate the value of $q$ for the virialization redshift $z_{vir}$ that would be interesting to observations. If for observationally interesting $z_{vir}$, $q$ will always be quite small, then we can conclude that the problem of energy non-conservation will not bother us too much in analyzing the effects of dark energy on the formation of non-linear structure. Unfortunately, this is not the case.
Fig. 2.— The dependence of $q$ on $z_{\text{vir}}$ from Eq. (18). The solid, dashed, dotted lines correspond to $\omega_Q = -0.7, -0.8, -0.9$, respectively.
Let’s begin by writing $q$ as

$$q = \frac{\rho_{Q,ta}}{\rho_{mc,ta}} = \frac{\rho_{Q,ta}}{\zeta \rho_{m,ta}} = \frac{\Omega_{Q_0}(1 + z_{ta})^{3\omega_Q}}{\zeta \Omega_{m0}},$$

(18)

where we have defined $\zeta = \rho_{mc,ta}/\rho_{m,ta}$.

First, after specified $z_{vir}$, $z_{ta}$ can be computed using the fact that $t_{vir} = 2t_{ta}$, which is due to the observation that collapse proceeds symmetrically to the expansion phase. Then, we use the fitting formula for $\zeta$ presented by Wang & Steinhardt (1998):

$$\zeta = \left(\frac{3\pi}{4}\right)^2 \Omega_{m,ta}^{-0.79+0.26\Omega_{m,ta}} - 0.06\omega_Q.$$

(19)

With those two inputs, we can get the dependence of $q$ on $z_{vir}$ from Eq. (18) shown in Fig. 2. It can be seen that $q$ will be of the order $10^{-2}$ when the virialization redshift is larger than roughly 2. Combining this with Fig. 1, we conclude that for observationally interesting clusters, i.e. clusters formed after redshift 2, the presence of dark energy will have sizable modifications to the standard result $x = 0.5$. Thus observational evidence for $x > 0.5$ would be a strong evidence in favor of dynamical dark energy.

An important parameter in fitting theoretical calculations to observation is the density contrast at virialization $\Delta_{vir} \equiv \rho_{mc,vir}/\rho_{m,vir} = \zeta y^3 x^{-3}$. Fig. 3 shows the dependence of $\Delta_{vir}$ on $z_{vir}$ for $\omega_Q = -0.6, -0.8, -1$ from top to bottom using Eq. (11) and Eq. (8). We can see that for $\omega_Q$ not too close to $-1$, there is obvious differences in the predicted $\Delta_{vir}$ using Eq. (11) and Eq. (8). Thus, whether including the effect of energy conservation may have large impact on fitting models to cosmological observation such as weak lensing (Weinberg & Kamionkowski 2003). It is worth commenting that if we ignore the $1 + 3\omega$ factor in $U_{mQ}$, as in Wang & Steinhardt (1998); Weinberg & Kamionkowski (2003), the difference will be even larger. For example, Weinberg & Kamionkowski (2003) computed that for $\omega_Q = -0.6$, $\Delta_{vir} \sim 420$ for $z_{vir} = 0$. Furthermore, as can be seen in Fig. 3, whatever scheme we use to find $x$, there is notable difference between the case of a true cosmological constant and dynamical dark energy. Thus, cluster observations may provide important information on the dynamical behavior of dark energy.

**III. CONCLUSIONS AND DISCUSSIONS**

To summarize, in this work we discussed the issue of energy non-conservation in the virialization process of spherical overdensity with homogeneous dark energy. We proposed that taking the dark energy density to be constant during the virialization process to obtain
Fig. 3.— Thick lines: the dependence of $\Delta_{\text{vir}}$ on $z_{\text{vir}}$ for $\omega_Q = -0.6, -0.8, -1$ from top to bottom using Eq. (11); thin lines: the dependence of $\Delta_{\text{vir}}$ on $z_{\text{vir}}$ for $\omega_Q = -0.6, -0.8$ from top to bottom using Eq. (8), the case of $\omega = -1$ is identical with that of using Eq. (11).
an estimate of the virialization radius. By comparing various schemes and estimating the parameter $q$, we conclude that there will be sizable effect of dark energy on virialization process. A general signature of dark energy is that the final virialization radius will be larger than half of the turn-around radius.

It should be emphasized that the analysis in this work is quite qualitative. More detailed numerical simulations and analysis of observational data are required to estimate quantitatively the effect of dark energy on spherical collapse models and answer the general question “can we constrain the evolution of dark energy by studying the structures of non-linear objects in our Universe”. Furthermore, establishing firmly the result $x > 0.5$ from observation is challenging. In practice, it is much easier to measure directly baryons in clusters. But there are some astrophysical processes leading to energy non-conservation in the virialization process of baryon in the dark matter halo (e.g. X-ray emission of the hot gas, conduction, AGN heating, dynamical friction, etc., see e.g. Peacock (1999b) and references therein). We should compare the effects of those processes to dark energy in realistic analysis. To achieve this, we need to know the concrete physical mechanism of virialization in both the dark matter and baryon sectors, which is now still not well-understood. Thus more works in this direction is needed and will be rewarding.

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