Phantom cosmology with general potentials

Valerio Faraoni

1 Physics Department, Bishop’s University
Lennoxville, Québec, Canada J1M 1Z7
vfaraoni@cs-linux.ubishops.ca

Abstract

We present a unified treatment of the phase space of a spatially flat homogeneous and isotropic universe dominated by a phantom field. Results on the dynamics and the late time attractors (Big Rip, de Sitter, etc.) are derived without specifying the form of the phantom potential, using only general assumptions on its shape. Many results found in the literature are quickly recovered and predictions are made for new scenarios.

PACS numbers: 98.80.-k, 04.90.+e
1 Introduction

There is now substantial agreement that the expansion of the universe is accelerated, with evidence from type Ia supernovae [1], WMAP data [2], and large scale structure surveys [3]. Two classes of models aim at explaining the observed cosmic acceleration: one modifies gravity on large scales by introducing corrections to the Einstein-Hilbert [4] or the Palatini [5] action (often there are instabilities [6] or incorrect post-Newtonian limits [7]), and the other advocates the existence of dark energy with density $\rho$ and exotic pressure $P < -\rho/3$.

Most dark energy candidates proposed thus far are scalar fields; there is marginal evidence [9] for an effective equation of state parameter of the dark energy $w \equiv P/\rho < -1$, which is equivalent to increasing Hubble parameter $\dot{H} > 0$. If confirmed, this measurement is important because an increasing $H$ cannot be explained by Einstein gravity with a canonical scalar field [8, 10, 11]. If the universe really superaccelerates it may end its existence in a finite time in an explosive expansion of the scale factor accompanied by diverging dark energy density, called a Big Rip [12, 13]. Other kinds of singularities at a finite time in the future discussed in the literature are called “sudden future singularities” and are characterized by finite scale factor and Hubble parameter, and diverging $\dot{H}$ and dark pressure [14].

Simple models of a superaccelerated universe employ a scalar field coupled nonminimally to the Ricci curvature [10, 15, 11, 16, 17] or a phantom field, i.e., a minimally coupled scalar field with negative kinetic energy [8, 12, 13]. Phantom fields and fields with non–canonical energy are present in string theories [18, 19, 20] and supergravity [21], are associated to bulk viscosity due to particle production [22], or arise in higher order theories [23]. It has also been proposed that early inflation and late time acceleration of the universe can be unified in a single theory based on a phantom field [24]. A phantom field poses several challenges: it is subject to severe instabilities, which may perhaps be avoided by thinking of the phantom as an effective field theory resulting from some fundamental theory with positive energies [24]. A fundamental quantum phantom is very difficult to stabilize [13, 25].

In general, a form of dark energy with $w < -1$ is worrisome because it violates the energy conditions cherished by most physicists and opens the door to disturbing exotica. For example, even a small amount of exotic matter violating the weak energy condition may be sufficient to open up a wormhole and make time travel possible [27]. On the other hand, even a simple classical scalar field coupled nonminimally to the Ricci curvature may violate all of the energy conditions [28], and one should probably not be too rigid in rejecting phantom fields a priori.

The peculiar dynamics arising from a negative kinetic energy density has been stud-
ied with the help of a toy model consisting of two mutually coupled oscillators, one with negative kinetic energy representing the phantom field, and the other with positive kinetic energy representing the gravitational field \[13\]. However, this toy model fails to properly describe the dynamics \[29\] and it is preferable to study the actual dynamical system. Furthermore, a number of papers in the literature study phantom models with different specific potentials \[30\]. Here we study the dynamics without toy models or approximations and we deduce the behaviour and the late time state of a phantom–dominated universe without specifying the form of the potential. The assumption of a spatially flat Friedmann–Lemaitre–Robertson–Walker (FLRW) universe plus general assumptions on the potential (such as boundedness, monotonicity, etc.) allow one to derive the asymptotic dynamics. In addition to the need to understand phantom dynamics with general potentials, our study is motivated by the question whether the universe will end in a Big Rip or will expand forever. In fact, a universe that super-accelerates may end its existence in a Big Rip or it may expand forever, depending on the dynamical equations. Often authors working with a specific choice of the phantom potential find a de Sitter regime as the final stage of evolution (late time attractor). We show that certain features of the phantom potential determine whether the final state is an indefinite expansion or a Big Rip, and the analysis needs not be repeated for all the possible potentials. We adopt the notations and conventions of \[32\] and units such that \(c = G = h = 1\).

2 The phase space of phantom cosmology

Based on the recent cosmological observations we adopt the FLRW line element
\[
d s^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right)
\] (2.1)
in comoving coordinates \((t, x, y, z)\). We consider the situation in which dark energy has already come to dominate the cosmic dynamics, hence a phantom field is the only form of matter in the Einstein equations. The energy density and pressure of the phantom are, respectively,
\[
\rho = -\frac{1}{2} \phi^2 + V(\phi) , \quad P = -\frac{1}{2} \phi^2 - V(\phi) .
\] (2.2)

The distinguishing negative kinetic energy is evident in eqs. \[2.2\] which exhibit the “wrong” sign for the kinetic term \(\dot{\phi}^2/2\). The Einstein equations for \(\phi(t)\) and the scale factor \(a(t)\) are
\[
H^2 = \frac{\kappa}{6} \left( -\dot{\phi}^2 + 2V \right) ,
\] (2.3)
\[
\dot{H} + H^2 = \frac{\kappa}{3} \left( \dot{\phi}^2 + V \right), \tag{2.4}
\]
\[
\ddot{\phi} + 3H\dot{\phi} - \frac{dV}{d\phi} = 0, \tag{2.5}
\]

where \( H \equiv \dot{a}/a \) is the Hubble parameter, an overdot denotes differentiation with respect to the comoving time \( t \), and \( \kappa \equiv 8\pi G \). The sign of the term \( dV/d\phi \) in the Klein–Gordon equation (2.5) is the opposite of the usual one. The potential is required to be positive by eq. (2.3), and this modification of the usual Klein–Gordon equation implies that a phantom falls up in an increasing potential, is repelled by a minimum, and settles in a maximum of \( V \) \cite{12, 33, 34}. In addition, a monotonically increasing unbounded potential may be expected to generate runaway solutions – the analogy with a canonical scalar field in a negative potential not bounded from below applies (these properties are proven later in this paper).

Eq. (2.3) implies that the phantom energy density is non–negative for any solution of eqs. (2.3)–(2.5). Only two equations in the set (2.3)–(2.5) are independent – when \( \dot{\phi} \neq 0 \) one can derive the Klein–Gordon equation (2.5) from the other two.

The field equations can be obtained from the Lagrangian \( L \) or the Hamiltonian \( \mathcal{H} \)

\[
L = 3a\dot{a}^2 + \kappa a^3 \left[ \frac{\dot{\phi}^2}{2} + V(\phi) \right], \quad \mathcal{H} = a^3 \left[ H^2 + \frac{\kappa}{6} \dot{\phi}^2 - \frac{\kappa V(\phi)}{3} \right], \tag{2.6}
\]

the Hamiltonian constraint (2.3) corresponding to \( \mathcal{H} = 0 \). The orbits of the solutions of eqs. (2.3)–(2.5) are constrained to the surface of constant energy \( \mathcal{H} = 0 \) in the phase space \( (a, \dot{a}, \phi, \dot{\phi}) \). We choose as dynamical variables the Hubble parameter and the scalar field \( (H, \phi) \) which correspond to physical observables\(^1\). The phase space accessible to the orbits of the solutions is the two-dimensional energy surface \( \mathcal{H} = 0 \) in the three-dimensional space \( (H, \phi, \dot{\phi}) \); this surface is in general curved (the appendix provides an example).

Once the values of \( H \) and \( \phi \) are given, one computes the corresponding value(s) of \( \dot{\phi} \) by rewriting the Hamiltonian constraint (2.3) as the algebraic equation for \( \dot{\phi} \)

\[
\dot{\phi}^2 - 2V + \frac{6H^2}{\kappa} = 0, \tag{2.7}
\]

\(^1\)Other authors choose as dynamical variables various combinations of \( H \) and \( \phi \) but this obscures the physical interpretation of their results.
with solutions

\[ \dot{\phi}_\pm (H, \phi) = \pm \sqrt{2 \left[ V(\phi) - \frac{3H^2}{\kappa} \right]} \equiv \pm \sqrt{F(H, \phi)}. \] (2.8)

Depending the form of \( V(\phi) \), there may be a region \( \mathcal{F} \) of the phase space forbidden to the dynamics and corresponding to a negative sign of \( F \equiv 2(V - 3H^2/\kappa) \):

\[ \mathcal{F} \equiv \left\{ \left( H, \phi, \dot{\phi} \right) : \quad V(\phi) < \frac{3H^2}{\kappa} \right\}. \] (2.9)

When \( F(H, \phi) > 0 \) there are two distinct values of \( \dot{\phi} \) for each value of \( H \) and \( \phi \), corresponding to the fact that the two-dimensional phase space consists of two sheets joining each other only at the points of the set

\[ \mathcal{B} \equiv \left\{ \left( H, \phi, \dot{\phi} \right) : \quad V(\phi) = \frac{3H^2}{\kappa}, \quad \dot{\phi} = 0 \right\}, \] (2.10)

which constitutes the boundary of the forbidden region \( \mathcal{F} \) and lies in the \( (H, \phi) \) plane. We denote the sheet corresponding to the positive sign of \( \dot{\phi} \) “upper sheet”, while the other is called “lower sheet”. This structure of the phase space is general in scalar–tensor cosmology [31] and is well known in the theory of a scalar field with canonical kinetic energy coupled non–minimally to the Ricci curvature [35, 36, 37]. Phantom cosmology can be seen as a special case of scalar–tensor theory for which there is always a forbidden region in the phase space, while this needs not be the case in other scalar–tensor theories. The equilibrium points of the system (2.4)–(2.5) consist of de Sitter spaces with constant scalar field \( (H_0, \phi_0) \). If they exist, these fixed points satisfy the constraints imposed by eqs. (2.8)–(2.9)

\[ H_0 = \pm \sqrt{\frac{\kappa V_0}{3}}, \quad V'_0 = 0, \] (2.11)

where \( V_0 \equiv V(\phi_0) \) and \( V'_0 \equiv \frac{dV}{d\phi}\bigg|_{\phi_0} \). The Hamiltonian constraint (2.8) can only be satisfied if \( V(\phi) \geq 0 \), which guarantees that \( H_0 \) is real – then there are de Sitter fixed points provided that \( dV/d\phi \) has zeros (the existence of equilibrium points is not guaranteed in all scalar–tensor theories). These fixed points lie on the boundary \( \mathcal{B} \) of the forbidden region\(^2\).

\(^2\)In other scalar–tensor theories the fixed points, if they exist, may be located anywhere on the energy surface \( \mathcal{H} = 0 \).
What happens to an orbit with initial conditions chosen exactly on $B$? To answer this question we consider the tangent to the orbits

$$\vec{T} = (\dot{H}, \dot{\phi}, \ddot{\phi}) = \left(\frac{\kappa}{2} \dot{\phi}^2, \dot{\phi}, \frac{dV}{d\phi} - 3H \dot{\phi}\right)$$

(2.12)

in the $(H, \phi, \dot{\phi})$ space. On $B$ it is $\dot{\phi} = 0$ and $\vec{T}_B = (0, 0, dV/d\phi)$. Hence if $dV/d\phi|_B > 0$, an orbit beginning on $B$ will move into the upper sheet, while if $dV/d\phi|_B < 0$, the orbit will move into the lower sheet. If instead $dV/d\phi|_B = 0$ one has a fixed point satisfying the properties $\dot{H} = 0$, $\dot{\phi} = 0$, $H^2 = \kappa V/3$, and $V' = 0$.

The following result is true for any form of the potential $V(\phi)$: $H$ always increases along the orbit of any solution except at points where $\dot{\phi} = 0$, at which also $\dot{H}$ vanishes.

This follows by combining eqs. (2.3) and (2.4) to obtain

$$\dot{H} = \frac{\kappa}{2} \dot{\phi}^2,$$

(2.13)

hence $\dot{H} > 0$ except at the points where $\dot{\phi} = 0$, in particular the fixed points $(H_0, \phi_0)$.

Eq. (2.13) is expressed by saying that the universe always superaccelerates (i.e., $\dot{H} > 0$ as opposed to acceleration defined by $\ddot{a} = \dot{H} + H^2 > 0$), or that the phantom field is a form of superquintessence [10, 11]. Eqs. (2.3) and (2.13) imply that

$$\dot{H} + 3H^2 = \kappa V.$$

(2.14)

A second result is that there are no limit cycles (periodic orbits).

In fact, $H(t)$ is non-decreasing and hence it cannot periodically return to its initial value apart from the trivial case of a fixed point. If $dV/d\phi$ has definite sign this possibility is also excluded, cf. eq. (2.11).

### 3 Bounded potential

If the potential $V(\phi)$ is bounded from above by a (positive) constant $V_0$, then the asymptotic solution of eqs. (2.3)–(2.5) at large times is such that $\dot{H} \to 0$ and $\dot{\phi} \to 0$. The resulting de Sitter attractor $(H_0, \phi_0)$ is a global attractor.

In fact, according to eq. (2.14), $\dot{H} = \kappa V - 3H^2$; assume that $\dot{H}$ does not tend to zero (or that it does not tend to zero faster than $t^{-1}$) as $t \to +\infty$. Then the limit

$$\lim_{t \to +\infty} H(t) = \lim_{t \to +\infty} \int_0^t \dot{H}(\tau) d\tau = +\infty$$

(3.1)

---

3Eq. (2.13) is equivalent to eq. (7) of [11] multiplied by $\dot{\phi}$. 

---
since $\dot{H} > 0$. But then $V(t) > 3H^2/\kappa \to +\infty$ as $t \to +\infty$ and this contradicts the fact that $V$ is bounded from above. Hence it must be $\dot{H} \to 0$ (faster than $t^{-1}$) as $t \to +\infty$.

Since $\dot{H} = \kappa \dot{\phi}^2/2$, then $\dot{\phi} \to 0$ (faster than $t^{-1/2}$) as well when $t \to +\infty$. The asymptotic state for a potential bounded from above is one with $\dot{H} \to 0$ (faster than $t^{-1}$) as $t \to +\infty$. Hence it must be $\dot{H} \to 0$ (faster than $t^{-1}$) as $t \to +\infty$. The asymptotic state for a potential bounded from above is one with $\dot{H} \to 0$ and $\dot{\phi} \to 0$.

Eq. (2.14) gives the asymptotic relation $H_0^2 = \kappa V / 3$ and the Klein–Gordon equation reduces, in this limit, to $dV/d\phi|_{\phi_0} = 0$ and the asymptotic state is the de Sitter fixed point $(H_0, \phi_0)$. The function

$$L(H, \phi) = (H - H_0)^2$$

(3.2)

defined on the entire phase space is a Ljapunov function – in fact $L(H, \phi) > 0 \forall (H, \phi) \neq (H_0, \phi_0)$, $L(H_0, \phi_0) = 0$, and $dL/dt = \kappa \dot{\phi}^2 (H - H_0) \leq 0 \forall (H, \phi) \neq (H_0, \phi_0)$ because $H$ tends to $H_0$ while non–decreasing, hence $H \leq H_0 \forall t$. The system is stable and the attraction basin of the de Sitter attractor is the entire phase space. This excludes the possibility of a Big Rip or a sudden future singularity.

If the potential has maxima, minima, or inflexion points, there will be fixed points of the system (2.3)–(2.5). The character of these points determines the stability of the fixed point. One can assess the stability by considering the perturbed equations (2.4)–(2.5) for homogeneous perturbations that depend only on time, as done in [38, 39]. It is more meaningful to study stability against inhomogeneous perturbations which depend on both space and time. Then the complication arises that a gauge-independent formalism is required. The study of inhomogeneous perturbations is carried out in [40] for generalized gravity theories described by the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} f(\phi, R) - \frac{\omega(\phi)}{2} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \right].$$

(3.3)

The result of [40] is that there is stability if and only if

$$\frac{f_{\phi\phi}(\phi_0)}{2} - V_{\phi\phi}(\phi_0) + \frac{6 f_{\phi R}(\phi_0) H_0^2}{f_{R}(\phi_0)} \leq 0.$$  

(3.4)

Phantom cosmology is the special case $f(\phi, R) = R, \omega = -1, f_{\phi R} = f_{\phi\phi} = 0, f_R = 1$ and stability corresponds to $V'' \leq 0$, where a prime denotes differentiation with respect to $\phi$. The perturbation analysis of [40] establishes that the de Sitter fixed point is stable against inhomogeneous linear perturbations if $V(\phi)$ has a maximum there. Here we recover this result in a different approach that does not rely on a linear perturbation analysis and it is valid to any order, extending and complementing the results of [40].
applied to phantom cosmology. Moreover, [40] does not draw conclusions about the size of the attraction basin, while here we establish that this attraction basin is the entire phase space. This agrees with, and extends, the result of Guo et al. [41] that, if a late time attractor exists in the phase space of phantom cosmology with bounded potential, it is unique.

The phantom field settling in a maximum and producing a late time de Sitter attractor is consistent with the literature selecting specific potentials $V(\phi)$ and with general conjectures [34] [13] [33]. Let us see some examples. Singh, Sami and Dadhich [33] consider the bell-shaped potential $V(\phi) = V_0 / \cosh (\kappa \phi)$ attaining its global maximum $V_0$ at $\phi = 0$, and they find a late time de Sitter attractor. The same type of attractor is found by Carroll, Hoffman and Trodden [13] for the Gaussian potential $V(\phi) = V_0 e^{-\phi^2/\sigma}$.

Hao and Li [39] consider the inverse power-law potential $V(\phi) = V_0 - \sigma (\phi/\phi_0)^2$ for $|\phi| \leq \phi_0 \sqrt{V_0/\sigma}$, with maximum $V_0$ at $\phi = 0$. They find again a de Sitter attractor empty of scalar field ($\phi = 0$) and with a truly constant cosmological constant $\Lambda = V_0$.

Another situation in which the potential is bounded from above occurs when $V(\phi)$ has a horizontal asymptote $V_0$ approached from below as $\phi \to \pm \infty$. Under these conditions the scalar field rolls up the slope of the potential but its (negative) kinetic energy is dissipated by friction – described by the term $3H\dot{\phi}$ in eq. (2.5) – while the force term $-dV/d\phi$ tends to zero approaching the asymptote, with $3H\dot{\phi} \simeq dV/d\phi$ and $\ddot{\phi} \to 0$. As a result the motion stops ($\dot{\phi} \to 0$) and a de Sitter regime with constant $\phi$ and $H$ is approached. An example is the potential studied by Sami and Toporensky [34] $V(\phi) = V_0 \left[1 - e^{-c\phi^2}\right]$, which has the shape of an inverted bell with $V_0$ as horizontal asymptote. These authors find a late time de Sitter attractor with cosmological constant $\Lambda = V_0$ [34].

Guo et al. [41] consider the bounded potential $V(\phi) = V_0 [1 + \cos (\phi/f)]$ and find a slow-climb solution in which $\phi$ settles in the maximum of the potential at $\phi = 0$ in a de Sitter regime with $(H_0, \phi_0) = \left(\sqrt{\kappa V_0/3}, 0\right)$.

### 4 Unbounded potential

If the phantom potential is not bounded from above it would seem that the asymptotic state could be either a de Sitter regime, or a very different state, possibly a Big Rip, depending on the shape of $V(\phi)$ and its slope. Physically, the motion of $\phi(t)$ is analogous to the motion of a ball with negative kinetic energy falling up a hill under the force $dV/d\phi$, damped by the term $3H\dot{\phi}$. If this friction becomes negligible in comparison with the force and inertial terms, $3H\dot{\phi} << dV/d\phi, \ddot{\phi}$, one has a “slow-climb” regime analogous
to the slow-roll, potential-dominated, regime of inflation for a canonical scalar field. This happens if the potential becomes sufficiently steep. If instead friction is comparable to the force term, motion could stop \((\dot{H} \sim 0, \dot{\phi} \sim 0)\) corresponding to a de Sitter regime. If \(V\) is strictly increasing this situation is forbidden and the expansion is super–exponential: if \(V(\phi)\) is not bounded from above and is strictly increasing the solution of eqs. (2.3)–(2.5) cannot approach a de Sitter fixed point at late times, but \(H\) and \(\phi\) go to infinity.

In fact, a de Sitter fixed point has \(V' = 0\) and in order to approach a de Sitter fixed point it must be \(V' = 0\), which is incompatible with our assumptions. Hence \(H\) and \(\phi\) go to infinity.

This result explains the examples available in the literature, in which an unbounded potential with \(dV/d\phi > 0\) never produces a late time de Sitter solution but rather gives a universe that expands faster than exponentially. Sami and Toporensky and Guo et al. consider the simple potential \(V(\phi) = m^2\phi^2/2\). Let us consider the region \(\phi > 0\) where \(dV/d\phi\) is positive – there \(\dot{\phi}\) and \(\dot{H}\) approach constant values at late times,

\[
\begin{align*}
\dot{\phi} & \approx \sqrt{\frac{2}{3\kappa}} m, \\
\dot{H} & \approx \frac{\kappa m}{\sqrt{3}} \dot{\phi}
\end{align*}
\]

as \(t \to +\infty\), and \(H, \phi \propto t\) asymptotically. The motion becomes potential–dominated at large times and the ratio of kinetic to potential energy \(-\frac{\dot{\phi}^2}{2V} \approx -\frac{\dot{\phi}^2}{m^2\phi^2 t^2} \to 0\) so that \(V \gg \dot{\phi}^2/2\) asymptotically. Both \(H\) and \(\phi\) diverge and the scale factor \(a(t) = a_0 e^{\alpha t^2/2}\) diverges faster than exponentially. However, it takes an infinite time for \(\phi\) and \(H\) to reach infinity and there is no Big Rip. The effective equation of state parameter

\[
w \equiv \frac{P}{\rho} = \frac{\dot{\phi}^2 + m^2\phi^2}{\phi^2 - m^2\phi^2} \to -1
\]

as \(t \to +\infty\). The asymptotic equation of state approximates \(P = -\rho\), which is the exact equation of state for de Sitter space, but the solution is not de Sitter space nor does it approach it. Asymptotically, \(\ddot{\phi} \to 0\) and the friction and force terms balance each other, \(3H\dot{\phi} \approx dV/d\phi\). The appendix is devoted to this example and shows that (4.1) is an attractor.

Sami and Toporensky also consider power-law potentials \(V(\phi) = V_0 \phi^\alpha\) (with \(\alpha > 0\)) and they find a “slow–climb” regime characterized by

\[
\begin{align*}
\dot{\phi} & \approx \frac{V'}{3H}, \\
H^2 & \approx \frac{\kappa V}{3}, \\
\frac{\dot{\phi}^2}{2V} & \approx \frac{1}{6\kappa} \left(\frac{V'}{V}\right)^2
\end{align*}
\]
if \( \alpha < 4 \). For \( \alpha = 4 \) they find exponential growth of both \( \phi \) and \( H \), while if the power-law potential is steeper \( (\alpha > 4) \), they find a Big Rip solution with \( \phi(t) \approx (t_0 - t)^{\frac{2}{3(1+w)}} \). This result is confirmed by Guo et al. \[41\] and agrees with the qualitative argument on the steepness of the potential \( V(\phi) \).

Hao and Li \[42\] consider the exponential potential \( V(\phi) = V_0 \exp(c\phi) \) (with \( c > 0 \)), which is unbounded and has \( dV/d\phi > 0 \) everywhere. They find an attractor with equation of state parameter \( w < -1 \) which makes the Big Rip unavoidable. In \[39\] the same authors consider again the exponential potential \( V(\phi) = V_0 \exp[\sqrt{3} \kappa A (\phi - \phi_0)] \) with \( V_0, A > 0 \) finding a Big Rip attractor in parametric form \( (a(\phi), t(\phi)) \). By eliminating the parameter \( \phi \) one finds the scale factor

\[
a(t) = \frac{a_0}{(\sqrt{V_0 A^2})^{\frac{2}{3(1+w)}}} \left[ \frac{2 \sqrt{A^2} + 2}{\sqrt{4V_0 \kappa A^2}} + t_0 - t \right]^{-\frac{2}{3A^2}}. \tag{4.4}
\]

In simple terms, the effective equation of state parameter is constant, \( w = -(1 + A^2) \), and the scale factor can be written as \( a \propto (t_* - t)^\frac{2}{3(1+w)} \). It is well known that a spatially flat FLRW universe filled with a fluid with constant equation of state \( P = w \rho \) has scale factor \( a \propto t^\frac{2}{3(1+w)} \) \[13\].

It seems intuitive that a potential \( V(\phi) \) unbounded and steeper than the exponential potential will always cause a sudden future singularity. This can be shown for extremely steep potentials: if \( V(\phi) \) has a vertical asymptote the universe evolves to a future singularity in a finite time.

In fact, since \( V(\phi) \rightarrow +\infty \) as \( \phi \rightarrow \phi_c \), it is \( dV/d\phi > 0 \). \( V \) is unbounded and strictly increasing and, due to our previous result, \( \dot{\phi} \) cannot tend to zero. The only two possibilities are that \( \dot{\phi} \) tends to a finite limit \( \dot{\phi}_c \) or that \( \dot{\phi} \) diverges. In both cases \( \dot{\phi} \) reaches the critical value \( \dot{\phi}_c \) in a finite time and \( V \) and \( dV/d\phi \) diverge. \( \dot{\phi} \) cannot have a finite limit \( \dot{\phi}_c \): in fact, if this happens, then \( \dot{H} = \kappa \dot{\phi}_c^2/2 \rightarrow H_c \equiv \kappa \dot{\phi}_c^2/2 \) and \( \ddot{\phi} \rightarrow 0 \). Then the modified Klein–Gordon equation \[2.5\] yields \( 3 \dot{H} \phi \approx dV/d\phi \rightarrow +\infty \) asymptotically and \( H \) must diverge in a finite time, which is in contradiction with \( \dot{H} \simeq \text{const.} \) Hence it must be \( \dot{\phi} \rightarrow +\infty \) and \( \dot{H} = \kappa \dot{\phi}_c^2/2 \rightarrow +\infty \). It may happen that both \( H \) and \( \dot{\phi} \) diverge or that they stay finite with their derivatives diverging at a sudden future singularity. Examples of this kind of singularities have been found in other models of dark energy based on the tachyonic field \[44\], the brane world \[15\], and cosmology with the Gauss–Bonnet term \[46\].

Out last result is consistent with the numerical examples of \[42,34\], who find a sudden future singularity for potentials with a vertical asymptote.
5 Discussion and conclusions

The dynamics of phantom cosmology are studied without assuming specific potentials $V(\phi)$, but assuming that the phantom dark energy has already come to dominate the cosmic dynamics. In the literature there are also scenarios in which the late time state of the universe is dominated again by cold (dark and ordinary) matter with zero pressure and the phantom energy decays [39]: such scenarios are a priori excluded from our analysis.

A clear and unified picture of the dynamics is obtained: the phase space is a two-dimensional surface in the $(H, \phi, \dot{\phi})$ space and is composed of two sheets. All the equilibrium points, which exist if and only if $dV/d\phi$ has zeros, are de Sitter spaces located on the curve where the two sheets touch each other – they are given by eqs. (2.11). $H$ is always increasing except when $\dot{\phi}$ vanishes, and there are no periodic orbits. In the presence of a bounded potential $V(\phi)$ the universe has a late time de Sitter attractor whose attraction basin is the entire phase space. If $V(\phi)$ is unbounded there cannot be a de Sitter attractor and the late time cosmic expansion is either super–exponential with infinite lifetime or else the universe ends in a Big Rip or other sudden future singularity. Unfortunately, in the general situation, we cannot provide a sharp criterion on the potential to discriminate between these two possibilities. Many results for specific potentials that are found in the literature are recovered and predictions are made for new scenarios featuring potentials with the shapes discussed.

It is worth remembering that there are serious doubts on whether phantom fields can be stable [13, 25, 26] and that the classical equations of motion considered have to be superseded by a semiclassical treatment near the Big Rip singularity. Indeed there is some evidence that semiclassical backreaction may avoid the Big Rip [19].

Finally, there are compelling arguments for a field with canonical kinetic energy to couple nonminimally to the Ricci curvature [50, 11]. In particular, conformal coupling is associated with an infrared fixed point of the renormalization group [51] and avoids causal pathologies [52]. It is not clear whether the same arguments apply to a phantom field and force it to couple nonminimally (a nonminimally coupled phantom is considered in [17, 29]). The slow–climb regime for a phantom would be altered by the inclusion of non–minimal coupling terms in eqs. (2.3)–(2.5), similarly to what happens in slow–roll inflation [53]. The dynamics of a nonminimally coupled phantom will be studied elsewhere.
Acknowledgments

The author thanks Werner Israel for pointing out Ref. [54]. This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

Appendix: massive phantom field

Here we consider a phantom field with potential $V(\phi) = \frac{m^2}{2} \phi^2$. The phase space is the two-dimensional surface $\frac{6}{\kappa} H^2 + \dot{\phi}^2 = \frac{m^2}{2} \phi^2$. By using the dimensionless variables

$$x \equiv \sqrt{\kappa} H, \quad y \equiv \frac{\kappa m}{\sqrt{6}} \phi, \quad z \equiv \frac{\kappa}{\sqrt{6}} \dot{\phi},$$

the equation of the surface can be written as $y = \pm \sqrt{x^2 + z^2}$, which describes a cone with axis along the $y$-axis and vertex in the origin. The upper sheet is described by $y = +\sqrt{x^2 + z^2}$ and the lower sheet by $y = -\sqrt{x^2 + z^2}$; the two sheets join on the plane $z = 0$ along the lines $y = \pm x$. Although we restrict our analysis to spatially flat ($K = 0$) universes, it can be shown that the cone separates trajectories belonging to an open ($K = -1$) universe lying inside the cone from orbits belonging to a closed ($K = +1$) universe lying outside the cone (cf. [54]).

The only equilibrium point satisfying eqs. (2.11) is the Minkowski space $(x, y, z) = (0, 0, 0)$. Our results imply that $H$ and $\phi$ go to infinity for any solution except the equilibrium point, and hence the equilibrium point is unstable. Since $V''(0) = m^2 > 0$ the fixed point (which coincides with the global minimum of the potential) is unstable – see the discussion of Sec. III.

This three-dimensional picture should be compared with fig. 3 of [41] showing a projection of the phase space (a two-dimensional cone) and of some orbits onto the $(\phi, \dot{\phi})$ plane, corresponding to our $(y, z)$ plane. A similar conical geometry of the phase space structure appears if one considers inflation realized by a massive scalar with the “right” sign of the kinetic energy density [54].

The trajectories of the solutions exhibit a late time attractor found analytically in [34] (eq. (4.1)). The authors of [34] [41] show numerically the convergence of many solutions to this asymptotic solution but do not prove analytically its attractor nature. The attractor can be obtained by requiring an asymptotic linear relationship between $H$ and $\phi$, as suggested by numerical integrations. By setting $H = \alpha \phi$ one obtains, using eq. (2.13), $\dot{\phi} = 2 \alpha / \kappa$. Eq. (2.3) yields $H \approx \pm \sqrt{\frac{2}{3\kappa}} \alpha m t$, while eq. (2.4) is identically satisfied and eq. (2.5) gives $\alpha = m \sqrt{\kappa / 6}$. Since $H \geq 0$ the negative sign is rejected. The
asymptotic solution found in this way coincides with the one of [34]

\[(H_*, \phi_*) = \left( \frac{m^2}{3} t, \sqrt{\frac{2}{3\kappa}} m t \right)\]  \hspace{1cm} (A.2)

The stability with respect to linear homogeneous perturbations is assessed by considering the perturbed Hubble parameter and scalar field \(H = H_\ast + \delta H, \phi = \phi_\ast + \delta \phi\), and inserting these expressions into the evolution equations (2.3)–(2.5). It is convenient to consider the contrasts \(\Delta_1 \equiv \delta H/H_\ast\) and \(\Delta_2 \equiv \delta \phi/\phi_\ast\). One finds \(\Delta_1 = \sqrt{\frac{3\kappa}{2} \frac{1}{mt}} \left( \delta \phi - \frac{\delta \dot{\phi}}{m^2 t} \right)\) and eq. (2.5) yields the evolution equation for \(\Delta_2\)

\[
\ddot{\Delta}_2 + \left( \frac{2}{t} - m + m^2 t \right) \Delta_2 + \left( m^2 - \frac{1}{t^2} \right) \Delta_2 = 0 \hspace{1cm} (A.3)
\]

with asymptotic solution \(\Delta_2 \simeq \text{const.}\) and \(\Delta_1 = \Delta_2 \left( 1 - \frac{1}{m^2 t^2} \right) - \frac{\dot{\Delta}_2}{\dot{\Delta}_1} \simeq \text{constant}\). The solution \((H_\ast, \phi_\ast)\) is stable and is an attractor, in agreement with the numerical results.

References


