Track fitting in slightly inhomogeneous magnetic fields

J. Alcaraz \(^{a,1}\)

\(^{a}\)CIEMAT, Avda. Complutense, 22, 28040-Madrid, SPAIN

Abstract

A fitting method to reconstruct the momentum and direction of charged particles in slightly inhomogeneous magnetic fields is presented in detail. For magnetic fields of the order of 1 T and inhomogeneity gradients as large as 1 T/m the typical momentum bias due to the proposed approximations is of the order of few MeV, to be compared with scattering components of the order of 20 MeV or even larger. This method is currently being employed in the reconstruction programs of the AMS experiment.

1 Introduction

The next generation of particle physics experiments will be characterized by an unprecedented accuracy in the determination of the positions and momenta of very energetic charged particles. The principle of momentum measurement is, in all cases, the linear relation between the track curvature and the inverse of the momentum in a plane perpendicular to the magnetic field direction. High energy physics experiments have successfully employed this principle for many decades, producing many relevant physics results and discoveries.

Track fitting in inhomogeneous magnetic fields involves the propagation of track parameters between consecutive detector layers. Typical approaches [1,2], use numerical methods of high order in order to integrate the equations of motion from layer to layer. This report describes a simple alternative algorithm, currently being employed in the AMS experiment [3]. All propagation operations are expressed in terms of path integrals, which are approximated with enough accuracy at initialization. The fitting step is also reduced to a linear approximation.

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problem. The simplicity of the algorithm allows fast refitting of tracks, even beyond the reconstruction phase, i.e. at the level of final physics analyses.

The report is organized as follows. The approximations that are the basis of the method and few formulae to estimate its accuracy are presented in section 2. The track fitting logic and a typical implementation are described in section 3. Section 4 discusses the inclusion of multiple scattering effects. The report is summarized in section 5.

2 Basis of the method

The trajectory of a particle with charge \( q \) in a static magnetic field \( \vec{B} \) is governed by the equation:

\[
\frac{d\vec{p}}{dt} = q (\vec{v} \times \vec{B})
\]

where \( \vec{p} \) and \( \vec{v} \) are the momentum and velocity of the particle at a given position \( \vec{x} \) and time \( t \) on the trajectory. An immediate conclusion is that \( v \equiv ||\vec{v}|| \) and \( p \equiv ||\vec{p}|| \) are constants of motion. Locally, the trajectory is a helix, with \( p \sin \theta_B = q ||\vec{B}|| R \), where \( R \) is the (signed) radius of curvature in a plane transverse to \( \vec{B} \) and \( \theta_B \) the angle between \( \vec{v} \) and \( \vec{B} \). Equation 1 can be rewritten in a different way:

\[
\frac{d\vec{u}}{dl} = \frac{q}{p} (\vec{u} \times \vec{B})
\]

\[
\vec{u} \equiv \frac{d\vec{x}}{dl} = \frac{\vec{v}}{v}
\]

where \( dl \equiv v \, dt \) is the differential length traversed by the particle. More visually, \( \vec{u} \) is a unitary vector tangent to the trajectory at the point \( \vec{x} \). Integration of this expression between two consecutive layers of a tracker detector, denoted by the subscripts 0 and 1, leads to:

\[
\vec{u}_1 = \vec{u}_0 + \frac{q}{p} \int_{l_0}^{l_1} \, dl' \left( \vec{u} \times \vec{B} \right) (l')
\]

The first approximation in our method consists in computing the previous equation as follows:
\[ \vec{u}_1 \approx \vec{u}_0 + \frac{q}{p} \int_0^1 d\alpha \ (\vec{x}_1 - \vec{x}_0) \times \vec{B} (\vec{x}_0 + \alpha[\vec{x}_1 - \vec{x}_0]) \]  

(4)

that is, computing the integral along the straight line connecting \( \vec{x}_0 \) and \( \vec{x}_1 \). The approximation is exact in two cases: a) when the magnetic field is homogeneous, and b) in the infinite momentum limit. For the homogeneous field case, \( \vec{B}(l) \equiv \vec{B}_0 \), and Equation 3 becomes:

\[ \vec{u}_1 = \vec{u}_0 + \frac{q}{p} \left( \int_{l_0}^{l_1} dl' \frac{\vec{x}_1 - \vec{x}_0}{\|\vec{x}_1 - \vec{x}_0\|} \right) \times \vec{B}_0 \]  

(5)

where the last equality is obtained by introducing the definition of \( \vec{u} \): \( d\vec{x} \equiv \vec{u} dl \). The result is identical to the one obtained using Equation 4. In the limit of very high momentum, case b), \( \vec{u} = \frac{\vec{x}_1 - \vec{x}_0}{\|\vec{x}_1 - \vec{x}_0\|} \) up to relative corrections of order \( 1/p \), leading trivially again to Equation 4.

In order to estimate the accuracy of the approximation in a general case, two additional expressions are necessary. First, the expression of the magnetic field as a series expansion around \( l_{1/2} \equiv (l_0 + l_1)/2 \):

\[ \vec{B}(l) = \vec{B}_0 + \vec{B}'_0 (l - l_{1/2}) + \ldots \]  

(6)

Second, the estimate \( \Delta(\vec{u}) \) of the difference between the true \( \vec{u} \) vector and \( \frac{\vec{x}_1 - \vec{x}_0}{\|\vec{x}_1 - \vec{x}_0\|} \):

\[ \vec{u} = \frac{\vec{x}_1 - \vec{x}_0}{\|\vec{x}_1 - \vec{x}_0\|} + \Delta(\vec{u}); \]

\[ \Delta(\vec{u}) = \frac{q}{p} \int_{l_{1/2}}^l d\ell' \frac{\vec{x}_1 - \vec{x}_0}{\|\vec{x}_1 - \vec{x}_0\|} \times \vec{B}_0 + \mathcal{O} \left( \frac{(q/p)^2 \|\vec{B}'_0\|}{\|\vec{B}_0\|} \right) \]  

(7)

Introducing the previous expressions in Equation 3 and comparing with Equation 4 one obtains the following correction at first order:

\[ \frac{q}{p} \int_{l_0}^{l_1} dl' \Delta(\vec{u})(l') \times \vec{B}'_0 (l' - l_{1/2}) \]

\[ = \left( \frac{q}{p} \right)^2 \int_{l_0}^{l_1} dl' \int_{l_{1/2}}^{l'} dl'' (l' - l_{1/2}) \left( \frac{\vec{x}_1 - \vec{x}_0}{\|\vec{x}_1 - \vec{x}_0\|} \times \vec{B} \right) \times \vec{B}'_0 \]

\[ \approx \frac{q}{p} \frac{(l_1 - l_0)^2 \|\vec{B}'_0\|}{12 R} \]  

(8)
where $R$ is the approximate radius of the trajectory from $l_0$ to $l_1$. The effect has to be compared with the corresponding term in Equation 4, of order $\frac{q}{p} \|B_0\| (l_1 - l_0)$. The difference translates into a relative momentum shift of order:

$$\frac{\Delta p}{p} \approx \frac{l_1 - l_0}{12 R} \times \frac{\Delta B}{\|B_0\|} \approx \frac{0.3 (l_1 - l_0)[m]}{12 p[GeV]} \Delta B[T]$$

where $\Delta B$ is the typical variation of the magnetic field between $l_0$ and $l_1$. For instance, inhomogeneities in the magnetic field of the order of 1 T/m imply uncertainties of order $\Delta p \approx \pm 1.6$ MeV, independent of the absolute value of the rigidity. This has to be compared with the typical contributions from multiple scattering in silicon detectors. For AMS-02 \cite{3}, optimized in this respect, the expected momentum resolution at the lowest momenta suggests multiple scattering effects of order $\Delta p \gtrsim 20$ MeV \cite{3}, safely beyond the accuracy of the approximation.

The second approximation concerns the extrapolation of the position vector onto the adjacent plane. Integrating Equation 2 twice we obtain:

$$\vec{x}_1 = \vec{x}_0 + \vec{u}_0 (l_1 - l_0) + \frac{q}{p} \int_{l_0}^{l_1} dx \int_0^x dy \left( \vec{u} \times \vec{B} \right) (y)$$

$$= \vec{x}_0 + \vec{u}_0 (l_1 - l_0) + \frac{q}{p} \int_{l_0}^{l_1} dy (l_1 - y) \left( \vec{u} \times \vec{B} \right) (y)$$

The $p \to \infty$ limit reads:

$$\vec{x}_1 = \vec{x}_0 + \vec{u}_0 ||\vec{x}_1 - \vec{x}_0|| + \frac{q}{p} \int_{l_0}^{l_1} dy (l_1 - y) \left[ \frac{\vec{x}_1 - \vec{x}_0}{||\vec{x}_1 - \vec{x}_0||} \times \vec{B}(y) \right]$$

The previous expression, which is linear in $\vec{x}_0$, $\vec{u}_0$ and $q/p$, does not coincide in general with the exact solution for the homogeneous magnetic field case. Nevertheless, we will prove that it is precise enough for most cases of interest. Let us define a convenient orthonormal reference system by the unitary vectors $\vec{u}_A$, $\vec{u}_B$ and $\vec{u}_C$:
\[ \vec{u}_A = \frac{(\vec{x}_1 - \vec{x}_0) - [\vec{u}_B(\vec{x}_1 - \vec{x}_0)] \vec{u}_B}{\| (\vec{x}_1 - \vec{x}_0) \| \sin \theta_B} \] (12)

\[ \vec{u}_B \equiv \frac{\vec{B}}{\| \vec{B} \|} \] (13)

\[ \vec{u}_C = \frac{(\vec{x}_1 - \vec{x}_0) \times \vec{u}_B}{\| (\vec{x}_1 - \vec{x}_0) \| \sin \theta_B} \] (14)

Note that \( \vec{u}_B \) is the unitary vector in the direction of the magnetic field, and \( \theta_B \) the angle between the vectors \( (\vec{x}_1 - \vec{x}_0) \) and \( \vec{B} \). In terms of these vectors, the trajectory in a homogeneous field corresponds to:

\[ \vec{u}(l) = \lambda_A(l) \vec{u}_A + \lambda_B(l) \vec{u}_B + \lambda_C(l) \vec{u}_C; \] (15)

\[ \lambda_A(l) = \sin \theta_B \cos \left[ \frac{\sin \theta_B}{R} (l - l_{ref}) \right] \] (17)

\[ \lambda_B(l) = \cos \theta_B \] (18)

\[ \lambda_C(l) = \sin \theta_B \sin \left[ \frac{\sin \theta_B}{R} (l - l_{ref}) \right] \] (19)

with \( \theta_B \) and \( l_{ref} \) constants. Let us also write \( (l_1 - l_0) \) as an expansion in powers of \( 1/R \):

\[ (l_1 - l_0) = \| \vec{x}_1 - \vec{x}_0 \| + \frac{\| \vec{x}_1 - \vec{x}_0 \|^3 \sin^2 \theta_B}{24 R^2} + \ldots \] (20)

Equation 10 for the homogeneous case can be then rewritten as:

\[ \vec{x}_1 = \vec{x}_0 + \vec{u}_0 (l_1 - l_0) \]
\[ + \frac{q}{p} \int_{l_0}^{l_1} dy (l_1 - y) \left( (\lambda_A(y) \vec{u}_A + \lambda_B(y) \vec{u}_B + \lambda_C(y) \vec{u}_C) \times \vec{B}_0 \right) \]
\[ = \vec{x}_0 + \vec{u}_0 \| \vec{x}_1 - \vec{x}_0 \| + \frac{q}{p} \int_{l_0}^{l_1} dy (l_1 - y) \frac{\vec{x}_1 - \vec{x}_0}{\| \vec{x}_1 - \vec{x}_0 \|} \times \vec{B}_0 \]
\[ + \frac{\| \vec{x}_1 - \vec{x}_0 \|^3 \sin^2 \theta_B}{24 R^2} [\vec{u}_0 - 2 \vec{u}_A] + \ldots \] (21)

Since the linear term in \( q/p \) gives a contribution of order \( \frac{(l_1 - l_0)^2}{2R} \), a naive calculation would suggest a relative shift in the momentum of order:

\[ \frac{\Delta p}{p} \approx \frac{l_1 - l_0}{12 R} \approx \frac{0.3 (l_1 - l_0) [m]}{B [T]} \] (22)
For a benchmark separation of 20 cm and a magnetic field of 1 T, we obtain a maximum possible shift of $\Delta p \approx \pm 5\text{ MeV}$. In practice, the effect is even smaller, since the missing correction affects coordinates in directions less sensitive to bending ($\vec{u}_0$ and $\vec{u}_A$). For most experiments, the fitting procedure is based on the minimization of a function in which position measurements in bending and non-bending directions are almost decoupled. In this configuration, the correction above will act in quadrature, effectively leading to a momentum shift of order:

$$\frac{\Delta p}{p} \approx \frac{1}{2} \left( \frac{l_1 - l_0}{12 R} \right)^2 \approx \frac{1}{2} \left[ \frac{0.3 (l_1 - l_0)[\text{m}]}{12 p[\text{GeV}]} \right]^2$$

(23)

The quoted shift is negligible, even for large magnetic fields, like those of LHC and future linear collider detectors. In fact, this conclusion is somehow equivalent to the one reached in Reference [4] in the context of homogeneous magnetic fields. There, only measurements along the direction of the impact parameter with respect to the track at each point (i.e. the sensitive “bending” direction) are considered. This assumption leads naturally to a linear problem in terms of the curvature parameter [4].

In summary, the following approximations are considered to be accurate enough for most practical cases:

$$\vec{u}_1 \approx \vec{u}_0 + \frac{q}{p} \frac{\|\vec{x}_1 - \vec{x}_0\|}{\|\vec{x}_1 - \vec{x}_0\|} \int_0^1 d\alpha \frac{\vec{x}_1 - \vec{x}_0}{\|\vec{x}_1 - \vec{x}_0\|} \times \vec{B}(\vec{x}_0 + \alpha [\vec{x}_1 - \vec{x}_0])$$

(24)

$$\vec{x}_1 \approx \vec{x}_0 + \vec{u}_0 \|\vec{x}_1 - \vec{x}_0\| + \frac{q}{p} \|\vec{x}_1 - \vec{x}_0\|^2 \int_0^1 d\alpha (1 - \alpha) \frac{\vec{x}_1 - \vec{x}_0}{\|\vec{x}_1 - \vec{x}_0\|} \times \vec{B}(\vec{x}_0 + \alpha [\vec{x}_1 - \vec{x}_0])$$

(25)

3 Track fitting

For simplicity, it is assumed that all tracker sensitive layers are parallel to the $z$ direction and that uncorrelated position measurements are performed along the $x$ and $y$ directions. The extension to more elaborated geometrical configurations is straightforward, since only simple rotations of the predictions are involved. An obvious example is that of a detector with a radial configuration. To deal with it, it is enough to substitute one of the $\chi^2$ terms in the expressions presented later by a sum of residues along the azimuthal direction.

We consider a scenario in which $z$ coordinates are known with infinite precision, so they can be fixed to their nominal values. The inclusion of an ad-
ditional z-term is, nevertheless, a trivial extension to the proposed scheme. Multiple scattering effects will be discussed in the next section.

We need to determine the position of the track at the first plane, \( \mathbf{x}_0 \equiv (x_0, y_0, z_0) \), the tangent vector at the first plane, \( \mathbf{u}_0 \equiv (u_{0x}, u_{0y}, u_{0z}) \), and the inverse of the rigidity, \( q/p \). From Equation 25 we obtain, on the second plane:

\[
\mathbf{x}_1 \approx \mathbf{x}_0 + \mathbf{u}_0 l_{10} + \frac{q}{p} \tilde{\beta}_{10} l_{10}^2
\]

where the following definitions have been introduced:

\[
l_{j,j-1} \equiv \| \mathbf{x}_j - \mathbf{x}_{j-1} \|\]

\[
\tilde{\beta}_{j,j-1} \equiv \int_0^1 d\alpha \, (1 - \alpha) \left[ \frac{\mathbf{x}_j - \mathbf{x}_{j-1}}{\| \mathbf{x}_j - \mathbf{x}_{j-1} \|} \times \mathbf{B}(\mathbf{x}_{j-1} + \alpha [\mathbf{x}_j - \mathbf{x}_{j-1}]) \right]
\]

The integrals \( \tilde{\beta}_{j,j-1} \) and the lengths \( l_{j,j-1} \) are stored in an initialization phase. They are determined from the measured positions and the magnetic field values on the line segment defined by \( \mathbf{x}_{j-1} \) and \( \mathbf{x}_j \).

At the third layer the extrapolation is given by:

\[
\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{u}_1 l_{21} + \frac{q}{p} \tilde{\beta}_{21} l_{21}^2
\]

\[
= \mathbf{x}_0 + \mathbf{u}_0 l_{10} + \frac{q}{p} \tilde{\beta}_{10} l_{10}^2 + \tilde{\beta}_{10} l_{10} l_{21} + \frac{q}{p} \tilde{\beta}_{21} l_{21}^2
\]

\[
= \mathbf{x}_0 + \mathbf{u}_0 (l_{10} + l_{21}) + \frac{q}{p} \left( \tilde{\beta}_{10} l_{10}^2 + \tilde{\beta}_{21} l_{21}^2 + \tilde{\gamma}_{10} l_{10} l_{21} \right)
\]

where the following path integral definition has been introduced (according to Equation 24):

\[
\tilde{\gamma}_{j,j-1} \equiv \int_0^1 d\alpha \left[ \frac{\mathbf{x}_j - \mathbf{x}_{j-1}}{\| \mathbf{x}_j - \mathbf{x}_{j-1} \|} \times \mathbf{B}(\mathbf{x}_{j-1} + \alpha [\mathbf{x}_j - \mathbf{x}_{j-1}]) \right]
\]

For many cases of interest (AMS-02 for instance), a Simpson method with just a few points is enough to calculate \( \tilde{\beta}_{j,j-1} \) and \( \tilde{\gamma}_{j,j-1} \) with sufficient accuracy.

In general, the extrapolation to layer \( i \) can be written as:

\[
\mathbf{x}_i = \mathbf{x}_0 + \mathbf{u}_0 l_{10} + \frac{q}{p} \left[ \sum_{k=1}^{i} \left( \tilde{\beta}_{k,k-1} l_{k,k-1}^2 + \tilde{\gamma}_{k,k-1} l_{k,k-1} l_{k,k-1} \right) \right]
\]
where the lengths $l_{ik}$ must be interpreted as follows:

$$l_{ik} = \sum_{m=k+1}^{i} l_{m,m-1}$$  \hspace{1cm} (32)

The five parameters $(x_0, y_0, u_{0x}, u_{0y}, q/p)$ are finally obtained by minimization of the following chi-square:

$$\chi^2 = \sum_{i=0}^{N-1} \frac{(x_{i,meas} - x_i(x_0, y_0, u_{0x}, u_{0y}, q/p))^2}{\sigma_x^2} + \sum_{i=0}^{N-1} \frac{(y_{i,meas} - y_i(x_0, y_0, u_{0x}, u_{0y}, q/p))^2}{\sigma_y^2}$$  \hspace{1cm} (33)

where $x_{i,meas}$ and $y_{i,meas}$ are the measured positions on layer $i$ of the tracker, and $\sigma_x$ and $\sigma_y$ are the tracker position resolutions in the sensitive directions. The minimization leads to a linear equation, which can be easily solved via matrix inversion.

Using an even more simplified notation, the $\chi^2$ can be written in a more convenient form:

$$\chi^2 = \sum_{i=0}^{N-1} \frac{(x_{i,meas} - \sum_{k=1}^{5} F_{ik} p_k)^2}{\sigma_x^2} + \sum_{i=0}^{N-1} \frac{(y_{i,meas} - \sum_{k=1}^{5} G_{ik} p_k)^2}{\sigma_y^2}$$  \hspace{1cm} (34)

where $p_j; j = 1, 5$ defines the vector of parameters to be determined, $\vec{p} \equiv (x_0, y_0, u_{0x}, u_{0y}, q/p)$. The components of the matrices $F_{ij}$ and $G_{ij}$ are:

$$F_{i1} = 1$$
$$F_{i2} = 0$$
$$F_{i3} = l_{i0}$$
$$F_{i4} = 0$$
$$F_{i5} = \sum_{k=1}^{i} \left( \beta_{k,k-1}^x l_{k,k-1}^2 + \gamma_{k,k-1}^x l_{k,k-1} l_{ik} \right)$$  \hspace{1cm} (35)

and:

[8]
\[ G_{i1} = 0 \]
\[ G_{i2} = 1 \]
\[ G_{i3} = 0 \]
\[ G_{i4} = l_{i0} \]
\[ G_{i5} = \sum_{k=1}^{i} \left( \beta_{k,k-1} l_{k,k-1}^2 + \gamma_{k,k-1} l_{k,k-1} l_{ik} \right) \]  \( (36) \)

with the upper indices \( x \) and \( y \) denoting the \( x \) and \( y \) components of the vector integrals \( \beta_{k,k-1} \) and \( \gamma_{k,k-1} \).

### 4 Multiple scattering treatment

It will be assumed that the amount of traversed material is reasonably small and that the momentum range of interest is such that energy losses can be safely neglected. In these conditions, multiple scattering between layers \( j-1 \) and \( j \) is taken into account by estimating the additional uncertainty induced on the director vector \( \vec{u}_j \). This uncertainty depends on: a) the amount of traversed material in radiation lengths, b) the particle momentum and c) its velocity.

From layer \( j-1 \) to layer \( j \) a particle is traversing the amount of material \( X_{j,j-1} \), measured in radiation lengths. The rms angular deviations in the \( xz \) and \( yz \) projections are equal. Denoting them by \( \Delta_{j,j-1} \), they can be approximately parametrized \([5,6]\) as follows:

\[ \Delta_{j,j-1} = \frac{0.0136}{\beta} \frac{q}{p[GeV]} \sqrt{X_{j,j-1}} [1 + 0.038 \ln(X_{j,j-1})] \]  \( (37) \)

where \( \beta \) is the velocity of the particle (in \( c \) units) and \( p \) its momentum expressed in GeV. Note also that the \( X_{j,j-1} \) thicknesses hide a dependence on the director vectors \( \vec{u}_j \). The previous expression, accurate at the few percent level in the range \( 0.003 \lessapprox X_{j,j-1} \lessapprox 0.01 \) \([6]\), does not admit a Gaussian treatment, in the sense that the expected additive property as a function of the amount of material is not satisfied: \( \Delta^2(X + Y) \neq \Delta^2(X) + \Delta^2(Y) \).

It is convenient to work with Gaussian uncertainties in order to keep a \( \chi^2 \) minimization scheme. In the case of a very small amount of traversed material a possible approach is to assume the previous formula to be correct for the total amount of traversed material and then distribute the remaining deviations in a linear way at any intermediate plane, i.e. such that the rms deviation is always proportional to \( \sqrt{X} \). If the total amount of material is \( X_{tot} \), the suggestion implies:
\[
\Delta_{j,j-1} \simeq 0.0136 \left[ 1 + 0.038 \ln(X_{tot}) \right] \frac{q}{p[GeV]} \sqrt{X_{j,j-1}}
\]  

(38)

The previous estimate is usually consistent with the quoted accuracy of Equation [6]. For the AMS-02 silicon tracker, it overestimates the \(\text{rms} \) deviations at the intermediate planes by at most 4%.

From the fitting point of view, multiple scattering just modifies the directions at the different layers as follows:

\[
\vec{u}_j = \vec{u}_j(NO\ MS) + \sum_{k=1}^{j} \vec{\epsilon}_{k,k-1}
\]  

(39)

where \(\vec{u}_j(NO\ MS)\) denotes the calculation in the absence of multiple scattering and \(\vec{\epsilon}_{j,j-1}\) is a deviation that follows a Gaussian of mean zero and width \(\approx (\Delta_{j,j-1}, \Delta_{j,j-1}, 0)\). The \(j\) dependence enters through the amount of accumulated radiation lengths \(X_{j,j-1}\) between the exit of layer \(j - 1\) and the exit of layer \(j\). At the level of position measurements the modified trajectories read:

\[
\vec{x}_j = \vec{x}_j(NO\ MS) + \sum_{k=1}^{j-1} \vec{\epsilon}_{k,k-1} \left( \sum_{m=k}^{j-1} l_{m+1,m} \right) \equiv \sum_{k=1}^{j-1} \vec{\epsilon}_{k,k-1} l_{jk}
\]  

(40)

There are two possible options to include these additional sources of uncertainty in the \(\chi^2\). The first one is to fit all these additional parameters \((2(\text{number of planes} - 2))\) with additional Gaussian constraints according to the expected widths. We will employ a second option, keeping the same number of fitted parameters, but building new covariance matrices according to the Gaussian uncertainties \(\Delta_{k,k-1}\). In the absence of multiple scattering, the covariance matrices for the \(x\) and \(y\) projections, \(V^0_{ij}\) and \(W^0_{ij}\) are given by:

\[
V^0_{ij} = \begin{pmatrix}
\sigma_x^2 & 0 & \ldots & 0 \\
0 & \sigma_x^2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \sigma_x^2
\end{pmatrix} ;
W^0_{ij} = \begin{pmatrix}
\sigma_y^2 & 0 & \ldots & 0 \\
0 & \sigma_y^2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \sigma_y^2
\end{pmatrix}
\]  

(41)

In the presence of multiple scattering, we need to take into account all fully correlated sources via \(\vec{\epsilon}_k\) terms, leading to the matrices:
\[ V_{ij} = V_{ij}^0 + \sum_{m=1}^{\min(i,j)-1} \Delta^2_{m,m-1} l_{im} l_{jm} \]

\[ W_{ij} = W_{ij}^0 + \sum_{m=1}^{\min(i,j)-1} \Delta^2_{m,m-1} l_{im} l_{jm} \]  

(42)

Finally, the \( \chi^2 \) reads:

\[ \chi^2 = \sum_{i,j=0}^{N-1} \left( x_{i,\text{meas}} - \sum_{k=1}^{5} F_{ik} p_k \right) V_{ij}^{-1} \left( x_{j,\text{meas}} - \sum_{m=1}^{5} F_{jm} p_m \right) + \sum_{i,j=0}^{N-1} \left( y_{i,\text{meas}} - \sum_{k=1}^{5} G_{ik} p_k \right) W_{ij}^{-1} \left( y_{j,\text{meas}} - \sum_{m=1}^{5} G_{jm} p_m \right) . \]  

(43)

or, in matrix form:

\[ \chi^2 = (\vec{x}_{\text{meas}} - F \vec{p})^T V^{-1} (\vec{x}_{\text{meas}} - F \vec{p}) + (\vec{y}_{\text{meas}} - G \vec{p})^T W^{-1} (\vec{y}_{\text{meas}} - G \vec{p}) . \]  

(44)

Formally, its minimization with respect to \( \vec{p} \) leads to the solution:

\[ \vec{p} = \left[ F^T V^{-1} F + G^T W^{-1} G \right]^{-1} \left[ F^T V^{-1} \vec{x}_{\text{meas}} + G^T W^{-1} \vec{y}_{\text{meas}} \right] \]  

(45)

Even if the \( \chi^2 \) to be minimized seems formally linear in the parameters \( \vec{p} \), multiple scattering introduces a dependence on \( q/p \) and \( \vec{u}_0 \) via the covariance matrices \( V \) and \( W \). A convenient way to solve the problem is to minimize the \( \chi^2 \) following an iterative procedure. In a first step, the \( \chi^2 \) is minimized using the diagonal covariance matrices \( V_{ij}^0 \) and \( W_{ij}^0 \). The minimization is then iterated several times, using the \( V_{ij} \) and \( W_{ij} \) matrices determined from the parameters of the previous step. The iterative procedure is rapidly convergent. It may be stopped either after a couple of iterations or when some convergence criteria are reached. If computing time is not an issue, a convenient choice is to stop when the difference in rigidity between two consecutive steps is smaller than the accuracy of the method (a few MeV).

5 Summary

We have presented a simple algorithm for track fitting of high energy particles traversing slightly inhomogeneous magnetic fields. The method is based on the
prior calculation of a few path integrals which depend just on the measured positions and a few values of the magnetic field. The minimization of a $\chi^2$, which presents a linear dependence on the track parameters, leads to a simple solution of the problem. Multiple scattering is considered in a straightforward and user-controlled way. This is particularly important when potential detector resolution problems have to be disentangled from trivial material budget effects. Compared to other methods, a few simple formulae (9, 22 and 23) allow for a fast estimate of the expected momentum uncertainties. These formulae use as inputs the average value of the magnetic field, the typical size of the field inhomogeneities and the distance between measuring layers. The uncertainties translate into a shift of the measured momentum which, for most cases of interest in present and future high energy experiments, are of the order of a few MeV, well below the uncertainties due to multiple scattering. The method discussed here is being employed in the context of the AMS experiment [3]. Thanks to its intrinsic simplicity, it is also being used for fast and reliable track fitting at the latest steps of data reconstruction and analysis.

References


