IS IT PHYSICALLY SOUND TO ADD A TOPOLOGICALLY MASSIVE TERM TO THREE-DIMENSIONAL MASSIVE ELECTROMAGNETIC OR GRAVITATIONAL MODELS?

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Abstract

The addition of a topologically massive term to an admittedly non-unitary three-dimensional massive model, be it an electromagnetic system or a gravitational one, does not cure its non-unitarity. What about the enlargement of avowedly unitary massive models by way of a topologically massive term? The electromagnetic models remain unitary after the topological augmentation but, surprisingly enough, the gravitational ones have their unitarity spoiled. Here we analyze these issues and present the explanation why unitary massive gravitational models, unlike unitary massive electromagnetic ones, cannot coexist from the viewpoint of unitarity with topologically massive terms. We also discuss the novel features of the three-term effective field models that are gauge-invariant.

Topologically massive models; massive electromagnetic models; massive gravitational models; unitarity; effective field models.

1 Introduction

In the last two decades much attention has been devoted to the study of the remarkable properties of gauge theories in (2 + 1) dimensions.\[1\]$^2$\[2\] Certainly, it would not be an exaggeration to claim that by now these properties are not only well-appreciated but also well-understood. Therefore, it should be natural, at least from a naive point of view, to expect that the addition of a Chern-Simons term to massive electromagnetic or gravitational models would produce systems endowed with properties that, in principle, should be as exciting as those concerning the well-known theories of Maxwell-Chern-Simons or Einstein-Chern-Simons. Our aim here is to analyze these massive, topologically massive, models.
Since, currently, there are two distinct non-topological mass-generating mechanisms for gauge fields: adding the well-known Proca/Fierz-Pauli, or the more sophisticated higher-derivative electromagnetic/higher-derivative gravitational, terms, our analysis will comprise topologically massive Proca electromagnetism (TMPE), topologically massive higher-derivative electromagnetism (TMHDE), which is also known as Podolsky-Chern-Simons planar electromagnetism,[3] topologically massive Fierz-Pauli gravity (TMFPG), and topologically massive higher-derivative gravity (TMHDG). These systems will be examined for both possible sign choices of the Maxwell/Einstein Lagrangian, as well as in its absence, which implies that they are the most general such models. Both TMPE and TMFPG are not gauge-invariant due the presence of an explicit mass term, but the three-term models with higher-derivatives are gauge-invariant.

We are particularly interested in two issues that are somewhat correlated:

i The compatibility—from the point of view of the unitarity—between massive electromagnetic or gravitational models and topologically massive terms.

ii The exciting physics resulting from the utilization of the gauge-invariant three-term systems as effective field models. We remark that it was recently shown that boson-boson bound states do exist in the framework of three-dimensional higher-derivative electromagnetism augmented by a topological Chern-Simons term.[3]

To probe the unitarity of the massive, topologically massive, models, we will make use of an uncomplicated and easily handling algorithm that converts the task of checking the unitarity, which in general demands much work, into a straightforward algebraic exercise. The prescription consists basically in saturating the propagator with external conserved currents, compatible with the symmetries of the system, and in examining afterwards the residues of the saturated propagator (SP) at each of their simple poles. We use natural units throughout.

2 Massive, Topologically Massive, Electromagnetic Models

The Lagrangian for TMPE is the sum of Maxwell, standard Proca mass, Chern-Simons, terms, namely,

$$\mathcal{L}_{\text{TMPE}} = -a F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu + \frac{s}{2} \varepsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho,$$

while the Lagrangian for TMHDE is the sum of Maxwell, higher-derivative, gauge-fixing (Lorentz-gauge), and Chern-Simons, terms, i.e.,

$$\mathcal{L}_{\text{TMHDE}} = -a F_{\mu\nu} F^{\mu\nu} + \frac{i^2}{2} \partial_\nu F^{\mu\nu} \partial^\lambda F_{\mu\lambda} - \frac{1}{2\lambda} (\partial_\nu A^\nu)^2 + \frac{s}{2} \varepsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho.$$


Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual electromagnetic tensor field, $l$ is a cutoff, $s > 0$ is the topological mass, and $\alpha$ is a convenient parameter that can take the values $+1$ (Maxwell’s term with the standard sign), $-1$ (Maxwell’s term with the “wrong sign”), or $0$ (absence of the Maxwell’s term). The corresponding propagators are given by

$$\mathcal{P}_{\text{TMPE}} = -\frac{ak^2 - m^2}{(ak^2 - m^2)^2 - s^2k^2} \theta + \frac{1}{m^2} \omega - \frac{s}{(ak^2 - m^2)^2 - s^2k^2} S, \quad (3)$$

$$\mathcal{P}_{\text{TMHDE}} = -\frac{l^2k^4 - ak^2}{(l^2k^4 - ak^2)^2 - s^2k^2} \theta - \frac{\lambda}{k^2} \omega - \frac{s}{(l^2k^4 - ak^2)^2 - s^2k^2} S, \quad (4)$$

where $\theta_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{c}$ and $\omega_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{c}$ are, respectively, the usual transverse and longitudinal vector projector operators, $S_{\mu\nu} \equiv \epsilon_{\mu\nu\rho} \partial^\rho$ is the operator associated with the topological term, and $\eta_{\mu\nu}$ is the Minkowski metric. Our signature conventions are $(+, -, -, -)$, $\epsilon^{012} = +1 = \epsilon_{012}$. Now, the algorithm from Ref. 4 says that all we have to do in order to check the unitarity of a massive, topologically massive, electromagnetic model is to verify whether the residues at each simple pole of the $\theta$-component of the propagator in the basis $\{\theta, \omega, S\}$ which, for short, we will designate as $f_\theta$, are $\leq 0$. We use this recipe in the following to test the unitarity of TMPE and TMHDE, in this order.

### 2.1 Checking the unitarity of TMPE

We start our analysis by setting the parameter $a$ in Eq. (1) equal to $+1$ because it must be positive both in the Proca ($s = 0$) and Maxwell-Chern-Simons ($m = 0$) limits[2]. Now, the $\theta$-component of the propagator in the basis $\{\theta, \omega, S\}$ is, according to Eq. (3), equal to $f_\theta = \frac{m^2k^2}{(k^2 - m_+^2)(k^2 - m_-^2)}$, where $m_\pm = \frac{1}{2} \left[ \sqrt{4m^2 + s^2} \pm s \right]$. Therefore, the model has two degrees of freedom with masses $m_+$ and $m_-$, which is precisely what Deser and Tekin have found using a rather different approach.[8] Our result is also in agreement with another works existing in the literature.[7] On the other hand, it is trivial to show that both $\text{Res } f_\theta |_{k^2 = m_+^2}$ and $\text{Res } f_\theta |_{k^2 = m_-^2}$ are less than zero. Thence, TMPE with the Maxwell’s term with the usual sign is unitary. Choosing $a = -1$, we see that if $s^2 > 4m^2$, then $f_\theta = \frac{m^2 + k^2}{(k^2 - m_+^2)(k^2 - m_-^2)}$, with $m_\pm = \frac{1}{2} \left[ s \pm \sqrt{s^2 - 4m^2} \right]$. A straightforward calculation allows us to conclude that $\text{Res } f_\theta |_{k^2 = m_+^2} > 0$, and $\text{Res } f_\theta |_{k^2 = m_-^2} < 0$, implying that, if $s^2 > 4m^2$, TMPE with Maxwell’s term with the wrong sign is non-unitary. It is worth mentioning that this system, despite having acceptable values for the masses, faces ghost problems. Of course, if $s^2 < 4m^2$ the two roots of $x^2 + x(2m^2 - s^2) + m^4 = 0$, where $k^2 \equiv x$, are imaginary; note also that for $s^2 = 4m^2$ the two hitherto complex roots coalesce and the masses are simply $m_+ = m_- = \frac{s}{2}$: these models are never viable. We focus, at last, on the case $a = 0$ (absence of the Maxwell’s term). This model
was analyzed long ago by Deser and Jackiw, who came to the conclusion that setting \( a = 0 \) yields just another version of the Maxwell-Chern-Simons theory and so it is equivalent to choosing \( m^2 = 0 \).

2.1.1 Discussion

Note that Proca electromagnetism (\( s = 0 \)) with \( a = -1 \) contains tachyons; however, TMPE with \( a = -1 \) and \( s^2 > 4m^2 \) is plagued by ghosts but not by tachyons: the particle content of the model is one non-tachyonic spin-1 ghost of mass \( m_+ \) and one massive spin-1 normal particle of mass \( m_- \). Thence, a field theory built from this model would not be satisfactory from the point of view of their fundamentals. It could regarded, perhaps, as an effective field theory, i.e., a low-energy approximation to a more fundamental theory. Nonetheless, the condition \( s > 2m \) is greatly discouraging as far as the possibility of applying this kind of model, for instance, to some condensed matter systems where one deals, in general, with low-energy excitations. Interesting enough, only the model with \( a = +1 \) may be viewed as physically sound. Why is this so? Because the aforementioned system has a Lagrangian that reproduces the Lagrangians of well-behaved physical models when the appropriate limits are taken. Indeed, if \( s = 0 \), we recover Proca electromagnetism; on the other hand, setting \( m = 0 \), we obtain Chern-Simons electromagnetism. It is remarkable that we also arrive at a nice physical model by removing the Maxwell’s term: the system with \( a = 0 \) and the “self-dual” model of Ref. 8 are equivalent.

2.2 Checking the unitarity of TMHDE

Based on the above informations, we have every reason to begin the unitarity analysis of TMHDE by setting \( a = +1 \) in Eq. (2). The calculations are now more complicated because, unlike the previous model, this represents in general three massive excitations rather than two massive ones. Since the \( \theta \)-component of the propagator concerning TMHDE with \( a = +1 \) can be written as \( f_\theta = \frac{M^2}{M^2 + M^2 - M^2} \), we have to analyze the nature, as well as the signs, of the roots of the cubic equation \( x^3 + a_2x^2 + a_1x + a_0 = 0 \), where \( a_2 = -2M^2 \), \( a_1 = M^4 \), and \( a_0 = -M^4s^2 \). Taking into account that we are only interested in those roots that are both real and unequal, we require \( D < 0 \), where \( D = Q^3 + R^2 \), with \( Q \) and \( R \) being, in this order, equal to \( \frac{3a_1-a_2^2}{3} \) and \( \frac{9a_1a_2-27a_0-2a_2^3}{54} \), is the polynomial discriminant. Performing the computations we get \( D = M^8s^2 \left[ \frac{s^2}{4} - \frac{M^2}{2m} \right] \), implying that only if only \( s^2 < \frac{4m^2}{2m} \) will the roots be real and unequal. Our next step is to verify whether or not these roots are positive. This can be accomplished by building the Routh-Hurwitz array,[9] namely,

Noting that there are three signs changes in the first column of the array above, we conclude that all the three roots are positive. In summary, if \( s^2 < \frac{4m^2}{2m} \), TMHDE with \( a = +1 \) is a model with acceptable values for the masses. Denot-
\[ \begin{bmatrix} 1 & M^4 \\ -2M^2 & -M^4s^2 \\ M^2 \left( M^2 - \frac{s^2}{2} \right) & 0 \\ -M^4s^2 & 0 \end{bmatrix} \]

ing these roots as \( x_1, x_2, \) and \( x_3, \) and assuming without any loss of generality that \( x_1 > x_2 > x_3, \) we get

\[
f_\theta = \frac{M^2(x_1 - M^2)}{(x_1 - x_2)(x_1 - x_3)} \frac{1}{x - x_1} + \frac{M^2(x_2 - M^2)}{(x_2 - x_1)(x_2 - x_3)} \frac{1}{x - x_2} \\
+ \frac{M^2(x_3 - M^2)}{(x_3 - x_1)(x_3 - x_2)} \frac{1}{x - x_3}.
\]

Hence, TMHDE with \( a = +1 \) will be unitary if the conditions \( x_1 - M^2 < 0, \) \( x_2 - M^2 > 0, \) and \( x_3 - M^2 < 0 \) hold simultaneously. Obviously, this will never occur, which allows us to conclude that TMHDE with the Maxwell’s term with the standard sign is non-unitary. If \( a = -1, \) \( f_\theta = \frac{M^2(x + M^2)}{x^2 + 2M^2x^2 + M^4x - M^4s^2}. \)

Since the polynomial discriminant, \( D = M^8s^2 \left[ M^4 + \frac{M^2}{27} \right], \) for the cubic equation \( x^3 + 2M^2x^2 + M^4x - M^4s^2 = 0 \) is greater than zero, one of the roots of the equation is real and the other two are complex conjugates, which means that the system with \( a = -1 \) is forbidden. To finish our analysis we set \( a = 0 \) in Eq. (2). In this case \( f_\theta = \frac{xM^2}{x^2 - M^4s^2}, \) and the polynomial discriminant related to \( x^3 - M^4s^2 = 0 \) is greater than zero. This model, as the previous one, is also forbidden.

2.2.1 Discussion

Should we expect intuitively that TMHDE with \( a = +1 \) faced unitary problems? The answer is affirmative. In fact, setting \( s = 0, \) for instance, in its Lagrangian, we recover the Lagrangian for the usual Podolsky electromagnetism which is non-unitary. Nonetheless, Podolsky-Chern-Simons (PCS) planar electromagnetism with \( a = +1 \) and \( s^2 < \frac{4M^2}{27}, \) despite being haunted by ghosts, has normal massive modes. Since the existence of these well-behaved excitations is subordinated to the condition \( s < \frac{2M}{\sqrt{27}}, \) we are really encouraged to regard this system as an effective field model. It is quite remarkable that the coupling of PCS planar electrodynamics with a charged scalar field, produces an attractive interaction between equal charge bosons. To see this we need to know beforehand the expression of the effective non-relativistic potential for the interaction of two charged-bosons in the center-of-mass frame. A somewhat involved computation, yields
Schrödinger equation associated with this potential is given by
\[ V(r) = -\frac{sQ^2}{\pi ml^4} \left[ \frac{l^4}{s^2 r^2} + \frac{1}{r} \sum_j B_j \sqrt{|x_j|} K_1(\sqrt{|x_j|} r) \right] L \]
\[ + \frac{Q^2}{2\pi l^2} \left[ \sum_j A_j K_0(\sqrt{|x_j|} r) \right], \]

where \( A_1 \equiv \frac{1+\alpha^2}{s^2(x_1-x_2)(x_1-x_3)} \), \( A_2 \equiv \frac{1+\alpha^2}{s^2(x_1-x_3)(x_2-x_3)} \), \( A_3 \equiv \frac{1+\alpha^2}{s^2(x_1-x_1)(x_3-x_3)} \), \( B_1 \equiv \frac{-1+\alpha^2}{s^2(x_1-x_2)(x_1-x_3)} \), \( B_2 \equiv \frac{-1+\alpha^2}{s^2(x_2-x_1)(x_2-x_3)} \), and \( B_3 \equiv \frac{-1+\alpha^2}{s^2(x_3-x_1)(x_3-x_2)} \), \( x_1, x_2, \) and \( x_3 \) are the roots of the equation \( x^3 + \frac{2\alpha^2}{r^2} + \frac{\alpha}{r} + \frac{\alpha^2}{r^3} = 0 \), \( L \) is the angular momentum, \( K \) is the modified Bessel function, and \( Q \) and \( m \) are, respectively, the charge and the mass of the scalar boson. On the other hand, the radial Schrödinger equation associated with this potential is given by

\[ \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right] R_{nl} + m \left[ E_{nl} - V_{\text{eff}}^r \right] R_{nl} = 0, \]

where

\[ V_{\text{eff}}^r(r) = -\frac{sQ^2}{\pi ml^4} \left[ \frac{l^4}{s^2 r^2} + \frac{1}{r} \sum_j B_j \sqrt{|x_j|} K_1(\sqrt{|x_j|} r) \right] \]
\[ + \frac{Q^2}{2\pi l^2} \left[ \sum_j A_j K_0(\sqrt{|x_j|} r) \right] + \frac{l^2}{ml^2}. \]

Here \( \tilde{l} \) denotes the eigenvalues of the operator \( \mathbf{L} \). In terms of the dimensionless parameters \( y \equiv sr \), \( \alpha \equiv \frac{Q^2}{s^2} \), \( b_j \equiv \frac{s^2}{\alpha} B_j \), \( X_j \equiv \frac{|x_j|}{s} \), \( \beta \equiv \frac{m}{s} \), \( a_j \equiv \frac{A_j}{s^4} \), and \( \tilde{E}_{nl} \equiv \frac{mE_{nl}}{s^4} \), Eq. (6) reads

\[ \left[ \frac{d^2}{dy^2} + \frac{1}{y} \frac{d}{dy} \right] R_{nl} + \left[ \tilde{E}_{nl} - \tilde{V}_{\text{eff}}^r \right] R_{nl} = 0, \]

with

\[ \tilde{V}_{\text{eff}}^r \equiv -\frac{\tilde{l}(\alpha - \tilde{l})}{y^2} + \frac{\alpha \beta}{2} \sum_j a_j K_0(X_j y) - \frac{\alpha \tilde{l}}{y} \sum_j b_j X_j K_1(X_j y). \]

We call attention to the fact that \( V_{\text{eff}}^r \) behaves as \( \frac{l^2}{y^2} \) at the origin and as \( \frac{\tilde{l}(\alpha - \tilde{l})}{y^{2(\alpha - 1)}} \) asymptotically. Now, in four dimensions, the anomalous factor of \( \frac{4}{9} \) in
the Abraham-Lorentz model for the electron does not show up if \( l > \frac{1}{2} r_e \), where \( r_e \) denotes the classical radius of the electron.\[10\] Therefore, we assume \( l \ll 1 \).

In this limit the derivative of the potential with respect to \( y \) reduces to

\[
\frac{d}{dy} V_{\text{eff}} \sim \frac{2\tilde{l}(\alpha-\bar{l})}{y^3} - \left[ \frac{2\alpha\tilde{l}}{y^2} + \frac{\alpha\beta}{2} \right] K_1(y) - \frac{\alpha\tilde{l}}{y} K_0(y).
\]

Supposing \( \bar{l} > 0 \), without any loss of generality, we promptly see that, if \( \bar{l} > \alpha \), the potential is strictly decreasing. The remaining possibility is \( \bar{l} < \alpha \). In this interval \( V_{\text{eff}} \) approaches +\( \infty \) at the origin and 0 for \( y \to +\infty \), which is indicative of a local minimum. Consequently, the existence of the attractive potential is subordinated to the conditions \( a \ll 1 \) and \( 0 < \bar{l} < \alpha \). One can show that the effective potential with \( l \ll 1 \) and \( 0 < \bar{l} < \alpha \) can bind a pair of identical charged-scalar bosons.\[3\] Accordingly, the addition of the topologically massive term to Podolsky’s electromagnetism with \( a = +1 \)—an admittedly non-unitary model—did not cure its non-unitary problem; nonetheless, the condition for the resulting three-term model to be free of tachyons gives rise to a constraint between the topological and Podolsky masses which is responsible for a scalar attractive interaction.

### 3 Massive, Topologically Massive, Gravitational Models (MTMG)

TMFPG is defined by the Lagrangian

\[
\mathcal{L}_{\text{TMFPG}} = \frac{a^2}{\kappa^2} \sqrt{g} \left( R - \frac{m^2}{2} (h_{\mu\nu}^2 - h^2) + \frac{1}{\mu} \epsilon^{\lambda\mu\nu} \Gamma_{\lambda\sigma} \left( \partial_\mu \Gamma_{\sigma \rho\nu} + \frac{2}{3} \Gamma_{\sigma \mu\beta} \Gamma_{\beta \rho\nu} \right) \right), \tag{7}
\]

at quadratic order in \( \kappa \), where \( \kappa^2 \) is a suitable constant that in four dimensions is equal to 24\( \pi G \), with \( G \) being Newton’s constant.\[11\] Here \( g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \), \( \mu > 0 \) is a dimensionless parameter, and \( h \equiv \eta_{\mu\nu} h^{\mu\nu} \). Indices are raised (lowered) with \( \eta^{\mu\nu} (\eta_{\mu\nu}) \). The Lagrangian related to TMHDG, in turn, is given by

\[
\mathcal{L}_{\text{TMHDG}} = \sqrt{g} \left( a^2 \frac{2R}{\kappa^2} + \frac{\alpha}{2} R^2 + \frac{\beta}{2} R_{\mu\nu} \right) + \frac{1}{\mu} \epsilon^{\lambda\mu\nu} \Gamma_{\lambda\sigma} \left( \partial_\mu \Gamma_{\sigma \rho\nu} + \frac{2}{3} \Gamma_{\sigma \mu\beta} \Gamma_{\beta \rho\nu} \right), \tag{8}
\]

where \( \alpha \) and \( \beta \) are suitable constants with dimension \( L \). For the sake of simplicity, the gauge-fixing term was omitted. Note that the parameter \( a \) appearing in Eqs. (7) and (8) allows for choosing the Einstein sign’s term or even removing it.
The propagator related to TMFPG is

\[ P_{\text{TMFPG}} = -\frac{1}{m^2} P^1 - \frac{M^2 (m^2 + a \Box)}{\Box^3 + M^2 a^2 \Box^2 + 2am^2M^2\Box + M^2m^4 P^2} \]

\[ - \frac{M}{\Box^3 + M^2 a^2 \Box^2 + 2am^2M^2\Box + M^2m^4 P} \]

\[ - \frac{m^2 + a \Box P^0}{2m^4} + \frac{1}{2m^2} P^0, \] (9)

where \( M \equiv \mu/\kappa^2 \). On the other hand, linearizing Eq. (8) and adding to the result the gauge-fixing term \( L_{\text{gf}} = \frac{1}{2} \left( \frac{1}{2} (h_{\mu\nu}, \nu - \frac{1}{2} h_{\mu\mu}) \right)^2 \) (de Donder gauge), we find that the propagator concerning TMHDG takes the form

\[ P_{\text{TMHDG}} = \frac{1}{\Box[a + b(\frac{3}{2} + 4c)\Box]} P^0 + \frac{2\lambda}{k^2} P^1 + \frac{1}{\Box[a + b(\frac{3}{2} + 4c)\Box]} P^0 \]

\[ + \frac{4M}{\Box[M^2b^2\Box^2 - 4(abM^2 - 1)\Box + 4M^2a^2]} P \]

\[ - \frac{2M^2(2a - b\Box)}{\Box[M^2b^2\Box^2 - 4(abM^2 - 1)\Box + 4M^2a^2]} P^2 \]

\[ + \left[ \frac{4\lambda}{\Box} + \frac{2}{\Box[a + b(\frac{3}{2} + 4c)\Box]} \right] P^0, \] (10)

where \( b \equiv \frac{\beta^2}{2} \), \( c \equiv \alpha \), and \( M \equiv \frac{\mu}{\kappa^2} \). Our conventions are \( R^\alpha_{\beta\gamma\delta} = -\partial_\delta \Gamma^\alpha_{\beta\gamma} + \ldots \), \( R_{\mu\nu} = \bar{R}^\alpha_{\mu\nu\alpha} \), \( \bar{R} = g^{\alpha\beta}R_{\mu\nu} \), where \( g_{\mu\nu} \) is the metric tensor, and signature (+, −, −). The propagators were calculated using the basis

\[ \{P^1, P^2, P^0, \overline{P}^0, \overline{P}^0, P\}, \]

where \( P^1, P^2, P^0, \overline{P}^0 \), and \( \overline{P}^0 \) are the usual three-dimensional Barnes-Rivers operators, namely,

\[ P^1_{\mu\nu, \rho\sigma} = \frac{1}{2} \left( \theta_{\mu\rho} \omega_{\nu\sigma} + \theta_{\mu\sigma} \omega_{\nu\rho} + \theta_{\nu\rho} \omega_{\mu\sigma} + \theta_{\nu\sigma} \omega_{\mu\rho} \right), \]

\[ P^2_{\mu\nu, \rho\sigma} = \frac{1}{2} \left( \theta_{\mu\rho} \theta_{\nu\sigma} + \theta_{\mu\sigma} \theta_{\nu\rho} - \theta_{\mu\nu} \theta_{\rho\sigma} \right), \]

\[ P^0_{\mu\nu, \rho\sigma} = \frac{1}{2} \theta_{\mu\nu} \theta_{\rho\sigma}, \quad \overline{P}^0_{\mu\nu, \rho\sigma} = \omega_{\mu\nu} \omega_{\rho\sigma}, \]

\[ \overline{P}^0_{\mu\nu, \rho\sigma} = \theta_{\mu\nu} \omega_{\rho\sigma} + \omega_{\mu\nu} \theta_{\rho\sigma}, \]

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and $P$ is the operator associated with the linearized Chern-Simons term, i.e.,

$$P_{\mu\nu, \rho\sigma} \equiv \frac{\Box \partial^\lambda}{4} \left[ \epsilon_{\mu \lambda \rho} \theta_{\nu \sigma} + \epsilon_{\mu \lambda \sigma} \theta_{\nu \rho} + \epsilon_{\nu \lambda \rho} \theta_{\mu \sigma} + \epsilon_{\nu \lambda \sigma} \theta_{\mu \rho} \right].$$

According to Ref. 4, the saturated propagator concerning the MTMG, is given by

$$SP_{\text{MTMG}} = \left[ T^\mu_\nu T^\mu_\nu - \frac{1}{2} T^2 \right] f_{P^2} + \frac{1}{2} T^2 f_{P^0},$$

(11)

where $T^\mu_\nu$ is the external conserved current that, obviously, is symmetric in the indices $\mu$ and $\nu$, $T \equiv \eta_{\mu\nu} T^{\mu\nu}$, and $f_{P^2}$ and $f_{P^0}$ are, respectively, the components $P^2$ and $P^0$ of the propagator in the basis $\{P^1, P^2, P^0, \overline{P}^0, \overline{P}^1, P\}$. Therefore, to find out whether or not the gravitational model is unitary, we must compute $SP_{\text{MTMG}}$ using Eq. (11) and determine afterwards the the residue at each simple pole of $SP_{\text{MTMG}}$: If all the residues are $\geq 0$, the model is unitary; however, if at least one of them is negative, the system is non-unitary. The unitarity analysis is greatly facilitated if we take into account that $[T^\mu_\nu T^\mu_\nu - \frac{1}{2} T^2]_{k^2=m^2} > 0$ and $[T^\mu_\nu T^\mu_\nu - T^2]_{k^2=0} = 0$, where $m \geq 0$ is the mass of a generic physical particle associated with the MTMG, and $k$ is the corresponding momentum exchanged.

Using this prescription, we check in the following the unitarity of TMFPG and TMHDE, in this order.

### 3.1 Checking the unitarity of TMFPG

To begin with, we set $a = -1$ in Eq. (8) because we want to recover the Einstein-Chern-Simons Lagrangian in the $m = 0$ limit—topologically massive gravity is a theory that requires $a = -1$ to be ghost-free. The corresponding saturated propagator is given by

$$SP_{\text{TMFPG}} = \left[ T^\mu_\nu T^\mu_\nu - \frac{1}{2} T^2 \right] \frac{M^2(m^2 + k^2)}{k^6 - M^2 k^4 - 2 m^2 M^2 k^2 - M^4 m^4}.$$

Our next step is to study the roots of the cubic equation $x^3 - M^2 x^2 - 2 m^2 M^2 x - M^2 m^4 = 0$. Since the discriminant, $D = M^4 m^6 \left[ \frac{m^2}{4} + \frac{M^2}{2} \right]$, related to this equation is greater than zero, the model at hand is unphysical and must be rejected. Consequently, we turn our attention to the system with $a = +1$. Now, we have to consider the roots of the equation $x^3 - M^2 x^2 + 2 m^2 M^2 x^2 - M^2 m^4 = 0$, whose polynomial discriminant can be written as $D = M^4 m^6 \left[ \frac{m^2}{4} - \frac{M^2}{2} \right]$. Therefore, if $\frac{m^2}{4} < \frac{M^2}{2}$, our equation has three distinct real roots. The corresponding Routh-Hurwitz array is
Accordingly, the system with \( a = +1 \) and \( \frac{m^2}{M^2} < \frac{4}{27} \) has acceptable values for the masses. Proceeding just as we have done for TMHDE with \( a = +1 \) and \( s^2 < \frac{4M^2}{27} \), we promptly obtain

\[
SP_{\text{TMFPG}} = \frac{F(k)(m^2 - x_1)}{(x_1 - x_2)(x_1 - x_3)k^2 - x_1} + \frac{F(k)(m^2 - x_2)}{(x_2 - x_1)(x_2 - x_3)k^2 - x_2} + \frac{F(k)(m^2 - x_3)}{(x_3 - x_1)(x_3 - x_2)k^2 - x_3},
\]

where \( F(k) \equiv \{T^\mu\nu(k)T^\mu\nu(k) - \frac{1}{2}[T(k)]^2\}M^2 \). From the above, we clearly see that this model will be unitary if \( m^2 > x_1, \ m^2 < x_2, \ \text{and} \ m^2 > x_3 \). We thus come to the conclusion that TMFPG with \( a = +1 \) and \( \frac{m^2}{M^2} < \frac{4}{27} \) is non-unitary, which means that the topological term is responsible for breaking down the unitarity of the harmless Fierz-Pauli gravity. If \( a = 0 \), the discriminant associated with the equation \( x^3 - M^2m^4 = 0 \) is greater than zero, which implies that this model is physically unsound. We remark that our conclusions are in complete agreement with those of Ref. 6 where a quite different approach to the unitarity problem was employed.

### 3.1.1 Discussion

The above results points to an important and at the same time interesting question: Why can unitary massive electromagnetic models coexist in peace with topologically massive terms, whereas unitary massive gravitational ones cannot? The root of the problem lies in the rather odd way Einstein-Chern-Simons theory is constructed: The presence of the ghosts in the dynamical field is avoided by choosing the Einstein’s term with the wrong sign. This is trivial to show. Indeed, writing the Einstein-Chern-Simons Lagrangian as

\[
\mathcal{L} = a\sqrt{g}\frac{2R}{\kappa^2} + \frac{1}{\mu}c^{\lambda\mu\nu\Gamma^\lambda_{\mu\nu}}(\partial_\mu \Gamma^\sigma_{\rho\nu} + \frac{2}{3}\Gamma^\sigma_{\mu\beta}\Gamma^\beta_{\nu\rho}),
\]

(12)

with \( a = \pm 1 \), we promptly see that the corresponding saturated propagator is given by

\[
SP = -\frac{1}{a}\left(T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right)\frac{1}{k^2 - M^2} - \frac{1}{a}\left(T^{\mu\nu}T_{\mu\nu} - T^2\right)\frac{1}{k^2}.
\]

(13)

Thus, to render the theory unitary we are obliged to set \( a = -1 \) in Eq. (13). Note that as far as these three-term systems are concerned, we are always in
a dilemma: Which value should we assign to \( a \), \(-1\) or \(+1\)? If we single out \( a = -1 \), for instance, we recover Einstein-Chern-Simons theory when the non-topological massive term is removed; however, in the absence of the topological term, we do not get a nice physical theory because now the Einstein’s term has the wrong sign. On the other hand, if we pick out \( a = +1 \), we do not recover Einstein-Chern-Simons theory when the non-topological massive term is removed. In other words, due to the unusual Einstein sign’s term in the Lagrangian concerning Einstein-Chern-Simons theory, the augmented systems do not reduce to well-behaved physical models in the suitable limits. Note that these idiosyncrasies do not occur in the framework of massive, topologically massive, electromagnetic models because the Maxwell sign’s term concerning Maxwell-Chern-Simons theory is the same as that of the usual Maxwell’s theory.

### 3.2 Checking the unitarity of TMHDG

Assuming \( a \neq 0 \), the \( SP \) concerning TMHDG can be written as

\[
SP_{\text{TMHDG}} = \frac{M^2 b}{2} \left[ \frac{T^\mu_\nu T^\mu_\nu - \frac{1}{2} T^2}{k^2 - M_1^2} - 1 + \sqrt{1 - 2abM^2} \right] \sqrt{1 - 2abM^2} \left[ 1 - abM^2 - \sqrt{1 - 2abM^2} \right] \\
+ \frac{M^2 b}{2} \left( T^\mu_\nu T^\mu_\nu - \frac{1}{2} T^2 \right) \frac{1}{k^2 - M_2^2} \sqrt{1 - 2abM^2} \left[ 1 - abM^2 + \sqrt{1 - 2abM^2} \right] \\
+ \frac{T^\mu_\nu T^\mu_\nu - T^2}{ak^2} + \frac{4T^2}{a(k^2 - m^2)},
\]

where

\[
M_1^2 = \left( \frac{2}{b^2 M^2} \right) \left[ 1 - abM^2 - \sqrt{1 - 2abM^2} \right], \\
M_2^2 = \left( \frac{2}{b^2 M^2} \right) \left[ 1 - abM^2 + \sqrt{1 - 2abM^2} \right], \\
m^2 = \frac{a}{b(3/2 + 4c)}.
\]

It is interesting to note that \( M_1^2 \to M^2, \) and \( M_2^2 \to +\infty, \) as \( b \to 0, \) which implies that Eq. (14) reduces to Eq. (13) when \( \alpha, \beta \to 0, \) as expected. We are now ready to analyze the excitations and mass counts concerning TMHDG for both allowed signs of \( a. \) To avoid needless repetitions, we restrict ourselves to presenting a summary of the main results in Table 1. The systems that do not appear in this table are tachyonic, \( i.e., \) unphysical.

In conclusion, we consider TMHDG with \( a = 0. \) In this case,

\[
SP_{\text{TMHDG}} = \frac{M^2 b}{2} \left[ \frac{T^\mu_\nu T^\mu_\nu - \frac{1}{2} T^2}{k^2} + \frac{T^\mu_\nu T^\mu_\nu - \frac{1}{2} T^2}{k^2 - \frac{4}{M^2 b^2}} \right] + \frac{1}{b(3/2 + 4c)} \frac{1}{k^4} T^2.
\]

The pole of order two at \( k^2 = 0 \) indicates that these models are unphysical.
Table 1: Unitarity analysis of the topologically massive higher-derivative gravity models

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$\frac{b}{2} + 4c$</th>
<th>excitations and mass counts</th>
<th>tachyons</th>
<th>unitarity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>spin-2 normal particles 1 massive spin-2 non-propagating particle 1 massive spin-0 ghost</td>
<td>no one</td>
<td>non-unitary</td>
</tr>
<tr>
<td>$-1$</td>
<td>$\frac{-1}{2M^2} &lt; b &lt; 0$</td>
<td>$&gt; 0$</td>
<td>spin-2 normal particle 1 massive spin-2 non-propagating particle 1 massive spin-2 ghost 1 massive spin-0 ghost</td>
<td>no one</td>
<td>non-unitary</td>
</tr>
<tr>
<td>$+1$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
<td>2 massive spin-2 ghosts 1 massive spin-2 non-propagating particle 1 massive spin-0 ghost</td>
<td>no one</td>
<td>non-unitary</td>
</tr>
<tr>
<td>$+1$</td>
<td>$0 &lt; b &lt; \frac{-1}{2M^2}$</td>
<td>$&gt; 0$</td>
<td>spin-2 normal particle 1 massive spin-2 non-propagating particle 1 massive spin-2 ghost 1 massive spin-0 normal particle</td>
<td>no one</td>
<td>non-unitary</td>
</tr>
</tbody>
</table>

3.2.1 Discussion

As intuitively expected, TMHDG is non-unitary for $a = \pm 1$; nonetheless, these models are in general non-tachyonic which means that under certain circumstances they may be viewed as effective field models. Our aim here is to investigate, in passing, the novel features of one of these non-unitary gauge-invariant three-term effective field models. To be more specific, we fix our attention on the first model of Table 1, i.e., TMHDG with $a = -1$, $b > 0$, and $\frac{b}{2} + 4c < 0$ Ref. 13. We have chosen the $a = -1$ system because it reduces, in the absence of the topologically massive term, to higher-derivative gravity with $a = -1$—an effectively multimass model of the fourth-derivative order with interesting properties of its own.[14] Now, it can be shown that the effective non-relativistic potential for the interaction of two scalars bosons in the framework of TMHDG with $a = -1$, $b > 0$, and $\frac{b}{2} + 4c < 0$, is given by[15]

$$V(r) = 2\bar{m}^2 \tilde{G} \left[ K_0(rm) \frac{K_0(rM_+)}{1 + \frac{bM_+^2}{2}} - \frac{K_0(rM_-)}{1 + \frac{bM_-^2}{2}} \right], \tag{15}$$

where $\bar{m}$ is the mass of one of the neutral bosons, $\tilde{G} = \frac{\kappa^2}{32\pi}$, and

$$M_\pm = \frac{1}{bM} \left[\sqrt{1 \pm 2bM^2} \pm 1\right].$$
Note that \( V(r) \) behaves as \( 2\bar{m}^2\bar{G}\ln\left(\frac{M_{1+} + \frac{bM^2}{r} - \frac{bM^2}{M_{1-}}}{m}\right) \) at the origin and as

\[
2\bar{m}^2\bar{G}\left[\sqrt{\frac{\pi}{2rm}}e^{-rm} - \frac{1}{1 + \frac{bM^2}{2}}\sqrt{\frac{\pi}{2rM}}e^{-rM} - \frac{1}{1 + \frac{bM^2}{2}}\sqrt{\frac{\pi}{2rM}}e^{-rM}\right].
\]

asymptotically. Accordingly, \( V(r) \) is finite at the origin and zero at infinity. The derivative of this potential with respect to \( r \) is in turn given by

\[
\frac{dV}{dr} \approx 2\bar{m}^2\bar{G}\left[-mK_1'(rm) + \frac{M_+}{1 + \frac{bM^2}{2}}K_1(rM+) + \frac{M_-}{1 + \frac{bM^2}{2}}K_1(rM-)\right],
\]

(16)

On the other hand, it was shown recently that in four dimensions the propagation of photons in the context of higher-derivative gravity (HDG) is dispersive. In other words, gravitational rainbows and semiclassical HDG can coexist without conflict. On the basis of the fact that the rainbow effect is currently undetectable, it is possible to show that \( |\beta| \leq 10^{60} \) Ref. 17. How reliable is this result? The aforementioned constraint is of the same order as that obtained by testing the gravitational inverse-square law in the submillimeter regime. Thence, we assume \( b \gg 1 \). As a consequence, Eq. (16) reduces to

\[
\frac{dV}{dr} \sim 2\bar{m}^2\bar{G}\left[-mK_1'(rm) + \sqrt{\frac{2}{b}}K_1\left(r\sqrt{\frac{2}{b}}\right)\right],
\]

implying that the potential \( V(r) \), which in this approximation may be expressed as

\[
V(r) \sim 2\bar{m}^2\bar{G}\left[K_0(rm) - K_0\left(r\sqrt{\frac{2}{b}}\right)\right],
\]

(17)

is everywhere attractive if \( \sqrt{\frac{2}{b}} > m \), is repulsive if \( m > \sqrt{\frac{2}{b}} \), and vanishes if \( m = \sqrt{\frac{2}{b}} \). If we appeal to the usual tools of Einstein’s geometrical theory, we arrive at the same conclusions. In fact, in the weak field approximation the gravitational acceleration, \( \gamma^l = \frac{\partial\gamma}{\partial r} \), of a slowly moving test particle is given by \( \gamma^l = -\kappa\left[\frac{\partial}{\partial r}h^t_0 - \frac{1}{2}\frac{\partial}{\partial r}h^0_0\right] \), which for time-independent fields reduces to \( \gamma^l = \frac{\kappa}{2}\frac{\partial}{\partial r}h^0_0 \). Now, taking into account that \( h^0_0 = \frac{2V'}{m^2c^2} \), we obtain

\[
\gamma^l = 2\bar{m}^2\bar{G}\frac{2l}{r}\left[-mK_1'(rm) + \sqrt{\frac{2}{b}}K_1\left(r\sqrt{\frac{2}{b}}\right)\right].
\]
Therefore, the gravitational force exerted on the particle,

\[ F^l = 2\tilde{m}^2\tilde{G}\frac{x^l}{r}\left[-mK_1(rm) + \sqrt{\frac{2}{b}}K_1\left(r\sqrt{\frac{2}{b}}\right)\right], \]

is everywhere attractive if \( \sqrt{\frac{2}{b}} > m \), is repulsive if \( m > \sqrt{\frac{2}{b}} \) (antigravity), and vanishes if \( m = \sqrt{\frac{2}{b}} \) (gravitational shielding). It is remarkable that this force does not exist in general relativity. It is peculiar to topologically massive higher-derivative gravity.

4 Final remarks

We have shown that topologically massive terms cannot be used as a panacea for curing the non-unitarity of massive electromagnetic/gravitational models. In truth, the addition of a Chern-Simons term to a massive electromagnetic/gravitational model is physically sound only if only the resulting three-term system reduces to well-behaved physical models in the suitable limits. A direct consequence of this fact is that we will never be able to construct an unitary, massive, topologically massive, gravitational model. Indeed, the fancy way Einstein-Chern-Simons theory is built, i.e., with the Einstein’s term with the opposite sign, precludes the existence of ghost-free, massive, topologically massive, gravitational models. Therefore, from a conceptual point of view, the addition of a topologically massive term to a massive gravitational model is a complete nonsense: On the one hand, it does not cure the non-unitarity of the original model; on the other hand, it spoils the unitarity of admittedly unitary models.

An interesting and elucidatory example is furnished by \( R + R^2 \) gravity, which is defined by the Lagrangian \( \mathcal{L} = \alpha \left[ \frac{\text{det} g}{2} + \frac{1}{2} R^2 \right] \sqrt{g} \), with \( \alpha = \pm 1 \). If \( \alpha > 0 \), this theory is non-tachyonic regardless of the sign of \( \alpha \); in addition, it is unitary if \( \alpha = +1 \), and non-unitary if \( \alpha = -1 \). Incidentally, \( R + R^2 \) gravity with \( \alpha = +1 \) is the only known gravity theory with higher-derivatives that is unitary. However, topologically massive \( R + R^2 \) gravity is non-unitary for both possible sign choices of \( \alpha \). Yet, a new and surprising physics emerges when we analyze the three-term effective field models that are both gauge-invariant and non-unitary. In the framework of the electromagnetic models, an attractive interaction between equal charge particles can be produced that leads to an unusual planar dynamics: scalar pairs can condense into bound states. In the framework of the gravity systems, in turn, unlike what occurs in the context of the insipid and odorless three-dimensional Einstein’s general relativity, we have a gravitational interaction that can be both attractive and repulsive. We can also have a null gravitational interaction, such as in three-dimensional gravity that is trivial outside the sources. Certainly, these effective field models deserve to be both much better known and further investigated.
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References


