How Required Reserve Ratio Affects
Distribution and Velocity of Money

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Abstract

In this paper the dependence of wealth distribution and the velocity of money on
the required reserve ratio is examined based on a random transfer model of money
and computer simulations. A fractional reserve banking system is introduced to
the model where money creation can be achieved by bank loans and the monetary
aggregate is determined by the monetary base and the required reserve ratio. It is
shown that monetary wealth follows asymmetric Laplace distribution and latency
time of money follows exponential distribution. The expression of monetary wealth
distribution and that of the velocity of money in terms of the required reserve ratio
are presented in a good agreement with simulation results.

Key words: Money creation, Reserve ratio, Statistical distribution, Velocity of
money, Random transfer

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1 Introduction

Recently two essential aspects of money circulation have been investigated
based on the random transfer models [1,2,3,4,5,6,7,8]. One is statistical distri-
bution of money, which is closely related to earlier Pareto income distribution
[9] and some recent empirical observations [10,11,12,13,14]. The other one is
the velocity of money, which measures the ratio of transaction volume to the

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money stock in an economic system. All the models which appeared in these
researches regarded the monetary system as being composed of agents and
money, and money could be transferred randomly among agents. In such a
random transferring process, money is always being held by agents, any single
agent’s amount of money may strongly fluctuate over time, but the overall
equilibrium probability distribution can be observed under some conditions.
The shape of money distribution in each model is determined by its transfer-
ning rule, for instance, random exchange can lead to a Boltzmann-Gibbs distri-
bution [3], transferring with uniform saving factor can lead to a Gaussian-like
distribution [4] and that with diverse saving factors leads to a Pareto distri-
bution [5]. On the other hand, the time interval between two transfers named
as holding time of money is also a random variable with a steady probability
distribution in the random transferring process. The velocity of money could
be expressed as the expectation of the reciprocal of holding time and the prob-
ability distribution over holding time was found to follow exponential or power
laws [7,8].

The amount of money held by agents was limited to be non-negative in the
models mentioned above except Ref. [3]. Allowing agents to go into debt and
putting a limit on the maximal debt of an agent, Adrian Drăgulescu and Vic-
tor Yakovenko demonstrated the equilibrium probability distribution of money
still follows the Boltzmann-Gibbs law. Although they devote only one section
to discussing the role of debt in the formation of the distribution, they are
undoubtedly pathfinders on this aspect. As cited in their paper, “debts create
money” [15]. Specifically, most part of the money stock is created by debts
through banking system, and this process of money creation plays a significant
role in performance of economy especially by affecting the aggregate output
[16]. Thus money creation should not be excluded from discussion on the is-
sues of monetary economic system. With cognition of this significance, Robert
Fischer and Dieter Braun analyzed the process of creation and annihilation
of money from a mechanical perspective by proposing analogies between as-
sets and the positive momentum of particles and between liabilities and the
negative one [17,18]. They further applied this approach into the study on
statistical mechanics of money [19].

As well known, the central bank plays an important role of controlling the
monetary aggregate that circulates in the modern economy. It issues the mon-
ey base which is much less than the monetary aggregate. The ratio of the
monetary aggregate to the monetary base is called the money multiplier. The
central bank controls the monetary aggregate mainly by adjusting the mon-
ey base and by setting the required reserve ratio which is a key determinant
of the multiplier. So the required reserve ratio is crucial in monetary economic
system. The aim of this work is to investigate the impacts of the required
reserve ratio on monetary wealth distribution and the velocity of money. Our
model is an extended version of that of Robert Fischer and Dieter Braun [19].
In their model, random transfer would increase the quantity of money without bounds unless some limits are imposed exogenously on the stock of assets and liabilities, which are given by specifying an aggregate limit or imposing a transfer potential. Compared with this, we introduce the monetary base and the required reserve ratio in our model by interpreting the process of money creation with the simplified money multiplier model. Thus the limit can be governed by setting the initial values of the monetary base and the required reserve ratio. In addition, we adopt the conventional economic definition of money instead of what they used. We think that the conventional definition of money is more appropriate to the analysis of realistic monetary system. We hope that our work can expose the role of the required reserve ratio in monetary circulation and is helpful to understand the effect of the central bank on monetary economic system.

This paper is organized as follows. In next section we make a brief presentation of money creation and the simplified multiplier model. In Section 3 we propose a random transfer model of money with a bank. And the shapes of monetary wealth distribution and latency time distribution are demonstrated. In Section 4 the dependence of monetary wealth distribution and the velocity of money on the required reserve ratio is presented quantitatively. We finish with some conclusions in Section 5.

2 Money Creation and Simplified Multiplier Model

Modern banking system is a fractional reserve banking system, which absorbs savers’ deposits and loans to borrowers. Generally the public holds both currency and deposits. As purchasing, the public can pay in currency or in deposits. In this sense, currency held by the public and deposits in bank can both play the role of exchange medium. Thus the monetary aggregate is measured by the sum of currency held by the public and deposits in bank in economics. When the public saves a part of their currency into commercial banks, this part of currency turns into deposits and the monetary aggregate does not change. Once commercial banks loan to borrowers, usually in deposit form, deposits in bank increase and currency held by the public keeps constant. So loaning behavior of commercial banks increases the monetary aggregate and achieves money creation.

Money creation of commercial banks is partly determined by the required reserve ratio. In reality, commercial banks always hold some currency as reserves in order to repay savers on demand. Total reserves are made up of ones that the central bank compels commercial banks to hold, called required reserves, and extra ones that commercial banks elect to hold, called excess reserves. Instead of appointing required reserves for each of commercial banks, the central
bank specifies a percentage of deposits that commercial banks must hold as reserves, which is known as the required reserve ratio. The role of the required reserve ratio in money creation is illuminated well by the multiplier model [20].

The multiplier model, originally developed by Brunner and Meltzer [21,22], has become the standard paradigm in the textbooks of macroeconomics. We introduce its simplified version here. In monetary economic system, the monetary aggregate can be measured by

\[ M = C + D, \quad (1) \]

where \( C \) denotes currency held by the public and \( D \) denotes total deposits. The monetary base \( M_0 \) is the sum of currency held by the public and reserves in the banking system, \( R \):

\[ M_0 = C + R. \quad (2) \]

Reserves, which are decomposed into required reserves \( RR \) and excess reserves \( ER \), can be given by

\[ R = RR + ER. \quad (3) \]

Required reserves can be calculated according to the required reserve ratio \( r \) and deposits in commercial banks \( D \):

\[ RR = rD. \quad (4) \]

So Equation (3) can be rewritten as

\[ R = ER + rD. \quad (5) \]

For simplicity, assume that the public holds no currency in hand and that excess reserves are always zero. With these two assumptions, combining Equations (1), (2) and (5) produces the monetary base-multiplier representation of the monetary aggregate:

\[ M = mM_0, \quad (6) \]

where \( m \), the money multiplier, is given by

\[ m = \frac{1}{r}. \quad (7) \]
According to this representation, an increment of one dollar in the monetary base produces an increment of $1/r$ dollars in the monetary aggregate. Since loans made by commercial banks create equal amount of money, its volume $L$ is the difference between the monetary aggregate and the monetary base, that is

$$L = \frac{M_0}{r} - M_0. \quad (8)$$

This equation shows clearly the relation between money creation and the required reserve ratio. As the required reserve ratio increases, the capability of money creation declines. Please note if the public holds currency in hand or commercial banks decide to keep some amount of currency as excess reserves, the amount of money $L$ created by the banking system is less than the value given by the right-hand side of Equation (8).

Although all factors involved in money creation except the required reserve ratio are ignored in the simplified multiplier model, it conveys us the essence of money creation in reality. This suggests that the role of money creation can be investigated by focusing on the impacts of the required reserve ratio on relevant issues. Thus we simply introduced a bank into the random transfer model to examine how the required reserve ratio affects monetary wealth distribution and the velocity of money.

3 Model and Simulation

Our model is an extension of the model in Ref. [19]. The economy turns into the one consisting of $N$ traders and a virtual bank. We postulate that all traders hold money only in deposit form throughout the simulations. At the beginning, a constant monetary base $M_0$ is equally allocated to $N$ traders and is all saved in the bank. As a result, total reserves held by the bank are $M_0$ at the beginning. Time is discrete. Each of the traders chooses his partner randomly in each round, and yield $N$ trade pairs. In each trade pair, one is chosen as “payer” randomly and the other as “receiver”. If the payer has deposits in the bank, he pays one unit of money to the receiver in deposit form. If the payer has no deposit and the bank has excess reserves, the payer borrows one unit of money from the bank and pays it to the receiver in deposit form. But if the bank has no excess reserve, the trade is cancelled. After receiving one unit of money, if the receiver has loans, he repays his loans. Otherwise the receiver holds this unit of money in deposit form.

Simulations are expected to show the results of two issues. One is monetary wealth distributions. Monetary wealth is defined as the difference between
deposit volume and loan volume of a trader. Thus the data of deposit and loan volumes of each trader need to be collected. The other is the velocity of money. When the transferring process of currency is a Poisson process, the velocity of money can be calculated by latency time, which is defined as the time interval between the sampling moment and the moment when money takes part in trade after the sampling moment for the first time. The distribution of latency time in this case takes the following form

$$P(t) = \frac{1}{T} e^{-\frac{t}{T}},$$  \hspace{1cm} (9)$$

where $1/T$ is the intensity of the Poisson process. It can be obtained by simple manipulation that the velocity of money is the same as the intensity [7]. Thus we have,

$$V = \frac{1}{T}. \hspace{1cm} (10)$$

As collecting latency time, each transfer of the deposits can be regarded as that of currency chosen randomly from reserves in the bank equivalently.

Since the initial settings of the amount of money and the number of traders have no impacts on the final results, we performed several simulations with $M_0 = 2.5 \times 10^5$ and $N = 2.5 \times 10^4$, while altering the required reserve ratio. It is found that given a required reserve ratio the monetary aggregate increases approximately linearly for a period, and after that it approaches and remains at a steady value, as shown in Figure 1. We first recorded the steady values of the monetary aggregate for different required reserve ratios and the results are shown in Figure 2. This relation is in a good agreement with that drawn from the simplified multiplier model. We also plotted the values of time when the monetary aggregate begins to be steady for different required reserve ratios in Figure 3. Since the maximal value among them is $1.2 \times 10^5$ or so, the data of deposit volume, loan volume and latency time were collected after $8 \times 10^5$ rounds. We are fully convinced that the whole economic system has reached a stationary state by that moment.

As shown in Figure 4, monetary wealth is found to follow asymmetric Laplace distribution which is divided into two exponential parts by Y Axis [23], which can be expressed as $p_- (m) \propto e^{\frac{m}{\mu_-}}$ and $p_+ (m) \propto e^{-\frac{m}{\mu_+}}$ respectively, where $\mu_+$ is the average amount of positive monetary wealth and $\mu_-$ is the average amount of negative monetary wealth. This asymmetry of the distribution arises from the non-zero monetary base set initially in our model which money creation can be achieved on the basis of. It is worth mentioning that in Ref. [19] the distribution with such a shape can also be obtained by imposing an asymmetric, triangular-shaped transfer potential. From simulation results, it is also seen
that latency time follows an exponential law, as shown in Figure 5. This result indicates that the transferring process of currency is indeed a Poisson type.

4 Results and Discussion

4.1 Monetary Wealth Distribution Versus the Required Reserve Ratio

We show monetary wealth distributions for different required reserve ratios in Figure 6. It is seen that both \( \overline{m}_+ \) and \( \overline{m}_- \) decrease as the required reserve ratio increases. When the required reserve ratio increases closely to 1, \( \overline{m}_- \) decreases closely to 0 and the distribution changes gradually from asymmetric Laplace distribution to Boltzmann-Gibbs law which is the result from the model of Adrian Drăgulescu and Victor Yakovenko.

The stationary distribution of monetary wealth can be obtained by the method of the most probable distribution [24]. In our model, if \( N \) traders are distributed over monetary wealth, with \( n_m \) traders holding monetary wealth \( m(\geq 0) \), \( n_{m'} \) traders holding monetary wealth \( m'(<0) \), this distribution can be done in

\[
\Omega = \frac{N!}{\prod_m n_m! \prod_{m'} n_{m'}!}. \tag{11}
\]

ways. It is also required that the total number of traders, the total amount of positive monetary wealth \( M_+ \) and that of negative monetary wealth \( M_- \) must be kept constant at stationary state, that is

\[
N = \sum_m n_m + \sum_{m'} n_{m'}, \tag{12}
\]

\[
M_+ = \sum_m n_m m = \frac{M_0}{r}, \tag{13}
\]

and

\[
M_- = \sum_{m'} n_{m'} m' = M_0 - \frac{M_0}{r}. \tag{14}
\]

The stationary distribution can be obtained by maximizing \( \ln \Omega \) subject to the constraints listed above. Using the method of Lagrange multipliers, we have

\[
d \ln \Omega - \alpha \ dN - \beta \ dM_+ - \gamma \ dM_- = 0, \tag{15}
\]
whose solutions can be given respectively by
\[ n_m = e^{-\alpha - \beta m} \] (16)
and
\[ n_{m'} = e^{-\alpha - \gamma m'}. \] (17)

So the stationary distribution can be expressed in continuous form as
\[ p_+(m) = \frac{N_0}{N} e^{-\beta m} \quad \text{for} \quad m \geq 0; \]
\[ p_-(m) = \frac{N_0}{N} e^{-\gamma m} \quad \text{for} \quad m < 0, \] (18)

where \( N_0 = e^{-\alpha} \) denotes the the number of traders with no monetary wealth.

Substituting Equations (16) and (17) into Equations (12), (13) and (14), and replacing summation symbol with integral one, we have
\[ \left( \frac{1}{\beta} - \frac{1}{\gamma} \right) e^{-\alpha} = N, \] (19)
\[ \frac{1}{\beta^2} e^{-\alpha} = \frac{M_0}{r}, \] (20)

and
\[ -\frac{1}{\gamma^2} e^{-\alpha} = M_0 - \frac{M_0}{r}, \] (21)

where Equation (19) holds only when \( \beta > 0 \) and \( \gamma < 0 \). Combining Equations (18), (19), (20) and (21), we can get
\[ \overline{m}_+ = \frac{1}{\beta} = \frac{1 + \sqrt{1 - r} M_0}{r} \] (22)

and
\[ \overline{m}_- = \frac{1}{\gamma} = \frac{1 - r + \sqrt{1 - r} M_0}{r}. \] (23)

It is seen that both \( \overline{m}_+ \) and \( \overline{m}_- \) decrease as the required reserve ratio increases, and the value of \( \overline{m}_+ \) is always larger than that of \( \overline{m}_- \) at the same required reserve ratio. These results are illustrated by the solid lines in Figure 7. They are in good agreement with simulation results denoted by dots.
4.2 The Velocity of Money Versus the Required Reserve Ratio

The formula of the velocity of money will be deduced here. It is known that the velocity of money is equal to the intensity of the Poisson process from Equation (10). The intensity of the Poisson process can be measured by average times a unit of money takes part in trades in each round. This suggests that the velocity of money is also the value of transaction volume in each round $A$ divided by the money stock, i.e.,

$$V = \frac{1}{T} = \frac{A}{M_0}.$$ \hspace{1cm} (24)

In order to obtain the expression of $V$ in terms of the required reserve ratio, the analysis of $A$ is required at first.

For convenience in manipulation, traders are now classified into two groups: the traders with positive monetary wealth and the ones with non-positive monetary wealth, whose numbers are denoted by $N_+$ and $N_-$ respectively. From the trading mode of our model, it can be reckoned out that each trader participates in trade averagely 2 times in one round. In each transfer of money, the probability of transferring a unit of money is $1/2$ for the traders with positive monetary wealth, and it must be less than $1/2$ for the traders with non-positive monetary wealth, for borrowing may fail due to the limitation of required reserves. Let $\omega$ denote this probability, from the detailed balance condition which holds in steady state, we have

$$\omega p_-(m_1)p_+(m_2) = \frac{1}{2} p_-(m_1 - 1)p_+(m_2 + 1).$$ \hspace{1cm} (25)

Substituting the expressions of monetary wealth distribution (18) into Equation (25), we obtain

$$\omega = \frac{1}{2} e^{-\frac{m_1 - m_2}{m_+} - \frac{1}{m_-}}.$$ \hspace{1cm} (26)

Thus the total trade volume in each round on average can be expressed as

$$A = N_+ + N_- e^{-\frac{m_1 - m_2}{m_+} - \frac{1}{m_-}}.$$ \hspace{1cm} (27)

Substituting Equation (27) into (24), the velocity of money can be given by

$$V = \frac{N_+}{M_0} + \frac{N_-}{M_0} e^{-\frac{m_1 - m_2}{m_+} - \frac{1}{m_-}}.$$ \hspace{1cm} (28)
Since in steady state the number of traders whose monetary wealth changes from 0 to 1 is equal to that of traders whose monetary wealth changes from 1 to 0, we have the following approximate relation

\[ \omega N_0 \frac{N_+}{N} + \frac{1}{2} N_0 \frac{N_+}{N} = \frac{1}{2} N_1 \frac{N_+}{N} + \frac{1}{2} N_1 \frac{N_-}{N}, \]  

(29)

where \( N_1 = N_0 e^{-\frac{1}{m+}} \) is the number of traders with monetary wealth 1. The left-hand side of Equation (29) represents the number of traders whose monetary wealth changes from 0 to 1 and the right-hand side denotes the number of traders whose monetary wealth changes from 1 to 0. Substituting Equation (26) into (29) and taking \( N = N_+ + N_- \) into account yield

\[ N_+ = \frac{e^{\frac{1}{m+}} - 1}{e^{\frac{1}{m+}} + e^{\frac{1}{m-}} - 1} N \]  

(30)

and

\[ N_- = \frac{e^{\frac{1}{m+}} + e^{\frac{1}{m-}} - e^{\frac{1}{m+}}}{e^{\frac{1}{m+}} + e^{\frac{1}{m-}} - 1} N. \]  

(31)

Combining Equations (28), (30) and (31), we can obtain

\[ V = \frac{N}{M_0} e^{-\frac{1}{m+}}. \]  

(32)

Figure 8 shows the relationships between the velocity of money and the required reserve ratio, from simulation results and from Equation (32) respectively. By measuring latency time for different required reserve ratios, the corresponding velocities of money are obtained from Equation (10). From Figure 8, it is seen that the velocity of money has an inverse relation with the required reserve ratio. This can be interpreted in this way. In each round, if every pair of traders could fulfill their transfer of money, the trade volume would be \( N \) in our model. However, in each round some transfers are cancelled because the payers with non-positive monetary wealth may not get loans from the bank. As indicated by Equation (32), the average realized transfer ratio can be expressed in the form of \( e^{-\frac{1}{m+}} \), which decreases as the required reserve ratio increases. Thus the trade volume in each round decreases, and as a result the velocity of money decreases.
5 Conclusion

In this paper, in order to see how money creation affects the statistical mechanics of money circulation, we develop a random transfer model of money by introducing a fractional reserve banking system. In this model, the monetary aggregate is determined by the monetary base and the required reserve ratio. Computer simulations show that the steady monetary wealth distribution follows asymmetric Laplace type and latency time of money obeys exponential distribution regardless of the required reserve ratio. The distribution function of monetary wealth in terms of the required reserve ratio is deduced. Likewise, the expression of the velocity of money is also presented. These theoretical calculations are in quantitative agreement with the corresponding simulation results. We believe that this study is helpful for understanding the process of money creation and its impacts in reality.

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References


**Figure Captions**

**Figure 1** Time evolution of the monetary aggregate for the required reserve ratio $r = 0.8$. The vertical line denotes the moment at which the monetary aggregate reaches a steady value.

**Figure 2** Steady value of the monetary aggregate versus the required reserve ratio obtained from simulation results (dots) and from the corresponding analytical formula $M = M_0/r$ derived from Equations (6) and (7) (continuous curve).

**Figure 3** The moment at which the monetary aggregate reaches a steady value versus the required reserve ratio.

**Figure 4** The stationary distribution of monetary wealth for the required reserve ratio $r = 0.8$. It can be seen that the distribution follows asymmetric Laplace distribution from the inset.

**Figure 5** The stationary distribution of latency time for the required reserve ratio $r = 0.8$. The fitting in the inset indicates that the distribution follows an exponential law.

**Figure 6** The stationary distributions of monetary wealth for different required reserve ratios. Note that the probability has been scaled by the corresponding maximum value.

**Figure 7** $\overline{m}_+$ (upper) and $\overline{m}_-$ (lower) versus the required reserve ratio obtained from simulation results (dots) and from the corresponding analytical formulas (continuous curves) given by Equations (22) and (23) respectively.

**Figure 8** The velocity of money versus the required reserve ratio obtained from simulation results (dots) and from the corresponding analytical formula (continuous curve) given by Equation (32).
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