Noncommutativity vs. Transversality in QED in a strong magnetic field*

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Quantum electrodynamics (QED) in a strong constant magnetic field is investigated from the viewpoint of its connection with noncommutative QED. It turns out that within the lowest Landau level (LLL) approximation the 1-loop contribution of fermions provides an effective action with the noncommutative $U(1)_{NC}$ gauge symmetry. As a result, the Ward-Takahashi identities connected with the initial $U(1)$ gauge symmetry are broken down in the LLL approximation. On the other hand, it is shown that the sum over the infinite number of the higher Landau levels (HLL's) is relevant despite the fact that each contribution of the HLL is suppressed. Owing to this nondecoupling phenomenon the transversality is restored in the whole effective action. The kinematic region where the LLL contribution is dominant is also discussed.

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1. Introduction

Quantum electrodynamics (QED) in a constant magnetic field has been thoroughly investigated since the classical papers. Following Ref.3 we here analyze this “old” subject from a “new” viewpoint connected with noncommutative QED. (For reviews of noncommutative field theories (NCFT), see Ref.4. Phenomenological issues of noncommutative QED have been studied, for example, in Ref.5.)

In this article, some sophisticated features of the dynamics in QED in a strong magnetic field are revealed. It is shown that in the approximation with the lowest Landau level (LLL) dominance the initial $U(1)$ gauge symmetry in the fermion determinant is transformed into the noncommutative $U(1)_{NC}$ gauge symmetry. In this regime, the effective action is intimately connected with that in noncommutative QED and the original $U(1)$ gauge Ward-Takahashi identities are broken. In fact, this dynamics yields a modified noncommutative QED in which the UV/IR

mixing is absent, similarly to the case of the NJL model in a strong magnetic field. However, it is not the end of the story. We show that the contribution of an infinite number of the higher Landau levels (HLL’s) plays very important role for restoration of the original $U(1)$ gauge symmetry. Although the contribution of each HLL is suppressed in an infrared region, their cumulative contribution is not (a nondecoupling phenomenon). The situation is dramatically changed when the contribution of the HLL’s is incorporated: The transversality is restored. We also indicate the kinematic region where the LLL approximation is reliable.

2. The LLL Approximation and Noncommutativity

Let us study a problem in QED in a strong magnetic field $B$. We consider the case with a large number of fermion flavors $N$ in order to justify the 1-loop approximation of fermions in the sense of $1/N$ expansion. We also choose the current mass $m$ of fermions satisfying the condition $m_{\text{dyn}} \ll m \ll \sqrt{|eB|}$, where $m_{\text{dyn}}$ is the dynamical mass of fermions generated in the chiral symmetric QED in a magnetic field [3]. The condition $m_{\text{dyn}} \ll m$ guarantees that there are no light (pseudo) Nambu-Goldstone bosons, and the only particles in low energy are photons in this model. As to the condition $m \ll \sqrt{|eB|}$, it implies that the magnetic field is very strong.

Integrating out fermions, we obtain the effective action for photons in the leading order in $1/N$:

$$\Gamma = \Gamma^{(0)} + \Gamma^{(1)},$$

with the tree level part,

$$\Gamma^{(0)} = -\frac{1}{4} \int d^4x f_{\mu\nu},$$

and the 1-loop part,

$$\Gamma^{(1)} = -iN\text{TrLn} \left[ i\gamma^{\mu} (\partial_\mu - ieA_\mu) - m \right],$$

where $f^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the vector field $A_\mu = A^{cl}_\mu + \tilde{A}_\mu$. The classical part $A^{cl}_\mu$ is $A^{cl}_\mu = \langle 0 | A_\mu | 0 \rangle$. For a constant magnetic field directed in the $+x^3$ direction, we may use the so called symmetric gauge for $A^{cl}_\mu$,

$$A^{cl}_\mu = \left( 0, \frac{Bx^2}{2}, -\frac{Bx^1}{2}, 0 \right).$$

In a constant magnetic field the fermion propagator is given by

$$S(x, y) = \exp \left[ \frac{ie}{2} (x - y)^{\mu} A^{cl}_\mu (x + y) \right] \tilde{S}(x - y),$$

The dynamical mass is $m_{\text{dyn}} \simeq \sqrt{|eB|} \exp (-N)$ for a large running coupling $\tilde{\alpha}_b \equiv N\alpha_b$ related to the magnetic scale $\sqrt{|eB|}$, and $m_{\text{dyn}} \sim \sqrt{|eB|} \exp \left[ -\frac{eN}{\tilde{\alpha}_b \ln(1/\tilde{\alpha}_b)} \right]$ when the coupling $\tilde{\alpha}_b$ is weak [3].
where the Fourier transform of the translationally invariant part $\tilde{S}$ can be decomposed over the Landau levels:

$$
\tilde{S}(k) = i \exp \left( -\frac{k^2}{|eB|} \right) \sum_{n=0}^{\infty} (-1)^n \frac{D_n(eB, k)}{k^2 - m^2 - 2n|eB|}
$$

(6)

with $k_\perp \equiv (k^1, k^2)$ and $k_\parallel \equiv (k^0, k^3)$. The functions $D_n(eB, k)$ are expressed through the generalized Laguerre polynomials $L_\alpha^m$:

$$
D_n(eB, k) = (k_\parallel \gamma_\parallel + m) \left[ (1 - i\gamma^1 \gamma^2 \text{sign}(eB)) L_n \left( 2 \frac{k^2}{|eB|} \right) 
- (1 + i\gamma^1 \gamma^2 \text{sign}(eB)) L_{n-1} \left( 2 \frac{k^2}{|eB|} \right) \right]
+ 4(k^1 \gamma^1 + k^2 \gamma^2) L_{n-1} \left( 2 \frac{k^2}{|eB|} \right),
$$

(7)

where $\gamma_\parallel \equiv (\gamma^0, \gamma^3)$. For a strong magnetic field $|eB| \gg m^2$, we expect that in the infrared region the LLL approximation should be reliable. Actually, the relation (6) seems to suggest that in the infrared region, $k_\perp, k_\parallel \ll \sqrt{|eB|}$, all the HLL’s with $n \geq 1$ decouple because of their heavy mass $\sqrt{2n|eB|}$ and only the LLL with $n = 0$ is relevant. Although the above argument is physically convincing, there may be a potential flaw due to an infinite number of the Landau levels. As will be shown, it is indeed the case in this problem: it turns out that the cumulative contribution of the HLL’s does not decouple.

Nevertheless, we first study the QED dynamics in the LLL approximation where the fermion propagator is replaced by the LLL one in the calculation of the effective action (3). Recently, the NJL model in a strong magnetic field has been analyzed within the LLL approximation. The extension of the analysis to the case of QED is straightforward. The effective action in the LLL approximation is given by

$$
\Gamma_{\text{LLL}} = \Gamma^{(0)} + \Gamma^{(1)}_{\text{LLL}},
$$

(8)

with

$$
\Gamma^{(1)}_{\text{LLL}} = -\frac{iN|eB|}{2\pi} \int d^2x_\perp \text{Tr}_\parallel [\mathcal{P} \ln[i\gamma^\parallel (\partial_\parallel - ieA_\parallel)| - m]_\ast],
$$

(9)

(compare with Eq. (54) in Ref. [7]. Here $\ast$ is the symbol of the Moyal star product, which is a signature of a NCFT, the spin projector $\mathcal{P}$ is defined by

$$
\mathcal{P} \equiv \frac{1}{2} \left[ 1 - i\gamma^1 \gamma^2 \text{sign}(eB) \right],
$$

(10)

and the longitudinal “smeared” fields $A_\parallel$ are defined as

$$
A_\parallel = e \frac{\nabla^2}{\sqrt{\text{det}}A_\parallel},
$$

(11)
where $\nabla^2_\perp$ is the transverse Laplacian. Notice that $\mathcal{P}$ is the projector on the fermion (antifermion) states with the spin polarized along (opposite to) the magnetic field and that the one-loop term $\Gamma_{\text{LLL}}^{(1)}$ in (8) includes only the longitudinal field $A_\parallel = (A_0, A_3)$. This is because the LLL fermions couple only to the longitudinal components of the photon field $\mathcal{A}$.

In the effective action (9), the trace $\text{Tr}|_\parallel$ of the longitudinal subspace should be taken in the functional sense and the star product relates to the space transverse coordinates. Therefore the LLL dynamics determines a NCFT with noncommutative transverse coordinates $\hat{x}_\perp^a, a = 1, 2$:

$$[\hat{x}_\perp^a, \hat{x}_\perp^b] = i\frac{1}{eB} \epsilon^{ab} \equiv i\theta^{ab}. \quad (12)$$

The structure of the logarithm of the fermion determinant in $\Gamma_{\text{LLL}}^{(1)}$ implies that it is invariant not under the initial $U(1)$ gauge symmetry but under the noncommutative $U(1)_{\text{NC}}$ gauge one (henceforth we omit the subscript $|_\parallel$ in gauge fields):

$$A_\mu \to U(x) \ast A_\mu \ast U^{-1}(x), \quad (\mu = 0, 3) \quad (13a)$$

$$F_{\mu\nu} \to U(x) \ast F_{\mu\nu} \ast U^{-1}(x), \quad (\mu, \nu = 0, 3) \quad (13b)$$

where $U(x) = (e^{i\lambda(x)})_*$ and the field strength $F_{\mu\nu}$ is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]_{\text{MB}}$$

with the Moyal bracket

$$[A_\mu, A_\nu]_{\text{MB}} \equiv A_\mu \ast A_\nu - A_\nu \ast A_\mu. \quad (15)$$

Therefore the derivative expansion of $\Gamma_{\text{LLL}}^{(1)}$ should be expressed through terms with the star product of the field $F_{\mu\nu}$ and its covariant derivatives:

$$\Gamma_{\text{LLL}}^{(1)} = a_0 S_{F^2} + a_1 S_{F^3} + a_2 S_{(DF)^2} + a_3 S_{D^2 F^2} + \cdots, \quad (16)$$

where

$$S_{F^2} \equiv -\frac{1}{4} \int d^2x_\perp d^2x_\parallel F_{\mu\nu} \ast F^{\mu\nu}, \quad (17)$$

$$S_{F^3} \equiv i e \int d^2x_\perp d^2x_\parallel F_{\mu\nu} \ast F^{\nu\lambda} \ast F^\mu_\lambda, \quad (18)$$

$$S_{(DF)^2} \equiv \int d^2x_\perp d^2x_\parallel D_\lambda F^{\lambda\mu} \ast D^\rho F_{\rho\mu}, \quad (19)$$

$$S_{D^2 F^2} \equiv \int d^2x_\perp d^2x_\parallel D_\lambda F_{\mu\nu} \ast D^\lambda F^{\mu\nu}, \quad (20)$$

and the covariant derivative of $F_{\mu\nu}$ is $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} - ie[A_\lambda, F_{\mu\nu}]_{\text{MB}}$. These are all independent operators which have the mass dimension four and six. In particular, by using the Jacobi identity,

$$[D_\mu, [D_\nu, D_\lambda]_{\text{MB}}]_{\text{MB}} + [D_\nu, [D_\lambda, D_\mu]_{\text{MB}}]_{\text{MB}} + [D_\lambda, [D_\mu, D_\nu]_{\text{MB}}]_{\text{MB}} = 0, \quad (21)$$
and the relation $F_{\mu\nu} = ie^{-1}[D_{\mu}, D_{\nu}]_{\text{MB}}$, one can easily check that the operator $\int d^2x_1 \cdots d^2x_n \lambda \Phi_{\mu\nu} * D^\mu F^{\nu\lambda}$ is not independent: it is equal to $-1/2 S_{D^2}$. The coefficients $a_i$, $(i = 0, 1, 2, 3, \cdots)$ in Eq. (16) can be found from the $n$-point photon vertices

$$T_{\text{LLL}}^{(n)} = \frac{(ie)^n N|eB|}{2\pi n} \int d^2x_1 \cdots d^2x_n \left[ \text{tr} \left[ S_{\parallel}(x_1 - x_2)A_{\parallel}(x_1) \cdots S_{\parallel}(x_n - x_1)A_{\parallel}(x_1) \right] \right]$$  \hspace{1cm} (22)

by expanding the vertices in powers of external momenta, where

$$S_{\parallel}(x) = \int \frac{d^2k}{(2\pi)^2} e^{-ik|x|} \frac{i}{k_{\parallel}|\gamma| - m} P$$  \hspace{1cm} (23)

and $A_{\parallel} \equiv |\gamma|A_{\parallel}$. In particular, from the vertices $T_{\text{LLL}}^{(2)}$ and $T_{\text{LLL}}^{(3)}$, we find the coefficients $a_0$, $a_1$, $a_2$, and $a_3$ connected with the operators of the dimension four and six in the derivative expansion (16) of $\Gamma_{\text{LLL}}$:

$$a_0 = \frac{\hat{\alpha}}{3\pi} \frac{|eB|}{m^2}, \hspace{1cm} a_1 = \frac{1}{60m^2} a_0, \hspace{1cm} a_2 = -\frac{1}{10m^2} a_0, \hspace{1cm} a_3 = 0,$$  \hspace{1cm} (24)

where $\hat{\alpha} \equiv Na = Ne^2/(4\pi)$ (since in the presence of a magnetic field the charge conjugation symmetry is broken\textsuperscript{b}, Furry’s theorem does not hold and thereby the 3-point vertex appears).

Notice that the action $\Gamma_{\text{LLL}}$ determines a conventional noncommutative QED only in the case of an induced photon field, when the Maxwell term $\Gamma^{(0)}$ is absent. When this term is present, the action also determines a NCFT, however, this NCFT is different from the conventional ones considered in the literature. In particular, expressing the photon field $A_\mu$ through the smeared field $A_{\parallel}$ as $A_\mu = e^{\frac{k_\mu}{\pi eB}} A_{\parallel}$, we find that the propagator of the smeared field rapidly, as $e^{\frac{k_\mu}{eB}}$, decreases for large transverse momenta. The form-factor $e^{-\frac{k_\mu}{eB}}$ built in the smeared field reflects an inner structure of photons in a magnetic field. This feature leads to removing the UV/IR mixing in this NCFT (compare with the analysis of the UV/IR mixing in Sec. 4 of Ref. [7]).

3. Nondecoupling Effect of The HLL and Transversality

The $U(1)$ gauge Ward-Takahashi identities imply that the $n$-point photon vertex $T^{\mu_1 \cdots \mu_n}(x_1, \cdots, x_n)$ should be transverse, i.e., $\partial_\mu T^{\mu_1 \cdots \mu_n}(x_1, \cdots, x_n) = 0$ ($j = 1, 2, \cdots, n$). It is easy to show that the 2-point vertex $T_{\text{LLL}}^{\mu \nu}$ yielding the polarization operator is transverse indeed. Now let us turn to the 3-point vertex and show that it is not transverse, i.e., the Ward-Takahashi identities connected with the initial gauge $U(1)$ are broken in the LLL approximation.

\textsuperscript{b}Noncommutative QED is also not $C$ invariant\textsuperscript{10}
In the LLL approximation, the 3-point vertex in the momentum space is given by

\[ T^{\mu_1 \mu_2 \mu_3}_{\text{LLL}}(k_1, k_2, k_3) = N e^\frac{|eB|}{2\pi} \sin \left( \frac{1}{2} \theta_{ab} k_{1\perp}^a k_{2\perp}^b \right) \Delta^{\mu_1 \mu_2 \mu_3}_{\text{LLL}}(k_1 \parallel, k_2 \parallel, k_3 \parallel) \]  

(25)

with

\[ \Delta^{\mu_1 \mu_2 \mu_3}_{\text{LLL}}(k_1 \parallel, k_2 \parallel, k_3 \parallel) = \int \frac{d^2\ell_{\parallel}}{i(2\pi)^2} \text{tr} \left[ \gamma^{\mu_1}_{\parallel} [(\ell - \ell_{1\parallel})_{\parallel} + m] \gamma^{\mu_2}_{\parallel} [(\ell + \ell_{3\parallel})_{\parallel} + m] \gamma^{\mu_3}_{\parallel} (\ell_{\parallel} + m) \right]. \]  

(26)

The argument of the sine in Eq. (25) is the Moyal cross product with \( \theta_{ab} = e^{ab}/eB \) (see Eq. (12)). It is easy to find that the divergence of the vertex (25) is not zero,

\[ k_1 \mu_1 T^{\mu_1 \mu_2 \mu_3}_{\text{LLL}}(k_1, k_2, k_3) = -2\frac{e}{i} \sin \left( \frac{1}{2} \theta_{ab} k_{1\perp}^a k_{2\perp}^b \right) \left[ \Pi^{\mu_2 \mu_3}_{\parallel}(k_2 \parallel) - \Pi^{\mu_2 \mu_3}_{\parallel}(k_3 \parallel) \right], \]  

(27)

with

\[ \Pi^{\mu
u}_{\parallel}(k_{\parallel}) = \frac{2\alpha |eB|}{\pi} \left( g^{\mu\nu} - \frac{k_{\mu} k_{\nu}}{k_{\parallel}^2} \right) \Pi(k_{\parallel}^2), \]  

(28)

and

\[ \Pi(k_{\parallel}^2) = 1 + \frac{2m^2}{k_{\parallel}^2 \sqrt{1 - \frac{4m^2}{k_{\parallel}^2}}} \ln \frac{1 + \sqrt{1 - \frac{4m^2}{k_{\parallel}^2}}}{1 - \sqrt{1 - \frac{4m^2}{k_{\parallel}^2}}}, \]  

(29)

where \( \Pi^{\mu
u}_{\parallel} \) is the polarization tensor (apparently it is transverse). We here defined \( g_{\parallel} = \text{diag}(1, -1) \). Hence the original \( U(1) \) gauge Ward-Takahashi identities are broken in the LLL approximation.

The origin of the violation of the transversality is obviously the change of the symmetry: the \( T^{(n)}_{\text{LLL}} \) vertices come from the 1-loop part \( \Gamma^{(1)}_{\text{LLL}} \) of the effective action which is invariant under the noncommutative \( U(1)_{\text{NC}} \) gauge symmetry. Therefore the Ward-Takahashi identities for the vertices \( T^{(n)}_{\text{LLL}} \) should not reflect the initial \( U(1) \) gauge symmetry, but \( U(1)_{\text{NC}} \).

However, it is clear that the full QED dynamics yields the transverse vertices. There should exist an additional contribution that restores the transversality broken in the LLL approximation. Surprisingly, we will find that heavy (naively decoupled) HLL’s play very important role for the restoration.

\(^c\)The analytic expression of \( \Delta^{\mu_1 \mu_2 \mu_3}_{\text{LLL}} \) is given in terms of the two dimensional version of the Passarino-Veltman functions.

\(^d\)The effective action \( \Gamma^{(1)}_{\text{LLL}} \) including both of the tree and 1-loop parts enjoys the longitudinal \( U(1)_\parallel \) gauge symmetry with gauge parameters \( \alpha(x_{\parallel}) \). This \( U(1)_\parallel \) is a subgroup of the initial \( U(1) \) and noncommutative \( U(1)_{\text{NC}} \).
The 3-point vertex with the full fermion propagator includes various contributions of the HLL’s. What kind of diagram is essential? We here note that the LLL diagram has a branch cut singularity above the threshold $k_{\parallel}^2 > 4m^2$. The relevant HLL contribution which we seek for should have the same analytic structure. We easily find that only the diagram shown in Fig.1 has such a branch cut. Therefore we consider the cumulative contribution of the particular HLL diagrams shown in Fig.2 where one of the LLL propagator is replaced by the full one (6) without $n = 0$. By using a common technique in Ref.[12] we can perform the loop integral with respect to $\ell_{\perp}$ and hence schematically obtain

$$\Delta_{\text{HLL}}^n \sim \frac{(-1)^n L_{\alpha} \left( \frac{2 \ell_{\perp}^2}{|eB|} \right)}{n|eB|}. \quad (30)$$

After the integral over $\ell_{\perp}$, we find that the contribution of each of the individual HLL with $n \geq 1$ is

$$\Delta_{\text{HLL}}^{\mu\nu\lambda}(p, q, k) = \frac{(-1)^n}{n|eB|} F^{\mu\nu\lambda}(p, q, k), \quad (31)$$

where $F^{\mu\nu\lambda}$ is some function of longitudinal and transverse momenta. As was expected, each HLL contribution $\Delta_{\text{HLL}}^{\mu\nu\lambda}$ is suppressed by powers of $1/|eB|$ in the infrared region. It is, however, quite remarkable that despite the suppression of individual HLL contributions, their cumulative contribution becomes relevant in the infrared region. In fact, by using the relation

$$(1 - z)^{-(\alpha+1)} \exp \left( \frac{xz}{z - 1} \right) = \sum_{n=0}^{\infty} L_n^{\alpha}(x) z^n \quad (32)$$

*From a technical viewpoint, it is convenient to sum over the Laguerre polynomials before performing the loop integral with respect to $\ell_{\perp}$. There is no subtlety concerning exchange of the ordering of the infinite sum and the loop integral, because the integral is essentially a 2-dimensional one and does not have any UV divergence.*
and integrating it with respect to $z$, we can perform explicitly the summation over the HLL contributions, $\sum_{n=1}^{\infty} (-1)^n L_n(x)/n$, and thereby obtain a transverse vertex:

$$T^{\mu_1\mu_2\mu_3}(k_1, k_2, k_3) = T^{\mu_1\mu_2\mu_3}_{\text{LLL}}(k_1, k_2, k_3) + T^{\mu_1\mu_2\mu_3}_{\text{HLL}}(k_1, k_2, k_3),$$

(33)

where

$$T^{\mu_1\mu_2\mu_3}_{\text{HLL}}(k_1, k_2, k_3) = \frac{2e}{i} \sin \left( \frac{1}{2} \theta_{ab} k_1^a k_2^b \right) \left[ \frac{-k_1^{\mu_1}}{k_1^2} \left( \Pi^{\mu_2\mu_3}_{\parallel}(k_{2\parallel}) - \Pi^{\mu_2\mu_3}_{\parallel}(k_{3\parallel}) \right) \right]$$

$$+ \frac{k_1^{\mu_1} k_2^{\mu_2} - k_2^{\mu_2} k_1^{\mu_1} - (k_1 \cdot k_2) g_1^{\mu_2\mu_3}}{k_1^2 k_2^2} k_{2\parallel} \cdot \Pi^{\mu_2\mu_3}_{\parallel}(k_{3\parallel})$$

$$+ \frac{-k_2^{\mu_2}}{k_2^2} \left( \Pi^{\mu_3\mu_1}_{\parallel}(k_{3\parallel}) - \Pi^{\mu_3\mu_1}_{\parallel}(k_{1\parallel}) \right)$$

$$+ \frac{k_2^{\mu_2} k_3^{\mu_3} - k_3^{\mu_3} k_2^{\mu_2} - (k_2 \cdot k_3) g_1^{\mu_2\mu_3}}{k_2^2 k_3^2} k_{3\parallel} \cdot \Pi^{\mu_2\mu_3}_{\parallel}(k_{1\parallel})$$

$$+ \frac{-k_3^{\mu_3}}{k_3^2} \left( \Pi^{\mu_1\mu_2}_{\parallel}(k_{1\parallel}) - \Pi^{\mu_1\mu_2}_{\parallel}(k_{2\parallel}) \right)$$

$$+ \frac{k_3^{\mu_3} k_1^{\mu_1} - k_1^{\mu_1} k_3^{\mu_3} - (k_3 \cdot k_1) g_1^{\mu_2\mu_3}}{k_3^2 k_1^2} k_{1\parallel} \cdot \Pi^{\mu_2\mu_3}_{\parallel}(k_{2\parallel})$$

$$+ (\text{transverse part}).$$

(34)

We here defined $g_\perp = \text{diag}(-1, -1)$, and $(p \cdot q)_\perp = p^1 q^1 + p^2 q^2$. Note that the vertex for the initial non-smeread fields $A_\mu$ is given by $e^{-k_1^2 + k_2^2 + k_3^2} T^{\mu_1\mu_2\mu_3}$. It is easy to check the transversality of the 3-point vertex $T^{\mu_1\mu_2\mu_3}$, (See Eq. 27) and also note that the transversality of the vacuum polarization tensor, $k_\parallel \cdot \Pi_{\parallel\parallel}(k_1) = 0$, and the momentum conservation, $k_1 + k_2 + k_3 = 0$.)

One might doubt whether or not there exists a kinematic region in which the LLL contribution is dominant. We find a positive answer, i.e., the region with momenta $k_\parallel^2 \gg |k_\parallel|^2$. In this region, the leading terms in the expansion of the LLL and HLL vertices in powers of $k_\parallel$ are:

$$T^{\mu_1\mu_2\mu_3}_{\text{LLL}}(k_1, k_2, k_3) = -\frac{2e\alpha |eB|}{3\pi m^2} \sin \left( \frac{1}{2} \theta_{ab} k_1^a k_2^b \right) \left[ (k_2 - k_3)^{\mu_1} g_{\parallel\mu_2\mu_3} \right.$$  

$$+ (k_3 - k_1)^{\mu_2} g_{\parallel\mu_3\mu_1} + (k_1 - k_2)^{\mu_3} g_{\parallel\mu_1\mu_2} \right].$$

(35)
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\[ T^{\mu_1 \mu_2 \mu_3}_{\text{HLL}} = \sum_{n=1}^{\infty} \begin{array}{ccc} n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n \end{array} \]

Fig. 2. The relevant contribution of the HLL's.

\[ T^{\mu_1 \mu_2 \mu_3}_{\text{HLL}}(k_1, k_2, k_3) = \]
\[-\frac{2e\alpha}{3\pi} \frac{|eB|}{m^2} \sin \left( \frac{1}{2} \theta_{ab} k_1^a \cdot k_2^b \right) \left[ -\frac{k_1^{\mu_1}}{k_{1 \perp}} \left( (k_2^2 - k_3^2) g_{\mu_2 \mu_3} - k_2^\mu \mu_3 - k_3^\mu \mu_2 + k_2^\mu k_3^\mu \right) \right] + \frac{k_1^{\mu_1} k_2^2 - k_1^{\mu_1} k_2^2}{k_{1 \perp}^2} - (k_1 \cdot k_2) g_{\mu_2 \mu_3} \left( k_3^\mu k_{2 \perp} - (k_2 \cdot k_3) k_{3 \perp} \right) + \text{permutations of } (k_1, \mu_1), (k_2, \mu_2), \text{ and } (k_3, \mu_3) \]
\[ + \frac{2e\alpha}{3\pi} \frac{|eB|}{m^2} \left\{ \exp \left( -\frac{(k_1 \cdot k_2)}{2|eB|} \right) - \cos \left( \frac{1}{2} \theta_{ab} k_1^a \cdot k_2^b \right) \right\} \]
\[ \times \left\{ k_1^{\mu_2} e_{ab} k_1^a k_2^b + (k_1 \cdot k_2) e_{ab} k_1^a k_2^b \left( k_3^\mu g_{\mu_2 \mu_3} - k_2^\mu k_3^\mu \right) \right\} \]
\[ + \frac{g_1^{\mu_1 \mu_2} e_{ab} k_1^a k_2^b + e_1^{\mu_1 \mu_2} (k_1 \cdot k_2) k_2^b}{k_{2 \perp}} \left( k_3^\mu g_{\mu_2 \mu_3} - k_3^\mu g_{\mu_2 \mu_3} \right) \]
\[ + \text{permutations of } (k_1, \mu_1), (k_2, \mu_2), \text{ and } (k_3, \mu_3) \} \right] \]

(36)

It is clear from these expressions that in that region the LLL contribution dominates indeed. This result is quite noticeable. The point is that as was shown in Ref[8] the region with momenta \( k_{1 \perp}^2 \gg |k_{\parallel}| \) yields the dominant contribution in the Schwinger-Dyson equation for the dynamical fermion mass in QED in a strong magnetic field. Therefore the LLL approximation is reliable in that problem.

We comment on the role of \( T^{\mu_1 \mu_2 \mu_3}_{\text{HLL}} \). Although the vertex \( T^{\mu_1 \mu_2 \mu_3}_{\text{HLL}} \) is subdominant...
in the region $k^2_{\perp} \gg |k^2_{\parallel}|$, it is crucial for restoration of the transversality. While the LLL vertex $T^{\mu_1 \mu_2 \mu_3}_{\text{LLL}}$ is multiplied only by a small longitudinal momentum $k_{\parallel}$, the HLL vertex $T^{\mu_1 \mu_2 \mu_3}_{\text{HLL}}$ includes certain terms multiplied by a large transverse momentum $k_{\perp}$. Owing to this nature, the divergence of the subdominant term can cancel out the nonvanishing divergence of the dominant one.

4. Summary

We found that the LLL approximation yields the effective action enjoying the noncommutative $U(1)_{NC}$ gauge symmetry. Hence the initial $U(1)$ gauge Ward-Takahashi identities are broken in the LLL approximation. We also showed that nondecoupling phenomenon of (heavy) HLL’s is the key point of the problem: the infinite sum over the HLL’s is relevant to restoration of the transversality. What physics underlines it? We believe that this phenomenon reflects the important role of a boundary dynamics at spatial infinity in this problem. The point is that the HLL’s are not only heavy states but their transverse size grows without limit with their gap $\sqrt{m^2 + 2n|eB|}$ as $n \to \infty$. This happens because the transverse dynamics in the Landau problem is an oscillator-like one. It implies that the role of the boundary dynamics at the transverse spatial infinity (corresponding to $n \to \infty$) is crucial. This is similar to the role of edge states in the quantum Hall effect: the edge states are created by the boundary dynamics and also restore the gauge invariance. Both these phenomena reflect the importance of a boundary dynamics in a strong magnetic field. It would be interesting to examine whether or not similar nondecoupling phenomena take place in noncommutative theories arising in string theories in magnetic background.

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References