A Hamiltonian functional for the linearized Einstein vacuum field equations

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By considering the Einstein vacuum field equations linearized about the Minkowski metric, the evolution equations for the gauge-invariant quantities characterizing the gravitational field are written in a Hamiltonian form by using a conserved functional as Hamiltonian; this Hamiltonian is not the analog of the energy of the field. A Poisson bracket between functionals of the field, compatible with the constraints satisfied by the field variables, is obtained. The generator of spatial translations associated with such bracket is also obtained.

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I. INTRODUCTION

As we know, the Lagrangian and the Hamiltonian formalisms employed in the study of mechanical systems with a finite number of degrees of freedom can be applied in the case of infinite-dimensional systems. The Hamiltonian formulation is usually obtained from the Lagrangian formulation by means of the Legendre transformation, but in the case of fields this canonical procedure presents difficulties since not always the momentum densities are independent of the field variables, which is usually mended by the introduction of constraints. Nevertheless, it is possible to avoid these complications and give a Hamiltonian formulation for a given continuous system, without making reference to the Lagrangian formulation, if its evolution equations can be written in the form

\[ \dot{\phi}_a = D_{ab} \frac{\delta H}{\delta \phi_b}, \]  

where the field variables \( \phi_a(a = 1, 2, ..., n) \) represent the state of the system, \( H \) is a suitable functional of the \( \phi_a \), \( \delta H/\delta \phi_b \) is the functional derivative of \( H \) with respect to \( \phi_b \), and the \( D_{ab} \) are, in general, operators that must satisfy certain conditions that allow the definition
of a Poisson bracket between functionals of the $\phi_a$ (see, e.g. Refs. [1] and [2]). Here
and henceforth a dot denotes partial differentiation with respect to the time and there is
summation over repeated indices.

In Ref. [3] the evolution equations for the gravitational field, given by the Einstein
vacuum field equations linearized about the Minkowski metric, are written in a Hamiltonian
form [1] in terms of gauge-invariant quantities only by using an analog of the energy of the
electromagnetic field as Hamiltonian.

In this paper we propose a conserved functional of the field as a new Hamiltonian and then
we find a Hamiltonian structure for the linearized Einstein theory. This conserved functional
is found in the Maxwell theory [5,6] and we use its analog in the linearized Einstein theory as
Hamiltonian. By contrast with the Hamiltonian structure found in Ref. [3], which involves
integral operators, in the present case the $D_{ab}$ turn out to be constants.

In the next section the linearized Einstein vacuum field equations are written in a nonco-
variant manner, emphasizing their analogy with Maxwell’s equations, in Sect. 3 we propose
the Hamiltonian, and in Sect. 4 such equations are written in a Hamiltonian form. A Poisson
bracket, compatible with the constraints imposed by the field equations, is obtained and it
is shown that it yields the expected relations between the Hamiltonian or the momentum
(which we give also) and any functional of the field. Throughout this paper Greek indices
run from 0 to 3 and Latin indices, $i, j, ..., $ from 1 to 3. Greek indices are raised and lowered
by means of the Minkowski metric. Repeated Latin lower indices are to be summed as
though a Kronecker delta $\delta^{ij}$ were present $a_i b_i = \delta^{ij} a_i b_j$.

II. THE LINEARIZED EINSTEIN VACUUM FIELD EQUATIONS

In the linearized Einstein theory it is assumed that, in a suitable coordinate system, the
metric of the space-time can be written in the form

$$ g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} $$

(2)

where $h_{\alpha\beta}$ represents a small deviation of the metric $g_{\alpha\beta}$ from the Minkowski metric

$$(\eta_{\alpha\beta}) \equiv \text{diag}(-1, 1, 1, 1).$$

(3)

The coordinate system in which expression (2) applies is not defined uniquely; under any
“infinitesimal coordinate transformation”, $x'^{\alpha} = x^{\alpha} + \xi^{\alpha}$, the metric has again the form (2)
with

\[ h'_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha, \]  

\[ (4) \]

where \( \partial_\alpha \equiv \partial / \partial x^\alpha \). The tensor field

\[ K_{\alpha\beta\gamma\delta} \equiv \frac{1}{2} \left\{ \partial_\alpha \partial_\gamma h_{\delta\beta} - \partial_\beta \partial_\gamma h_{\delta\alpha} + \partial_\beta \partial_\delta h_{\gamma\alpha} - \partial_\alpha \partial_\delta h_{\gamma\beta} \right\}, \]

\[ (5) \]

which is the curvature tensor corresponding to the metric \( g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \) to first order in \( h_{\alpha\beta} \), is invariant under the gauge transformations (4). From its definition it is clear that \( K_{\alpha\beta\gamma\delta} \) possesses the symmetries

\[ K_{\alpha\beta\gamma\delta} = -K_{\beta\alpha\gamma\delta} = -K_{\alpha\beta\delta\gamma} = K_{\gamma\delta\alpha\beta}, \]

\[ (6) \]

\[ K_{\alpha\beta\gamma\delta} + K_{\alpha\delta\beta\gamma} + K_{\alpha\gamma\delta\beta} = 0, \]

\[ (7) \]

and that it also satisfies the identities

\[ \partial_\alpha K_{\beta\gamma\delta\epsilon} + \partial_\epsilon K_{\beta\gamma\alpha\delta} + \partial_\delta K_{\beta\gamma\epsilon\alpha} = 0. \]

\[ (8) \]

Conversely, Eqs. (6,7) imply that, locally, \( K_{\alpha\beta\gamma\delta} \) has the form (5) where \( h_{\alpha\beta} \) is some symmetric tensor field defined up to the transformations (4).

In terms of the right dual \( K^{\ast}_{\alpha\beta\gamma\delta} \) of \( K_{\alpha\beta\gamma\delta} \) defined by

\[ K^{\ast}_{\alpha\beta\gamma\delta} \equiv \frac{1}{2} K_{\alpha\beta}^{\rho\sigma} \epsilon_{\rho\sigma\gamma\delta} \]

\[ (9) \]

where \( \epsilon_{\alpha\beta\gamma\delta} \) is completely antisymmetric with \( \epsilon_{0123} = 1 \), Eqs. (7) and (8) are equivalent to

\[ K^{\ast}_{\alpha\beta\gamma} = 0 \]

\[ (10) \]

and

\[ \partial^\gamma K^{\ast}_{\alpha\beta\gamma\delta} = 0, \]

\[ (11) \]

respectively. From Eqs. (9) and (10) it follows that

\[ K^{\ast}_{\alpha\beta\gamma\delta} = -K^{\ast}_{\beta\alpha\gamma\delta} = -K^{\ast}_{\alpha\beta\delta\gamma} \]

\[ (12) \]

which are analogous to the first two equalities in (6). Nevertheless, in general, \( K^{\ast}_{\alpha\beta\gamma\delta} \) does not possess all the symmetries of \( K_{\alpha\beta\gamma\delta} \) [Eqs. (6,7)]. In fact, from the definition (8) one obtains that

\[ K^{\ast}_{\alpha\beta\gamma\delta} - K^{\ast}_{\gamma\delta\alpha\beta} = \frac{1}{2} \left\{ \epsilon_{\alpha\beta\rho\delta} K^{\rho}_{\gamma} + \epsilon_{\beta\alpha\gamma\rho} K^{\rho}_{\delta} + \epsilon_{\gamma\beta\delta\rho} K^{\rho}_{\alpha} + \epsilon_{\delta\alpha\gamma\rho} K^{\rho}_{\beta} \right\}, \]

\[ (13) \]
where the tensor field
\[ K_{\alpha\beta} \equiv K_{\alpha\gamma\beta}^\gamma, \]  
which is symmetric as a consequence of Eqs. [13], has been introduced. Similarly, one finds that
\[ K_{\alpha\beta\gamma\delta}^* + K_{\alpha\delta\beta\gamma}^* + K_{\alpha\gamma\delta\beta}^* = -\epsilon_{\beta\gamma\delta\rho} K_{\alpha}^\rho. \]  
(15)

On the other hand, from the identities [12] it follows that
\[ \partial^\alpha K_{\beta\alpha\delta\varepsilon} = -\partial_\varepsilon K_{\beta\alpha}^\alpha \delta - \partial_\delta K_{\beta\alpha}^\alpha \varepsilon = \partial_\varepsilon K_{\beta\delta} - \partial_\delta K_{\beta\varepsilon}, \]  
(16)

The linearized Einstein field equations are
\[ K_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} K_{\gamma}^\gamma = -\frac{8\pi G}{c^4} T_{\alpha\beta}, \]  
(17)
or, equivalently
\[ K_{\alpha\beta} = -\frac{8\pi G}{c^4} \left( T_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} T_{\gamma}^\gamma \right), \]  
(18)
where \( T_{\alpha\beta} \) is the energy-momentum tensor of the matter to first order in \( h_{\alpha\beta} \). Therefore, the linearized Einstein vacuum field equations are \( K_{\alpha\beta} = 0 \) and from Eqs. [6,7] and [13,14] one sees that \( K_{\alpha\beta\gamma\delta}^* \) satisfies the same relations as \( K_{\alpha\beta\gamma\delta} \) if and only if \( K_{\alpha\beta} = 0 \).

In what follows it will be assumed that the conditions \( K_{\alpha\beta} = 0 \) hold. This implies that all the components \( K_{\alpha\beta\gamma\delta} \) can be expressed in terms of the fields \( E_{ij} \) and \( B_{ij} \) defined by
\[ E_{ij} \equiv K_{0i0j}, \quad B_{ij} \equiv -K_{0i0j}^*, \]  
(19)
where the minus sign is introduced for later convenience. As a consequence of Eqs. [9], [10], [13] and [14], the fields \( E_{ij} \) and \( B_{ij} \) are symmetric and have vanishing traces, hence each of them has five independent components. Equations [11] and [16] amount to
\[ \partial_i E_{ij} = 0, \quad \partial_i B_{ij} = 0 \]  
(20)
and
\[ \frac{1}{c} \dot{E}_{ij} = \epsilon_{ikm} \partial_k B_{mj}, \quad \frac{1}{c} \dot{B}_{ij} = -\epsilon_{ikm} \partial_k E_{mj}, \]  
(21)
where \( \epsilon_{ijk} \) is completely antisymmetric with \( \epsilon_{123} = 1 \), which are analogous to the source-free Maxwell equations. It is easy to see that, due to Eqs. [21] and to the fact that \( E_{ij} \) and \( B_{ij} \) have vanishing trace, the right-hand sides of Eqs. [21] are symmetric in the indices \( i \) and \( j \). Equations [20], which do not involve time derivatives, can be regarded as constraints on the fields \( E_{ij} \) and \( B_{ij} \).
III. HAMILTONIAN FUNCTIONAL

As it was already mentioned in the introduction, in Ref. [3] a Hamiltonian structure for the linearized Einstein theory is found by using as Hamiltonian density the analog of the energy of the electromagnetic field, \( H = \int \kappa (E_{ij}E_{ij} + B_{ij}B_{ij})dv/2 \) (where \( \kappa \) is a constant), having to introduce ad hoc modifications in order to get consistency with the constraints imposed by the field variables. This Hamiltonian structure involves integral operators.

In the Maxwell theory we can see easily that the functional \( H = \int c \epsilon_{ijk} (E_i \partial_j E_k + B_i \partial_j B_k)dv/2 \) is a conserved functional [5, 6, 7], and it can be used as Hamiltonian in that theory. By analogy with the electromagnetic field one can introduce the conserved functional

\[
H = \int \mathcal{H} dv = \frac{c}{2} \int \epsilon_{ikm} (E_{ij} \partial_k E_{mj} + B_{ij} \partial_k B_{mj}) dv
\]

(22)

as Hamiltonian in the linearized Einstein theory (we suppose that the fields vanish at infinity).

We point out that a functional \( F \) is a conserved functional if and only if \( dF/dt = 0 \). Therefore, to check if a functional of the field is a conserved functional one needs to use the evolution equations for the field only without having to choose a particular Hamiltonian \( H \) and its corresponding Hamiltonian structure \( D_{ab} \). Of course, if one makes a choice of the pair \( (H, D_{ab}) \), then one can also use this knowledge to check it.

IV. HAMILTONIAN STRUCTURE

Equations (21) can be written in the Hamiltonian form

\[
\dot{E}_{ij} = D_{ijkm} \frac{\delta H}{\delta B_{km}}, \quad \dot{B}_{ij} = -D_{ijkm} \frac{\delta H}{\delta E_{km}}
\]

(23)

where

\[
D_{ijkm} = \frac{1}{2} (\delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk})
\]

(24)

and \( H \) is given by (22) [cf. Eq. (1)], which is a conserved functional. In the functional derivatives (23) the 18 components \( B_{km} \) and \( E_{km} \) are treated as if they were independent. This is due to the fact that the right-hand sides of Eqs. (23), restricted to the submanifold \( N \) defined by the conditions \( E_{ij} = E_{ji}, E_{ii} = 0, B_{ij} = B_{ji} \) and \( B_{ii} = 0 \), are symmetric in the
indices \( i \) and \( j \) and have vanishing trace; therefore the evolution curves given by Eqs. (23) are tangent to \( N \).

Making use of the \( D_{ijkm} \) given by Eq. (24), a Poisson bracket between any pair of functionals of the field \( F \) and \( G \) can be defined as

\[
\{F, G\} \equiv \int \left( \frac{\delta F}{\delta E_{ij}} D_{ijkm} \frac{\delta G}{\delta B_{km}} - \frac{\delta F}{\delta B_{ij}} D_{ijkm} \frac{\delta G}{\delta E_{km}} \right) dv = \int \left( \frac{\delta F}{\delta E_{km}} \frac{\delta G}{\delta B_{km}} - \frac{\delta F}{\delta B_{km}} \frac{\delta G}{\delta E_{km}} \right) dv.
\]

(25)

The bracket (25) is antisymmetric and satisfies the Jacobi identity due to the fact that the \( D_{ijkm} \) are constants [1]. Hence the \( D_{ijkm} \) define a hamiltonian structure. From Eq. (25) one finds that

\[
\{E_{ij}(r', t), E_{km}(r'', t)\} = 0 = \{B_{ij}(r', t), B_{km}(r'', t)\}
\]

(26)

and that

\[
\{E_{ij}(r', t), B_{km}(r'', t)\} = D_{ijkm} \delta(r' - r'')
\]

(27)

which are consistent with Eqs. (20) since \( \partial_i D_{ijkm} = 0 \). Furthermore, with respect to the Hamiltonian structure given by \( D_{ijkm} \), one can find easily that the functionals

\[
P_k = \int \frac{1}{2} \left( E_{ij} \partial_k B_{ij} - B_{ij} \partial_k E_{ij} \right) dv = - \int B_{ij} \partial_k E_{ij} dv
\]

(28)

(see e.g. Ref.[2]) are the components of the momentum and they are conserved by the invariance of \( H \) in any direction.

If \( F \) is any functional of the field that does not depend explicitly on the time then Eqs. (28) and (25) give

\[
\{F, H\} = \int \left( \frac{\delta F}{\delta E_{ij}} \dot{E}_{ij} - \frac{\delta F}{\delta B_{ij}} \dot{B}_{ij} \right) dv = \dot{F}
\]

(29)

Similarly, by using Eqs. (25), (28) and (20), and the fact that the right-hand sides of Eqs. (21) are symmetric in the indices \( i \) and \( j \) one obtains

\[
\{F, P_k\} = - \int \left( \frac{\delta F}{\delta E_{ij}} \partial_k E_{ij} + \frac{\delta F}{\delta B_{ij}} \partial_k B_{ij} \right) dv
\]

(30)

which means that, with respect to the Hamiltonian structure associated with the bracket (25), the functional \( P_k \), is, in effect, the generator of the translations in the direction of the axis \( x^k \). It is in this sense that the \( P_k \) are the components of the momentum of the field.

In contrast with Eq. (28), when \( H \) is given by the analog of the energy of the field, the functionals

\[
P_k = \int \frac{\kappa}{c} \epsilon_{km} E_{ij} B_{mj} dv
\]

(31)

are the components of the momentum of the field [3].
The Hamiltonian employed in this example is a conserved functional of the field, however it is not known if there exist additional conditions for a conserved functional to be a Hamiltonian, with the corresponding Poisson bracket satisfying the Jacobi identity (when the $D_{ab}$ are constants the Jacobi identity is always satisfied, but in other cases one has to verify that this identity is satisfied [1]). In the case of a mechanical system with a finite number of degrees of freedom in classical mechanics, any constant of motion can be used as Hamiltonian by defining appropriately the symplectic structure of the phase space (or, equivalently, the Poisson bracket) [4].

The example considered here shows a different form to the traditional canonical formalism to write the evolution equations for an infinite-dimensional system, in which sometimes there are constraints. In the present case the $D_{ab}$ are constants because $H$ depends on the $\partial_i \phi_a$ (and on the $\phi_a$, of course), this is an advantage since one can obtain the components of the momentum of the field immediately [2].

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