TOPICS ON ELEMENTARY THEORY OF SCATTERING

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I. SCATTERING BY A CENTRAL POTENTIAL

1. Phase shift analysis

In this section we give a short review on the scattering of a particle by a fixed centre of force described by a potential \( V(r) \). (For a more detailed account see e.g.: L. Schiff, Quantum Mechanics).

The problem of elastic collision between two particles (with masses \( m_1, m_2 \)) is easily reduced to the scattering of a single particle (with "reduced mass" \( \mu = \frac{m_1 m_2}{m_1 + m_2} \)) by a potential \( V(r) \), by considering the relative motion of the two particles.

We shall use, in general, the coordinate system of the centre of mass of the two particles.

The incident particle (moving along the \( z \) axis in the positive direction) is described by a plane wave \( e^{ikz} \); the scattered particle is described, at a great distance from the scattering centre, by an outgoing spherical wave

\[
\psi(\theta) = \frac{ikr}{r},
\]

where \( f(\theta) \) is the scattering amplitude (\( \theta \) is the angle between the \( z \) axis and the direction of motion of the scattered particle; \( r \) is the distance of the particle from the scattering centre); thus, the complete solution of the stationary Schrödinger equation

\[
-\frac{\hbar^2}{2\mu} \nabla^2 \psi + V(r)\psi = E\psi
\]

is represented, at large distances from the scattering centre, by the asymptotic form
\[ \psi(r, \theta) = e^{i k z} + f(\theta) \frac{e^{i k r}}{r}. \]  

(2)

The solution of Eq. (1) can be expanded in terms of the Legendre polynomials \( P_\ell(\cos \theta) \):

\[ \psi(r, \theta) = \frac{1}{r} \sum_{\ell=0}^{\infty} u_\ell(r) P_\ell(\cos \theta) \]

(3)

where \( u_\ell(r) \) is the solution of the radial Schrödinger equation

\[ \frac{d^2}{dr^2} u_\ell + \left[ k^2 - U(r) - \frac{\ell(\ell+1)}{r^2} \right] u_\ell = 0 \]

(4)

with

\[ k^2 = \frac{2 \mu E}{h^2}, \quad U(r) = \frac{2 \mu V(r)}{h^2}. \]

In the case of a potential with a finite range \( R \), the \( u_\ell \)'s have the asymptotic behaviour (for \( r \gg R \)):

\[ u_\ell(r) \to a_\ell \frac{1}{kr} \sin \left( kr - \frac{1}{2} \ell \pi + \delta_\ell \right) \]

(5)

where the quantity \( \delta_\ell \) (which depends, in general, on the energy \( E \)) is the phase shift for the scattering in the state of angular momentum \( \ell \). The general asymptotic form of the solution Eq. (4) can then be written as

\[ \psi(r, \theta) \to \frac{1}{kr} \sum_{\ell=0}^{\infty} a_\ell \sin \left( kr - \frac{1}{2} \ell \pi + \delta_\ell \right) P_\ell(\cos \theta). \]

(6)

We have now to chose the quantities \( a_\ell \) in such a way that the two expressions Eqs. (2) and (6) coincide. Using the expansion

\[ e^{i k z} = \frac{1}{kr} \sum_{\ell=0}^{\infty} i^{2 \ell+1} (2 \ell + 1) \sin \left( kr - \frac{1}{2} \ell \pi \right) P_\ell(\cos \theta) \]

(7)

we can write
\[ f(\theta) \frac{ikr}{kr} = \psi(r, \theta) - e^{ikz} = \]

\[
= \frac{1}{kr} \sum_{\ell=0}^{\infty} \left\{ \alpha_\ell \sin \left( kr - \frac{1}{2} \ell \pi + \delta_\ell \right) - i(2\ell + 1) \sin \left( kr - \frac{1}{2} \ell \pi \right) \right\} P_\ell(\cos \theta)
\]

from which the following relations are easily derived:

\[
a_\ell = i(2\ell + 1) e^{i\delta_\ell}
\]

(8)

\[
f(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell + 1) (e^{2i\delta_\ell} - 1) P_\ell(\cos \theta)
\]

(9)

The formula Eq. (9) solves the problem of expressing the scattering amplitude in terms of the phase shifts \( \delta_\ell \). We shall sometimes use the notation

\[
f(\theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) f_\ell(k) P_\ell(\cos \theta)
\]

(10)

where

\[
f_\ell(k) = \frac{e^{i\delta_\ell} - 1}{2ik} = \frac{e^{i\delta_\ell} \sin \delta_\ell}{k}
\]

(11)

represents the scattering amplitude of the \( \ell \)-th partial wave.

The **differential cross-section** is given by

\[
\frac{d\sigma}{d\Omega} = |f(\theta)|^2
\]

(12)

and the **total cross-section** is obtained by integration over all the angles:

\[
\sigma(k) = 2\pi \int_0^{\pi} |f(\theta)|^2 \sin \theta \, d\theta = \frac{4\pi}{k} \sum_{\ell=0}^{\infty} (2\ell + 1) \sin^2 \delta_\ell.
\]

(13)

It is easy to derive from Eqs. (8) and (13) the relation

\[
\text{Im} \, f(0) = \frac{k}{4\pi} \sigma(k).
\]

(14)
This relation between the total cross-section and the forward scattering amplitude is known as "optical theorem".

2. Determination of the phase shifts

The determination of the phase shifts depends on the specific form of the potential \( V(r) \). We are interested here in the general properties of the phase shifts valid for a potential of finite range \( R \); we refer again to textbooks, such as Schiff, for detailed analyses.

The radial solution of Eq. (4) for \( r > R \) (in this region \( V(r) = 0 \)) can be written as:

\[
u_\ell(r>R) = \cos \delta_\ell \cdot j_\ell(kr) + \sin \delta_\ell \cdot \eta_\ell(kr)
\]  

(15)

where \( j_\ell \) and \( \eta_\ell \) are the spherical Bessel and Neuman functions. The phase shifts are determined by joining at the boundary \( r = R \) the wave function and its first derivative, i.e. by joining the logarithmic derivative of the wave function. We write:

\[
\phi_\ell(k) = R \left( \frac{u_\ell}{u_\ell'} \right)_{r=R}
\]

(16)

where the quantity \( \phi_\ell(k) \) is \( R \) times the logarithmic derivative of the wave function valid inside the potential region evaluated at \( r = R \). Clearly \( \phi_\ell \) depends on the form of the potential; but we shall consider it here as a given parameter.

Suppose now that \( kR \ll 1 \); in this case we can use the asymptotic expressions

\[
j_\ell(x) \approx \frac{x^\ell}{(2\ell+1)!!}; \quad \eta_\ell(x) \approx \frac{(2\ell-1)!!}{x^{\ell+1}} \quad (x < 1)
\]

(17)

in the above relations Eqs. (15) and (16). We obtain in this way:

\[
\tan \delta_\ell = \alpha_\ell \frac{(kR)^{2\ell+1}}{(2\ell+1)!! (2\ell-1)!!}
\]

(18)

with

\[
\alpha_\ell = \frac{i - \phi_\ell}{i + \phi_\ell}
\]

(19)
For a regular behaviour of $\alpha_\ell$, $t_\ell \delta_\ell$ decreases strongly with the increasing angular momentum $\ell$. For very small values of $kR$, we may neglect all the terms except the first. In this approximation (equivalent to keep only the $S$ wave) one gets:

\[ t_\ell \delta_\ell \approx \delta_\ell \approx \alpha_\ell k \]  

and

\[ f(\theta) \approx \alpha_\ell ; \quad \sigma \approx 4\pi \alpha_\ell^2 . \ [21]\]

Then, for small velocity, the scattering is isotropic, and the cross-section is independent of the energy. The same result is obtained for the low-energy scattering by an impenetrable sphere of radius $R = \alpha_\ell$.

3. Resonance scattering

We have assumed, in the previous section, that for very small $kR$ the quantities $\alpha_\ell$ are also small. This is obviously not true in the special case

\[ \rho_\ell (k_R) = -(\ell + 1) . \ [22]\]

In this case $t_\ell \delta_\ell$ goes to infinity and $\delta_\ell = \pi/2$ at the value $k = k_R$.

We can expand $\rho_\ell (k)$ for values of $k$ close to $k_R$ (or equivalently for values of the energy $E$ close to $E_R = h^2 k_R^2 / 2\mu$):

\[ \rho_\ell (k) \approx -(\ell + 1) + (E_R - E) \left( \frac{d\rho_\ell}{dE} \right)_{E=E_R} . \ [23]\]

We can then re-write Eq. (18) as

\[ t_\ell \delta_\ell = \frac{\Gamma_\ell / 2}{E_R - E} \]  

with

\[ \Gamma_\ell = \frac{2(kR)^2 \ell + 1}{(d\rho_\ell / dE)_{E=E_R} [(2\ell - 1)!!]} . \ [25]\]
This behaviour corresponds to a resonance which occurs in the scattering of the \(l\)-th partial wave at the energy \(E_R\).

If we write for the total cross-section:

\[
\sigma = \sum_{l=-\infty}^{\infty} \sigma_l
\]

we see that for \(E\) close to \(E_R\), the "partial wave cross-section" \(\sigma_l\) becomes:

\[
\sigma_l(E) = \frac{4\pi}{k^2} (2l+1) \left(\frac{E}{2}\right)^2 \frac{1}{(E-E_R)^2 + \left(\frac{\gamma}{2}\right)^2}.
\]

It is clear that at the resonance \(E = E_R\), \(\sigma_l\) reaches the maximum value

\[
\sigma_l = \frac{4\pi}{k^2} (2l+1)
\]

which is called the geometrical value. The quantity \(\gamma\) is the width of the resonance; at \(E = E_R \pm \gamma/2\) the cross-section decreases to \(\frac{1}{2}\) of the maximum value.

4. Graphical representation

We give here a graphical representation for the scattering amplitude given by Eq. (11), which can also be written as

\[
f_l(k) = \frac{1}{k \cot \delta_k - ik}.
\]

We also give the following relations

\[
\text{Im } f_l^{-1} = -k
\]

\[
\text{Re } f_l^{-1} = k \cot \delta_k
\]

the first of which corresponds to the "unitarity" of the S matrix, (see following).
In the case of a resonance in the \( \ell \) partial wave, we get from Eq. (24):

\[
f'_\ell(k) = \frac{1}{k} \frac{\sqrt{2}}{(E_R - E) - i\pi/2}.
\]

(31)

It is easy to show that the quantity

\[
F_\ell(k) = k f'_\ell(k) = \frac{1}{\cot \delta_\ell - 1}
\]

(32)

is represented by a circle in the complex plane of \( F_\ell \) (\( \text{Im } F_\ell \) versus \( \text{Re } F_\ell \)). In fact, by writing

\[
\epsilon = \cot \delta_\ell
\]

\[
x = \text{Re } F_\ell = \frac{\epsilon}{\epsilon^2 + 1}
\]

\[
y = \text{Im } F_\ell = \frac{1}{\epsilon^2 + 1}
\]

we get

\[
x^2 = (1 - y)y
\]

which is the equation of a circle with centre at the point \( x = 0, y = \frac{1}{2} \), and radius \( \frac{1}{2} \) (see Fig. 1). For \( \epsilon = \text{const} \), one gets straight lines of equation: \( y = \frac{1}{\epsilon} x \).

In the case of a resonance \( \epsilon \) is given by

\[
\epsilon = \frac{2}{n} (E_R - E).
\]

(33)

At the resonance \( E = E_R \): \( \epsilon = 0, x = 0, y = 1 \); for \( E < E_R \): \( \epsilon > 0 \); for \( E > E_R \): \( \epsilon < 0 \). The point \( x = y = 0 \) corresponds to \( \delta_\ell = 0, \pi, \ldots \). (We assume \( \delta_\ell \to 0 \) as \( E \to 0 \)). Thus, as the energy \( E \) passes through \( E_R \), the representative point on the circle in Fig. 1 passes through the point \( x = 0, y = 1 \) in an anticlockwise sense.

Fig. 1
II. SCATTERING OF PARTICLES WITH SPIN

1. Scattering of a spin-0 particle by a spin-$1/2$ particle

We generalize now the previous results to the case in which the particles have spins different from zero, and the potential depends on the spin orientation, as e.g. in the case of the spin-orbit potential.

In this case the solution of the Schrödinger equation can be expanded in terms of the eigenstates \( \gamma_{J\ell}^{M}(\theta, \varphi) \) of \( J, M, \ell \) (\( J \) is the total angular momentum and \( M \) is its component along the \( z \) axis):

\[
\psi(r, \theta, \varphi) = \frac{1}{\mathcal{R}_{J, \ell}} \sum_{M} u_{J, \ell}^{M}(r) \gamma_{J\ell}^{M}(\theta, \varphi) .
\]  

(34)

The \( \gamma_{J\ell}^{M} \) are expressed in terms of the usual spherical harmonics \( Y_{\ell}^{m} \), and the spin eigenfunctions \( \chi_{S}^{m_{S}} \):

\[
\gamma_{J\ell}^{M}(\theta, \varphi) = \sum_{m, m_{S}} C_{J, \ell; m, m_{S}}^{S} \chi_{S}^{m_{S}} Y_{\ell}^{m}(\theta, \varphi)
\]

(35)

where the \( C_{J, \ell; m, m_{S}}^{S} \)'s are the Clebsch-Gordan coefficients (\( m, m_{S} \) are the \( z \) components of \( \ell \) and \( S \)).

The radial Schrödinger equation is now

\[
\frac{d^{2}}{dr^{2}} u_{J, \ell}^{M} + \left[ k^{2} - U_{J, \ell}^{M}(r) - \frac{\ell(\ell + 1)}{r^{2}} \right] u_{J, \ell}^{M} = 0 .
\]

(36)

We consider here the simple case in which the two particles have spin zero and spin $1/2$; for a given value of $\ell$, the total angular momentum $J$ can have the two values $J = \ell \pm 1/2$ and the potential splits into two terms (we assume $U_{\ell \pm 1/2} \neq U_{\ell - 1/2}$; clearly, if the potential does not depend on $J$, the results of the preceding sections remain unchanged).

Since for the incident wave $m = 0$ (there is symmetry around the $z$ axis), the $z$ component of the total angular momentum is $M = m_{S} = \pm 1/2$ in
the initial state, and then it will be always \( \tilde{\mu} = \pm \frac{1}{2} \). On the other hand, if the incident particle has a given \( m_s = \pm \frac{1}{2} \), the outgoing particle can have both values \( \mu' = \pm \frac{1}{2} \), \( m'_s = -\frac{1}{2} \).

The asymptotic wave function can then be written as follows:

\[
\psi_{\pm \frac{1}{2}}(\theta, \varphi) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} e^{ikz} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \frac{e^{ikr}}{r} f(\theta, \varphi) + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \frac{e^{ikr}}{r} g(\theta, \varphi)
\]

(37)

where we have used the notation \( \alpha = x\frac{1}{2}, \beta = x\frac{i}{2} \). The quantity \( g \) is called spin-flip amplitude, and \( f \) non-spin-flip amplitude. These amplitudes determine the differential cross-section

\[
\frac{d\sigma}{d\Omega} = |f(\theta, \varphi)|^2 + |g(\theta, \varphi)|^2.
\]

(38)

The asymptotic form of the general solution is given by Eq. (34) with

\[
u_{J, \ell}(r) \approx \frac{1}{r} a_{J, \ell} \sin \left( kr - \frac{1}{2} \ell \pi + \delta_{J, \ell} \right), \quad (J = \ell \pm \frac{1}{2})
\]

(39)

In this specific case we give the explicit expressions for the \( u_{J, \ell} \):

\[
u_{\ell + \frac{1}{2}} = \sqrt{\frac{\ell + \frac{1}{2}}{2\ell + 1}} \alpha_{\ell}^0 + \sqrt{\frac{\ell}{2\ell + 1}} \beta_{\ell}^{-1}
\]

\[
u_{\ell - \frac{1}{2}} = -\sqrt{\frac{\ell}{2\ell + 1}} \alpha_{\ell}^0 + \sqrt{\frac{\ell + 1}{2\ell + 1}} \beta_{\ell}^1
\]

Using again the expansion Eq. (7), written in the form

\[
e^{ikz} = \frac{1}{kr} \sum_{\ell} \sqrt{4\pi(2\ell + 1)} i^\ell \sin \left( kr - \frac{1}{2} \ell \pi \right) Y_\ell
\]

(40)

one obtains by comparison of Eqs. (37) and (34):

\[
a_{\ell + \frac{1}{2}, \ell} = \sqrt{\frac{4\pi}{\ell + 1}} \frac{i^\ell}{\ell} e^{i\delta_{\ell + \frac{1}{2}, \ell}}
\]

(41)

\[
a_{\ell - \frac{1}{2}, \ell} = \sqrt{4\pi} i^\ell e^{i\delta_{\ell - \frac{1}{2}, \ell}}
\]
\[ f(\sigma, \varphi) = \frac{1}{12\pi} \sum_{\ell} \sqrt{\frac{2\ell + 1}{2\ell + 1}} \sqrt{(\ell + 1) f_{\ell + \frac{1}{2}} + f_{\ell - \frac{1}{2}}} Y_{\ell}^{m}(\sigma) \]

\[ g(\sigma, \varphi) = \frac{1}{12\pi} \sum_{\ell} \sqrt{\frac{2\ell + 1}{2\ell + 1}} \sqrt{(\ell + 1)} \left[ f_{\ell + \frac{1}{2}} - f_{\ell - \frac{1}{2}} \right] Y_{\ell}^{m}(\sigma, \varphi) \]

where

\[ f_{\ell \pm \frac{1}{2}} = \frac{\sin \delta_{\ell \pm \frac{1}{2}}}{\sin \delta_{\ell \pm \frac{1}{2}}} \sin \delta_{\ell \pm \frac{1}{2}} \cdot \]

It is clear that the differential cross-section does not depend on the angle \( \varphi \). By integration over all the angles one can obtain the total cross-section

\[ \sigma = \frac{4\pi}{k^2} \sum_{\ell = 0}^{\infty} (\ell + 1) \sin^2 \delta_{\ell + \frac{1}{2}} + \ell \sin^2 \delta_{\ell - \frac{1}{2}} = \]  

\[ = \frac{4\pi}{k^2} \sum_{J = \frac{1}{2}}^{\infty} (J + \frac{1}{2}) \sin^2 \delta_{J,\ell} \cdot \]

This formula can be easily extended to the general case in which the two particles have spins \( s_1 \) and \( s_2 \). The result is the following

\[ \sigma = \frac{4\pi}{k^2} \left( 2s_1 + 1 \right) \left( 2s_2 + 1 \right) \sum_{J, S} (2J + 1) \sin^2 \delta_{J, S} \]

where \( S \) is the total spin \( (S = s_1 + s_2) \).

These results can be applied to the case of a resonance occurring at a particular value of \( J \).

2. **Example: the \( \pi^+ p \) scattering at low energy**

We consider here as an example the \( \pi^+ p \) scattering at low-energy (\( \pi^+ \) kinetic energy up to \( \approx 200 \text{ MeV} \)).

Assuming that only the S and P waves are important and that the higher waves can be neglected, we get from Eq. (42):
\[ f = \frac{1}{k} \left[ f_{0, \gamma/2} + (2f_{1, \gamma/2} + f_{1, \gamma/2}) \cos \theta \right] \]

\[ g = \frac{1}{k} (f_{1, \gamma/2} - f_{1, \gamma/2}) \sin \theta \exp i\phi. \]

The differential cross-section is

\[ \frac{d\sigma}{d\Omega} = \frac{1}{k^2} (A + B \cos \theta + C \cos^2 \theta) \]

with

\[ A = |f_{0, \gamma/2}|^2 + |f_{1, \gamma/2} - f_{1, \gamma/2}|^2 \]

\[ B = 2 \text{Re} f_{0, \gamma/2}^* (2f_{1, \gamma/2} + f_{1, \gamma/2}) \]

\[ C = 3|f_{1, \gamma/2}|^2 + 6 \text{Re} f_{1, \gamma/2}^* f_{1, \gamma/2} \]

In the case of pure \( P_{3/2} \) (\( \ell = 1, J = \frac{3}{2} \)) state, which is the state of the well-known resonance in \( \pi^+ p \) at \( \approx 200 \) MeV, one has

\[ A = |f_{1, \gamma/2}|^2, \quad B = 0, \quad C = 3|f_{1, \gamma/2}|^2 \]

and the differential cross-section is

\[ \frac{d\sigma}{d\Omega} = |f_{1, \gamma/2}|^2 (1 + 3 \cos^2 \theta). \]

The angular distribution is then symmetric around \( \theta = \pi/2 \) for the pure \( P_{3/2} \) resonant state. The presence of the S wave and its interference with the \( P_{3/2} \) wave gives rise to the asymmetric term

\[ B \cos \theta = 4 \text{Re} f_{0, \gamma/2}^* f_{1, \gamma/2} \cos \theta. \]

This is, in fact, the experimental feature.
III. INELASTIC COLLISIONS

1. Elastic and inelastic cross-sections

We call inelastic a collision which produces a change in the nature or in the number of the colliding particles. Examples of inelastic collisions between elementary particles are the following: \( K^- + p \rightarrow \Sigma ^+ + \pi ^+ \), \( \pi ^- + p \rightarrow p + \pi ^- + \pi ^0 \). In general, several final states are possible for a given initial state. For example, considering only the two particle states, the initial system \( K^- + p \) can go into the final states: \( K^- + \pi ^- \), \( K^0 + n \), \( \Lambda ^+ + \pi ^0 \), \( \Sigma ^+ + \pi ^+ \), \( \Sigma ^0 + \pi ^0 \). Each different state is called a different channel.

The asymptotic form of the wave function of the system of the two colliding particles will be a sum of different terms, each representing a possible channel. Among these terms there is always, in particular, a term corresponding to the elastic scattering. In addition there is, of course, a term describing the particles before the collisions.

For the moment we consider only the global effect of the inelastic processes, specifically on the elastic scattering, and devote Section IV to the study of superimposed inelastic channels. We consider, for the sake of simplicity, spin-zero particles.

The asymptotic expressions of the radial functions \( u_\ell (r) \) are now modified with respect to the pure elastic case. The expression (5) can be considered as the sum of an incident and outgoing waves with the same amplitudes. In the present case, since several channels are present in the final states, the amplitude of the final outgoing wave must be less than that of the ingoing wave. We write then:

\[
    u_\ell (r) \approx \frac{a_\ell e^{-i(kr - \frac{1}{2} \ell \pi)} - e^{-i(kr - \frac{1}{2} \ell \pi)}}{2\ell kr}
\]

where \( a_\ell \) is in general a complex quantity, with modulus less than unity; it can be written as: \( a_\ell = \eta_\ell e^{i\delta_\ell} \) \((\delta_\ell, \eta_\ell \) real numbers).
The asymptotic expression of the wave function \( \psi(r, \theta) \) is then given by:

\[
\psi(r, \theta) \to \sum_{\ell} i^{\ell} (2\ell + 1) \frac{a_\ell e^{i(kr - \ell \pi)} - e^{-i(kr - \ell \pi)}}{2ikr} P_\ell(\cos \theta). \quad (51)
\]

Using the same procedure of Section 1.1, one can find by comparison of Eqs. (2) and (51):

\[
f(\ell) = \sum_{\ell} (2\ell + 1) f_\ell(k) P_\ell(\cos \theta)
\]

with

\[
f_\ell(k) = \frac{a_\ell - 1}{2ik} = \frac{\eta_\ell e^{i\delta_\ell} - 1}{2ik}. \quad (52)
\]

The total elastic cross-section is given by:

\[
\sigma_{el} = \frac{\pi}{k^2} \sum_{\ell} (2\ell + 1) \left| 1 - a_\ell \right|^2 = \frac{4\pi}{k^2} \sum_{\ell} (2\ell + 1) \left| \frac{\eta_\ell e^{i\delta_\ell} - 1}{2i} \right|^2. \quad (53)
\]

The total inelastic cross-section (for all possible final states of the particles) can also be expressed in terms of the quantities \( a_\ell \).

For each value of \( \ell \) the intensity of the outgoing wave is reduced by the ratio \( |a_\ell|^2 < 1 \) with respect to the intensity of the ingoing wave. This reduction is entirely due to inelastic scattering:

\[
\sigma_{in} = \frac{\pi}{k^2} \sum_{\ell} (2\ell + 1) \left( 1 - |a_\ell|^2 \right) = \frac{4\pi}{k^2} \sum_{\ell} (2\ell + 1) \left( 1 - \eta_\ell^2 \right). \quad (54)
\]

Obviously, for \( \eta_\ell = 1 \), \( \sigma_{el} \) becomes identical to Eq. (13), and \( \sigma_{in} \) vanishes.

We note that only \( \sigma_{el} \) contains the complex term \( a_\ell \) (while \( \sigma_{in} \) contains only the modulus \( |a_\ell| \)): this corresponds to the fact that in the elastic scattering the incident and outgoing wave are coherent.

It is easy to check that the optical theorem expressed by Eq. (14) holds for the total cross section \( \sigma_{tot} = \sigma_{el} + \sigma_{in} \).
Let us denote by $\sigma^{(el)}_c$, $\sigma^{(in)}_c$ the terms in Eqs. (53) and (54) and define:

$$\kappa_{el} = k^2 \sigma^{(el)}_c / (2 \delta + 1) \frac{1}{\eta_c \cdot t^2}$$

$$\kappa_{in} = k^2 \sigma^{(in)}_c / (2 \delta + 1) \frac{1}{\eta_c}$$

(56)

For a given value of $\delta$,$\kappa_{el}$ can be expressed in terms of $\kappa_{in}$. We plot in Fig. 2 $\kappa_{el}$ versus $\kappa_{in}$ for a few typical values of $\delta$. The boundary ($\delta = 0$, $\delta = \pi/2$) corresponds to the minimum and maximum values of the elastic cross-section, as a function of the inelastic one. For $\eta = 1$, $\kappa_{el}$ reaches the maximum value $|e^{\frac{i\pi}{16}2\delta} - 1|^2$, while $\kappa_{in} = 0$ (pure elastic scattering). The maximum value of $\kappa_{in}$ is obtained for $\eta = 0$ (complete absorption); in this case $\kappa_{el} = \kappa_{in} = 1$.

The previous results can be generalized to the case of spin different from zero, (see e.g. Blatt and Weisskopf, Theoretical Nuclear Physics, Chapter VIII). Usually a channel is characterized also by a given value of the total spin $S = \vec{s}_1 + \vec{s}_2$ (we note the scattering without a change of $S$ is coherent); this channel has the statistical weight

$$\delta(S) = \frac{2J+1}{(2s_1+1)(2s_2+1)}$$

(57)

For given values of $S$ and $J$ the elastic and inelastic cross-sections are given by:

$$\sigma_{el}(S,J) = \frac{\pi}{k^2} \frac{2J+1}{2S+1} \sum_{\ell, \ell'} |\delta_{\ell \ell'} - a_{\ell \ell'}(S,J)|^2$$

(58)

$$\sigma_{in}(S,J) = \frac{\pi}{k^2} \frac{2J+1}{2S+1} \sum_{\ell, \ell'} (|\delta_{\ell \ell'} - a_{\ell \ell'}(S,J)|^2)$$

(59)
where one has to take into account that the orbital angular momentum $\ell'$ in the final state can be different from the corresponding value $\ell$ in the initial state, for given values of $J$ and $S$ (the conservation of parity implies that $\ell$ and $\ell'$ differ by an even number).

The total cross-sections for a given $S$, $\sigma_{el}(S)$, $\sigma_{in}(S)$ are obtained by summing $\sigma_{el}(S,J)$, $\sigma_{in}(S,J)$ over all the values of $J$. For unpolarized beams one has to sum over all possible values of $S$, multiplying $\sigma_{el}(S)$, $\sigma_{in}(S)$ by the statistical weight $g(\ell)$. The results obtained in this way are the complete generalization of the formulae Eqs. (53) and (54).

2. Scattering by a black sphere

We consider here, as an example, the scattering by a "black" sphere, i.e. a sphere which absorbs all the particles which strike on it. We consider the case in which the wave length of the incident particle $1/k$ is much smaller than the radius $R$ of the sphere.

These assumptions can be expressed as follows:

$$\alpha_\ell = 0 \quad \text{for} \quad \ell \leq kR \quad (\text{no outgoing wave})$$
$$\alpha_\ell = 1 \quad \text{for} \quad \ell > kR \quad (\text{no scattering})$$

From Eqs. (53) and (54) one gets

$$\sigma_{el} = \sigma_{in} = \frac{\pi}{k^2} \sum_{\ell=0}^{\ell_{\text{max}}} (2\ell + 1) \approx \frac{\pi}{k^2} \sum_{\ell=0}^{kR} (2\ell + 1) \approx \frac{\pi R^2}{k^2}. \quad (60)$$

Both the elastic and inelastic cross-sections are equal to the geometrical cross-section. The elastic scattering, which is called shadow scattering, is due to the diffraction effects which occur at the edge of the target. The scattering angles of this diffraction scattering are rather small and of the order of $\ell^{-1} \approx (kR)^{-1}$; the differential cross-section presents a very narrow peak in the forward direction.

We can give some estimate of the shape of the diffraction peak. The scattering amplitude $f(\theta)$ is pure imaginary, as one can see from Eq. (52)
\[
\ell_{\text{max}} \sum_{\ell=0}^\ell (2\ell+1) P_\ell^2(\cos \theta).
\]

For small angles and large values of \( \ell \) one can write

\[
P_\ell^2(\cos \theta) \approx J_0(\ell \sin \theta)
\]

and the scattering amplitude, by replacing the sum over \( \ell \) by an integral, \( f(\theta) \) becomes

\[
\text{Im} f(\theta) \approx \frac{1}{2k} \int_0^{2\ell} 2\ell J_0(\ell \sin \theta) d\ell = \frac{R}{\sin \theta} J_1(kR \sin \theta).
\]

The behaviour of \( \text{Im} f(\theta) \) versus \( \theta \) is represented in Fig. 3. The value at \( \theta = 0 \) is given by

\[
\text{Im} f(\theta) = \lim_{\theta \to 0} \left( \frac{R}{\sin \theta} \cdot \frac{kR \sin \theta}{2} \right) = \frac{kP^2}{2}
\]

as given also by the optical theorem Eq. (55). The first zero of \( J_1(kR \sin \theta) \) is at \( \theta \approx 3.8/kR \).

IV. MULTI-CHANNEL FORMALISM

1. The scattering matrix

We have considered in the previous chapter the case of multi-channel processes, and we have examined in particular the elastic channel. We present here a more general approach, which allows us to obtain the cross-sections for the transition from a given initial channel to a different final channel.
We consider, for the sake of simplicity, only two spin-zero particles in each channel.

The asymptotic wave function describing the relative motion of the two particles in a given channel \( i \), with orbital angular momentum \( \ell_i \), is represented by

\[
\psi_{\ell_i}(r) = \frac{1}{2 i \ell_i r} \left[ e^{-i(k_1 r - \frac{1}{2} \ell_i \pi)} - S_{ii} e^{i(k_1 r - \frac{1}{2} \ell_i \pi)} \right].
\] (63)

This expression represents the ingoing and outgoing spherical waves in the same channel \( i \). It corresponds to the \( \ell \)-th term of Eq. (51) with \( a_i \) replaced by \( S_{ii} \).

However, with an ingoing wave in the channel \( i \), there will be outgoing waves also in all the other different channels which are open (a channel is said to be open when the available energy is greater than its threshold). The outgoing wave in the \( f \) channel is written as

\[
\psi_{\ell_f}(r) = \frac{1}{2 i \sqrt{k_f k_i} r} S_{fi} e^{i(k_f r - \frac{1}{2} \ell_f \pi)}. \] (64)

The ratio of the flux of the outgoing wave in the channel \( f \) to the incident plane wave flux gives the cross-section

\[
\sigma_{\ell_i}^{(i \rightarrow f)} = (2\ell_i + 1) \frac{\pi}{k_i^2} |S_{fi}|^2. \] (65)

This formula is valid for \( i \neq f \); in order to obtain the scattered wave in the elastic channel, we have to subtract from the outgoing wave in Eq. (63) the \( \ell \)-th term of the outgoing part of the plane wave Eq. (6):

\[
\sigma_{\ell_i}^{(i \rightarrow i)} = (2\ell_i + 1) \frac{\pi}{k_i^2} |1 - S_{ii}|^2. \] (66)

The expressions Eqs. (65) and (66) can be written in a single formula:

\[
\sigma_{\ell_i}^{(i \rightarrow f)} = (2\ell_i + 1) \frac{\pi}{k_i^2} |\delta_{fi} - S_{fi}|^2. \] (67)
The complex quantities $S_{fi}$ ($i, f = 1, 2, \ldots n; n = \text{number of channels}$) can be ordered in an $n \times n$ matrix $S$, which is called the scattering matrix:

\[
S = \begin{pmatrix}
S_{11} & S_{12} & S_{13} & \ldots \\
S_{21} & S_{22} & S_{23} & \ldots \\
S_{31} & S_{32} & S_{33} & \ldots \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\]

We quote here two important properties of the $S$ matrix (for the proof see e.g.: Blatt and Weisskopf):

1) $S$ is a unitary matrix: $SS^+ = I$ ($S^+$ is the hermitian conjugate of $S$:
$S^+_{ij} = S_{ji}^*$).

2) $S$ is a symmetrical matrix: $S = S$ ($S$ is the transposed matrix of $S$:
$S^+_{ij} = S_{ji}$).

The first property is connected with the conservation of flux, the second with time reversal invariance.

In the case of one channel, the $S$ matrix, which reduces to one element, can be written, using the first property:

\[
S_{i} = e^{2i\delta_i}
\]

(68)

where $\delta_i$ is the usual phase shift. In fact, the cross-section Eq. (67) with Eq. (68) becomes:

\[
\sigma_i = \frac{A_\pi}{k^2} (2i + 1) \sin^2 \delta_i
\]

which agrees with the formula Eq. (13).

In the case of two channels, using both unitarity and symmetry, the $S$ matrix can be written as follows:

\[
S_i = \begin{pmatrix}
\eta_i e^{2i\delta_i} & i\sqrt{1 - \eta_i^2} e^{i(\delta_i^{(1)} + \delta_i^{(2)})} \\
\eta_i e^{2i\delta_i} & \eta_i e^{2i\delta_i}
\end{pmatrix}
\]

(69)
where the quantities $\delta_c^{(1)}$, $\delta_c^{(2)}$, $\eta_c$ are real. In this case Eq. (67) gives:

$$\sigma^{(i\rightarrow i)} = (2\ell + 1) \frac{\pi}{k_i^2} |1 - \eta_c e^{i\delta_c^{(1)}}|^2$$

$$\sigma^{(i\rightarrow f)} = (2\ell + 1) \frac{\pi}{k_f^2} (1 - \eta_c^2)$$

which agree with the results obtained in Eqs. (53) and (54).

For more than two channels, the situation is much more complicated. Since the $S$ matrix is unitary and symmetrical, the number of independent real parameters in the case of $n$ channels is $\frac{1}{2}n(n+1)$.

We close this section by giving, without proof, a general expression for the differential cross-section, by which one can evaluate the angular distribution for a reaction of particles with spin:

$$\frac{d\sigma}{d\Omega}^{(i\rightarrow f)} = \frac{(2k_i)^{-1}}{(2S_{i1} + 1)(2S_{i2} + 1)} \sum_{S_i, S_f} \left| \sum_{J, i, f} \sqrt{4\pi (2\ell_i + 1)} C_{i^J S_i} (J, m_{S_i}; 0, m_{S_i}) \right|^2 \cdot C_{i^J S_f} (J, m_{S_f}; m_{S_i}, m_{S_f}) \left[ \delta_{f_i} - S_{f_i} (J, S_i, S_f, \ell_i, \ell_f) \right] y_{\ell_f}^{m_{S_f}} (\theta, \phi) \right|^2 .$$

(70)

The $S$ matrix depends, in this case, for a given total angular momentum $J$, on the total spin $\vec{S} = \vec{S}_1 + \vec{S}_2$ and angular momenta $\ell_i, \ell_f$ of the initial and final channels.

2. The scattering amplitude and the reaction matrix

It is useful to introduce two other matrices, besides the $S$ matrix. We define a matrix $T$, which is the generalization of the scattering amplitude for a multi-channel system, by:

$$S_{fi} = \delta_{fi} + 2i k_i^{1/2} k_f^{1/2} T_{fi} .$$

(71)
In the matrix notation:

\[ S = 1 + 2i \, k^{\frac{1}{2}} \, T \, k^{\frac{1}{2}} \]  

or

\[ T = k^{-\frac{1}{2}} \, S - 1 \, 2i \, k^{-\frac{1}{2}} \]

where \( k \) is a diagonal matrix.

In terms of the elements \( T_{fi} \), the cross-section Eq. (67) becomes

\[ \sigma^{(i\rightarrow f)}_{ci} = 4\pi \, (2l_i + 1) \, \frac{k_f}{k_i} \, |T_{fi}|^2 \]  

(74)

One sees immediately that, in the case of one channel, \( T \) coincides with the scattering amplitude

\[ T_i = \frac{\alpha_i \delta_i - 1}{2ik} \]  

(75)

The expressions Eqs. (63) and (64), by use of Eq. (71), can be replaced by

\[ \psi^{(i\rightarrow f)}_{l_f} (r) = \delta_{ri} \, \frac{\sin (k_f r - \frac{1}{2} \ell_f \pi)}{k_f r} + T_{fi} \, \frac{e^{i(k_f r - \frac{1}{2} \ell_f \pi)}}{r} \]  

(76)

which is a generalization of the \( i \)-th term of Eq. (2). The expression Eq. (76) represents than an incident wave of unit amplitude in the \( i \) channel, together with an outgoing wave of amplitude \( T_{fi} \) in the \( f \) channel.

The situation in which the outgoing waves are replaced by standing waves (for all channels) is described by

\[ \psi^{(i\rightarrow f)}_{l_f} (r) = \delta_{ri} \, \frac{\sin (k_f r - \frac{1}{2} \ell_f \pi)}{k_f r} + K_{fi} \, \frac{\cos (k_f r - \frac{1}{2} \ell_f \pi)}{k_f r} \]  

(77)

The quantities \( K_{fi} \) are the elements of a matrix \( K \), called the reaction matrix.

It is possible to show that the following relation holds between the matrices \( T \) and \( K \):
\[ T = K (1 - iK)^{-1} = (1 - iK)^{-1} K \] 

or, equivalently:

\[ T^{-1} = K^{-1} - iK. \] 

We will show that the definition Eq. (77) leads to the relation Eq. (78) in the particular case in which only the S waves are present in all channels, (see also: H. H. Dalitz: Strange particles and strong interactions; Oxford University Press, 1962). For S waves, Eq. (76) and (77) become:

\[
\psi^{(i\rightarrow f)}(r) = \delta_{f\bar{i}} \frac{\sin k_f r}{k_f r} + T_{f\bar{i}} \frac{e^{ik_f r}}{r}.
\] 

\[
\psi^{(i\rightarrow f)}(r) = \delta_{f\bar{i}} \frac{\sin k_f r}{k_f r} + K_{f\bar{i}} \frac{\cos k_f r}{k_f r}.
\]

The wave function \( \psi(r) \) satisfies the following condition at \( r = 0 \):

\[
\left. \left( r \psi^{(i\rightarrow f)} \right) \right|_{r=0} = K_{f\bar{i}} = \Sigma K_{f\bar{\ell}} \delta_{f\bar{\ell}} = \Sigma K_{f\bar{\ell}} \left[ \frac{3}{r} \right] \psi^{(i\rightarrow f)} \right|_{r=0}.
\] 

This is a linear condition which is valid for all channels \( i \), and therefore it is valid for any linear combination of \( \psi^{(i\rightarrow f)} \). Since the \( \psi \)'s form a complete set, we can write in general the relation Eq. (82) for any wave function, in particular for the \( \psi^{(i\rightarrow f)} \) given in Eq. (80). In this way, one gets:

\[
T_{f\bar{i}} = \Sigma \frac{K_{f\bar{\ell}}}{k_{f\bar{\ell}}} \left( \delta_{f\bar{\ell}} + i k_{f\bar{\ell}} T_{f\bar{i}} \right).
\]

or in matrix notation

\[
T = K (1 + iK T)
\]

which is equivalent to Eq. (78).
By means of Eqs. (72) and (78) we can also express the $K$ matrix in terms of the $S$ matrix:

$$K = ik^{-1/2} \ (1 - S) \ (1 + S)^{-1} \ k^{1/2}. \quad (84)$$

It is easy to show that, since the $S$ matrix is unitary and symmetric, the $K$ matrix is Hermitian ($K = K^*$) and symmetric, i.e. it is a real matrix.

Of course, the number of parameters in both matrices is the same, and given by $\frac{1}{2} n(n+1)$ for an $n$-channel system.

In the one channel case, using Eq. (22) we get:

$$K = \frac{1}{k \cot \delta}. \quad (85)$$

We consider now a two channel system, limiting ourselves to $S$ waves. Following Dalitz (see above reference), we write:

$$K = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \quad (86)$$

where $\alpha, \beta, \gamma$ are real quantities.

For an incident $S$ wave in channel 1 we have

$$\psi(1 \rightarrow 1)(r) = \sin k_1 r + T_{11} \frac{e^{ik_1 r}}{r}$$

$$\psi(1 \rightarrow 2)(r) = T_{21} \frac{e^{ik_2 r}}{r}. \quad (87)$$

The application of the condition Eq. (83) to (86) and (87) gives:

$$T_{11} = \alpha \ (1 + i k_1 T_{11}) + i \beta k_2 \ T_{21}$$

$$T_{21} = \beta \ (1 + i k_1 T_{11}) + i \gamma k_2 \ T_{21}. \quad (88)$$

from which one gets

$$T_{11} = \frac{A}{1 - i k_1 A} \quad (89)$$

with
\[ A = x + \frac{ik_2 \beta^2}{1 - ik_2 \gamma} \]  
\[ (90) \]

and

\[ T_{2,1} = \frac{\beta}{(1 - ik_2 \gamma)(1 - ik_1 A)} \]  
\[ (91) \]

From Eqs. (69) and (73), using a complex phase shift \( e^{i\alpha} = \eta e^{i\beta} \), one can write

\[ T_{1,1} = \frac{e^{i\alpha_1}}{k_1} \frac{\sin \alpha_1}{\sin \alpha} \]

from which it follows

\[ k_1 \cot \delta_1 = \frac{1}{A}. \]  
\[ (92) \]

The complex quantity \( A \) is called the \textit{scattering length} for channel 1. At low energy \( A \) can be considered independent of the energy. This approximation applied by Dalitz (see Rev. Mod. Phys. \textbf{33}, 471 (1961)) to the study of the \( K^-p \) interactions at low energy, corresponds to a zero \textit{effective range}. This nomenclature is taken from the effective range theory for one channel scattering, which is well known in Nuclear Physics (see e.g., Blatt and Weisskopf, Chapter II). In this case one expands \( k \cot \delta \) in powers of \( k^2 \):

\[ k \cot \delta = \frac{1}{a} + \frac{1}{2} x_0^2 k^2 + \ldots \]  
\[ (92') \]

The first term "\( a \)" is a constant which gives the cross-section at zero energy \( (\sigma_0 = 4\pi a^2) \); it is called \textit{scattering length}. The quantity \( n \), which depends only on the form of the potential, is called \textit{effective range}.

An expansion similar to Eq. (92) is performed for the matrix \( K^{-1} \) (which for one channel coincides with \( k \cot \delta \)) in the case of multi-channel processes. This approximation, with effective range different from zero, has been applied, for instance, to the \( K^-p \) interactions (see Ross and Shaw, Ann. Phys. \textbf{2}, 391 (1960)).

We go back now to the two channel system, whose \( K \) matrix is given in Eq. (86). For an incident \( S \) wave in channel 2, we have, in analogy with Eq. (87):
\[ \psi^{(x \rightarrow y)}(r) = T_{12} \frac{e^{ik_1 r}}{r} \]

\[ \psi^{(x \rightarrow 2)}(r) = \sin \frac{k_2 r}{k_2 r} + T_{22} \frac{e^{ik_2 r}}{r} \]

(93)

In a similar way, one gets:

\[ T_{22} = \frac{\mathcal{B}}{1 - ik_2 \mathcal{B}} ; \quad k_2 \cot \alpha_2 = \frac{1}{\mathcal{B}} \]

(94)

with

\[ \mathcal{B} = \gamma + \frac{ik_4 \beta^2}{1 - ik_1 \alpha} \]

(95)

and

\[ T_{12} = \frac{\beta}{(1 - ik_1 \alpha)(1 - ik_2 \mathcal{B})} \]

(96)

One can prove, by comparison, that the relation \( T_{12} = T_{21} \) is satisfied.

The corresponding cross-sections are evaluated by means of Eq. (74).

3. The reduced reaction matrix

We have considered in the previous section only open channels. It is instructive to include in our consideration the case of a closed channel.

Going back to the two channel example, suppose that the energy at which we consider the process is below the threshold for channel 1. We can still use the expressions Eqs. (94) and (96) for the amplitudes in channel 2, provided we now take \( k_1 \) pure imaginary: we take \( k_1 = i|k_1| \), so that the wave function \( \psi^{(x \rightarrow 0)}(r) \) behaves like \( e^{-|k_1| r} \) and is always finite. The amplitude \( T_{22} \) is again given by:

\[ T_{22}^{-1} = \mathcal{B}^{-1} - ik_2 \]

(97)

with

\[ \mathcal{B} = \gamma - \frac{|k_1| \beta^2}{1 + |k_1| \alpha} \]

(98)
These considerations can be extended to the case of \( n \) channels, by writing

\[
K = \begin{pmatrix}
\alpha & \beta \\
\widetilde{\beta} & \gamma
\end{pmatrix}
\]  

(99)

with

\[
\alpha = K_{11}, \quad \beta = (K_{12} \quad K_{13} \ldots K_{1n})
\]

\[
\tilde{\beta} = \begin{pmatrix}
K_{12} \\
K_{13} \\
\vdots \\
K_{1n}
\end{pmatrix}
\]

\[
\gamma = \begin{pmatrix}
K_{22} & K_{23} & \ldots & K_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
K_{n2} & K_{n3} & \ldots & K_{nn}
\end{pmatrix}
\]

Suppose again that the channel 1 is closed (one can extend these considerations to the case of more channels closed, replacing \( \alpha \) by an \( m \times m \) matrix, and \( \beta \) by an \( m \times n \) matrix). Then, for the \( T \) matrix which contains only the open channels, one gets the same formal relation Eq. (97):

\[
T_r^{-1} = K_r^{-1} - i k
\]

(100)

with

\[
K_r = \gamma - \tilde{\beta} \left| k \right| \frac{1}{1 + \left| k \right|^2} \beta.
\]

(101)

The matrix \( K_r \), which is called reduced reaction matrix, plays the role of a \( K \) matrix for the open channels, since the expression (100) has exactly the same form as Eq. (79). The expression (101) relates the elements of \( K_r \) to all the elements of \( K \), connecting in this way the open with the closed channels.

4. Resonances in a multi-channel system

We extend now the formula Eq. (31) obtained for the resonant one-channel scattering amplitude to the case of many channels.

For the sake of simplicity, we consider a system of \( n \) channels all open, and replace the \( K \) and \( T \) matrices by:

\[
K' = k^{1/2} K k^{1/2},
\]

\[
T' = k^{1/2} T k^{1/2}.
\]

(102)
The following considerations hold also if there are some closed channels: one has simply to replace the K and T matrices, by the reduced K and T matrices (see also: Dalitz, Strange particles, etc.).

The $K'$ matrix can be diagonalized by the eigenvalue equation

$$K'|\alpha> = (\cot \delta_\alpha)^{-1}|\alpha> .$$

(103)

The eigenvalues $(\cot \delta_\alpha)^{-1}$ are all real, since $K'$ is real and symmetric. The same set of eigenstates $|\alpha>$ diagonalizes also the $T$ matrix, as one can see from Eq. (78):

$$T'|\alpha> = (\cot \delta_\alpha - i)^{-1}|\alpha> = e^{i\delta_\alpha}\sin \delta_\alpha|\alpha> .$$

(104)

The matrix elements $T_{\alpha\beta}'$ given in Eqs. (74) and (79) correspond, however, to the representation $<f|T|i>$ for the $T$ matrix, where $|i>,|f>$ are the momentum eigenstates in the $i,f$ channels. Since the eigenstates $|\alpha>$ form a complete orthogonal set, one can write for each channel

$$|i> = \sum_\alpha C_{i\alpha}|\alpha>$$

(105)

where the coefficients $C_{i\alpha}$ are also real and satisfy the condition

$$\sum_\alpha C_{i\alpha}C_{i\alpha}^* = \delta_\alpha\delta_\beta .$$

(106)

By means of Eqs. (104), and (105) one gets

$$T_{\alpha\beta}' = <f|T'|i> = \sum_\alpha C_{f\alpha}C_{i\alpha} <\beta|T'|\alpha> =$$

$$= \sum_\alpha C_{f\alpha}C_{i\alpha} e^{i\delta_\alpha}\sin \delta_\alpha = \sum_\alpha \frac{C_{f\alpha}C_{i\alpha}}{\cot \delta_\alpha - i} .$$

(107)

Suppose now that one of the eigenvalues of $K'$ becomes infinite at the energy $E = E_R$, i.e. that $\delta_\alpha$ passes through $\pi/2$ at $E_R$. In analogy with the situation examined in the one channel case, we say that the amplitude $T_{\alpha\beta}'$ has a resonance at the energy $E_R$. If this resonance is
isolated, or in other words if the other phase shifts $\delta_\alpha$ ($\alpha \neq R$) are small around $E_R$, one can write approximately in this region

$$T'_{fi} \approx \frac{C_f R C_i R}{\cot \delta_R - i} \cdot$$

(108)

In the same region we can make a linear approximation for $\cot \delta_R$ and write, in analogy with Eq. (24):

$$\cot \delta_R \approx \frac{2}{i} (E_R - E) \cdot$$

(109)

Then, we get

$$T'_{fi} = \frac{C_f R C_i R \, i/2}{E_R - E - i \, \Gamma/2} \cdot$$

(110)

If we define:

$$\Gamma_i = C^2_{i R} \Gamma$$

since by Eq. (106):

$$\sum \Gamma_i = \Gamma$$

the resonant amplitude can also be written as:

$$T'_{fi} = \frac{i/2 \, \Gamma_i ^{1/2} \Gamma_f ^{1/2}}{E_R - E - i \, \Gamma/2} \cdot$$

(111)

The cross-section is obtained by means of Eqs. (74) and (102):

$$\sigma_{i \rightarrow f}^{(i \rightarrow f)} = \frac{\pi}{k_i^2} (2t_i + 1) \frac{\Gamma_i \Gamma_f}{(E - E_R)^2 + \frac{1}{4}} \cdot$$

(112)

This is the **Breit-Wigner** formula, valid for isolated resonances. The quantities $\Gamma_i, \Gamma_f$ are the partial widths for the $i, f$ channels: they are in general functions of the energy.

The graphical representation described for the amplitude Eq. (32) can be extended to the present multi-channel case. Defining:
the amplitude (111) can be re-written as

\[ T'_{fi} = \frac{X_{fi}}{\epsilon - 1} \]  

(115)

The real and imaginary parts of \( T'_{fi} \) are given by:

\[ x = \text{Re} \ T'_{fi} = X_{fi} \frac{\epsilon}{\epsilon^2 + 1} \]  
\[ y = \text{Im} \ T'_{fi} = X_{fi} \frac{1}{\epsilon^2 + 1} \]  

(114)

from which we get

\[ x^2 = (X_{fi} - y) \frac{y}{x} \]  

(115)

which is the equation of a circle of radius \( R = \frac{1}{2} X_{fi} \) and centre at \( x = 0, y = \frac{1}{2} X_{fi} \). The situation is described graphically in Fig. 4 where we have considered a few different values for \( X_{fi} \); clearly for \( X_{fi} = 1 \) one gets the limiting case valid for one channel. The lines \( \epsilon = \text{const} \) are also here the straight lines \( y = 1/\epsilon \ x \).

Before closing this section, we want to point out that the resonances in a multi-channel system can be originated in two different ways:

1) The complete reaction matrix \( \mathbf{R} \) can have a pole at a real value \( E_R \) of the energy, in the sense that each element of \( \mathbf{R} \) has a pole at \( E_R \). In this case the resonance is present in all the channels and it appears in all the reactions.

An example is given by the \( P_{3/2} \) resonance of the \( \pi - N \) system, which appears also in the photoproduction process \( \gamma + p \rightarrow \pi^+ + n \ (\pi^0 + \bar{p}) \) and in the
Compton scattering $\gamma + p \rightarrow \gamma + p$ (these processes can, in fact, be easily related by multi-channel formalism; (see e.g. Gell-Mann and Watson, Ann. Rev. of Nucl. Sci. 4, 213 (1954)).

2) A pole can be generated at $E'_{\mathbb{R}}$ in the reduced reaction matrix $K_{\mathbb{R}}$ by its connection with a closed channel. This pole does not appear, in general, in the complete $K$ matrix. One example is given by Eq. (94), with $B$ given by Eq. (98). $B$ goes to infinity for

$$1 + \alpha |k_1| = 0$$

(116)

$k_2 \cot \alpha_2$ passes through zero and a resonance appears in channel 2.

Suppose now that the (closed) channel 1 is weakly coupled with channel 2 ($\beta \approx 0$). The condition Eq. (82) gives in this case

$$(r^\psi(1 \rightarrow t)|_{r=0} = \alpha \left[ \frac{d}{dr}(r^\psi(1 \rightarrow t)|_{r=0} \right]$$

(117)

which coincides with Eq. (116) in the case of a bound state.

We can interpret such a resonance in channel 2, as due to the effect of a strong interaction below threshold in channel 1. If the coupling between the two channels were rigorously zero, there would be a bound state in channel 1; in fact, it is a virtual bound state, since it can go into the open channel 2.

V. REGGE POLES

1. Regge poles in potential scattering

We again consider scattering by a central potential and go back to expression Eq. (10), which we re-write here giving explicitly the dependence on the energy $E$ and on $z = \cos \theta$;

$$f(z,E) = \sum_{\ell=c}^{\infty} (2\ell + 1) f_{\ell}(E) P_{\ell}(z).$$

(118)
It is possible to transform the sum over \( \ell \) into an integral, considering \( \ell \) as a continuous complex variable (Stieltjes-Cauchy transformation):

\[
F(z, \lambda) = \frac{i}{2} \int \frac{(2\lambda + 1)}{\sin \pi \lambda} f(\lambda, E) P_\lambda(-z) \, d\lambda.
\]

(119)

The partial wave amplitude \( f_\ell(E) \) has been replaced by \( f(\lambda, E) \) and \( \lambda \) replaces the discrete variable \( \ell \). The integral is performed in the complex \( \lambda \)-plane along the path \( \gamma \) taken around the real axis (closed at infinity), as shown in Fig. 5.

In order to prove that the two expressions Eqs. (118) and (119) are indeed the same, we use the Cauchy theorem, which states that the integral of a meromorphic function along a closed path (in the clockwise sense) is equal to \((2\pi i)\) times the sum of the residues of the poles which are included in the path. For instance, in the case of only one pole at \( z = z_0 \) inside the closed region, one has for the analytic function \( f(z) \):

\[
\int \frac{f(z)}{z - z_0} \, dz = 2\pi i \, f(z_0).
\]

One sees that the expression Eq. (119) has poles at all the integral values of \( \lambda = \ell = 0, 1, \ldots \) (\( \sin \pi \ell = 0 \)), (other possible poles of \( f(\lambda, E) \) on the real axis have to be excluded by the path). Around one of these poles, the integrand can be written as

\[
\frac{i(2\lambda + 1) f(\lambda, E) P_\lambda(-z)}{(-1)^\ell 2\pi (\lambda - \ell)}
\]

and the residue is

\[
\frac{i}{2\pi} (2\ell + 1) f_\ell(E) P_\ell(z)
\]

which gives back the formula Eq. (118).
We consider the expression Eq. (119), but we take now the path $C'$ represented in Fig. 6, where the radius of the semi-circle tends to infinity.

![Fig. 6](Image)

The poles on the real axis for integer $\lambda$ are now outside the path, so that the integral along the new path $C'$ will be equal to $(2\pi i)$ times the residues of the eventual poles of $f(\lambda, E)$. Assuming that there are $n$ such poles at $\lambda_i$ ($i = 1, \ldots, n$), we can write in general:

$$
\frac{i}{2} \int_{C'} \frac{(2\lambda + 1)}{\sin \pi \lambda} f(\lambda, E) P_\lambda(-z) d\lambda = \sum_{i=1}^{n} \frac{(2\lambda_i + 1) \varphi_i(E)}{\sin \pi \lambda_i} P_{\lambda_i}(-z) \quad (120)
$$

where the $\varphi_i$ are the residues of $f(\lambda, E)$. It has been proven (see e.g. T. Regge, in the Proc. of the 1961 Herceg Novi Summer School) that for a rather general class of potentials (superposition of Yukawa potentials), the integral over the semi-circle at infinity vanishes, so that Eq. (120) can be re-written as:

$$
\frac{i}{2} \int_{C} \frac{(2\lambda + 1)}{\sin \pi \lambda} f(\lambda, E) P_\lambda(-z) dz = \int_{-\frac{1}{2}\pi i}^{\frac{1}{2}\pi i} \frac{(2\lambda + 1)}{\sin \pi \lambda} f(\lambda, E) P_\lambda(-z) d\lambda = 
\frac{(2\lambda + 1) \varphi(E)}{\sin \pi \lambda_i} P_{\lambda_i}(-z) \quad (121)
$$

and by use of Eq. (119):

$$
f(z, E) = \int_{-\frac{1}{2}\pi i}^{\frac{1}{2}\pi i} \frac{(2\lambda + 1)}{\sin \pi \lambda} f(\lambda, E) P_\lambda(-z) d\lambda = \sum_{i=1}^{n} \frac{(2\lambda_i + 1) \varphi(E)}{\sin \pi \lambda_i} P_{\lambda_i}(-z). \quad (121')
$$
It has also been proved that the roots of \( f(\lambda, E) \) lie in the upper half plane of \( \lambda \), and their number is limited. The position of such poles, which are called Regge poles, depends on the energy \( \lambda_{\ell}(E) \).

In order to clarify the meaning of such poles, we consider the contribution of a single Regge pole on the scattering amplitude, by writing simply:

\[
f(z, E) \simeq (2\lambda + 1) \frac{\varphi(E)}{\sin \pi \lambda(E)} \lambda(E) (-z).
\]

(123)

If one projects out from \( f(z, E) \) the partial wave amplitude \( f_{\ell}(E) \), one obtains

\[
f_{\ell}(E) = \frac{1}{2} \int f(z, E) P_{\ell}(z) \, dz = \frac{1}{\pi} \frac{2\lambda(E) + 1}{\lambda(E) + \ell + 1} \frac{\varphi(E)}{(\lambda(E) - 1)}.
\]

(124)

Suppose now that at \( E = E_R \): \( \text{Re} \lambda(E_R) = \ell \) and \( \text{Im} \lambda(E_R) \) is small; around \( E_R \) we can then expand \( \text{Re} \lambda(E) \) as follows:

\[
\text{Re} \lambda(E) \simeq \ell + \left( \frac{\text{d} \text{Re} \lambda}{\text{d}E} \right) \frac{E - E_R}{E = E_R}.
\]

(125)

and Eq. (124) becomes

\[
f_{\ell}(E) \simeq \frac{1}{\pi} \frac{\varphi(E_R)}{\left( \frac{\text{d} \text{Re} \lambda}{\text{d}E} \right) \left( E - E_R \right) + i \text{Im} \lambda(E_R)}.
\]

(126)

By comparison of this relation with the formula Eq. (31) for a resonant amplitude, we see that Eq. (126) can be interpreted as describing a resonance in the \( \ell \)-th partial wave at the energy \( E_R \) and with the width

\[
P = 2 \text{Im} \lambda(E_R) \left( \frac{\text{d} \text{Re} \lambda}{\text{d}E} \right)_{E = E_R}.
\]

(127)
It is interesting to note that in Eq. (31) where $l$ is a real integer, the resonance appears as a pole in the scattering amplitude at the complex energy

$$E = E_R - i \frac{\Gamma}{2}.$$ 

On the other hand, in Eq. (126) the resonance appears, for a real value of the energy, as a pole at the complex value of $\lambda$:

$$\lambda(E_R) = l + i \text{Im} \lambda(E_R).$$

In general, the quantity $\lambda(E)$ moves with the energy on the complex $\lambda$-plane, describing a trajectory (Regge trajectory). We can say that a Regge pole represents a resonance whenever $\lambda(E)$ goes close to real integral values $l$. One sees that a Regge pole can then connect different resonances occurring at different values of $l$. When the energy becomes negative, one can show that $\text{Im} \lambda = 0$, and to an integral value of $\lambda$ there corresponds a bound state.

2. **Classification of the elementary particles on Regge trajectories**

The concepts related to the Regge poles, derived in potential theory, have been extended to the strong interactions of the elementary particles by Chew and Frautschi (Phys.Rev.Lett. 8, 41 (1962)). Their conjecture is that all baryons and mesons (stable and unstable) are associated with Regge poles (poles of an $S$ matrix which describes the strong interactions) which move on the complex angular momentum plane as functions of the energy ($\lambda$ represents now the complex values of the spin $J$ of the particle). The trajectory of a particular pole is characterized by a set of quantum numbers (isotopic spin, hypercharge, etc.) and by the evenness or oddness of the physical values of $J$ for mesons and $J - \frac{1}{2}$ for baryons. Below the threshold for the lowest channel with the given set of quantum numbers, one has: $\text{Im} \lambda = 0$. The stable and unstable particles occur when $\text{Re} (E_R) = J$, where $E_R$ is the rest energy of the particle. The quantity $\text{Im} \lambda \neq 0$ gives the width for an unstable particle, according to Eq. (117).
Using these conjectures, all the strongly interacting particles can be classified on Regge trajectories. We report in Figs. 7 and 8 the Chew-Frautschi plot (Re $\lambda$ is drawn versus the squared mass) as given by Rosenfeld (Proc. of the 1962 Int. Conf. on High-Energy Physics at CERN).

All the trajectories corresponding to the known particles lie below a trajectory which passes through $J = 0$, $\mu_1^2 = 0$: this corresponds to the limiting trajectory consistent with the unitarity and analyticity of the S matrix. The physical state on this trajectory at $J = 2$ has been attributed to the $f^0$ meson ($\pi\pi$ resonance at $\sim 1250$ MeV).
Fig. 8
3. High energy behaviour of the cross-sections

We will briefly consider here the implications of the Regge conjecture to the high-energy behaviour of the cross-sections.

We use here an invariant notation for the scattering amplitudes and the related variables (see also: G. Chew, in "Dispersion Relations", 1960 Scottish University Summer School). It is clear that two independent kinematical variables are sufficient for the scattering process

\[ A + B \rightarrow A' + B' \quad (I) \]

which we represent graphically in Fig. 9. The two variables can be taken, for example, as the momentum \( k \) and the scattering angle \( \theta \) in the c.m. system. The following set of variables is useful (we use the metric \( k = (k_0, ik) \)):

\[
\begin{align*}
    s &= (k_1 + k_2)^2 = (k_1^- + k_2^-)^2 \\
    t &= (k_1 - k_1^-)^2 = (k_2^- - k_1^-)^2 \\
    u &= (k_1^- - k_2^-)^2 = (k_2^- - k_1^-)^2
\end{align*}
\]

(128)

In the case of identical particles (with mass \( \mu \)) they can be written as:

\[
\begin{align*}
    s &= 4(k^2 + \mu^2) \\
    t &= -2k^2 \left( 1 - \cos \theta \right) \\
    u &= -2k^2 \left( 1 + \cos \theta \right)
\end{align*}
\]

(129)

Of course only two of these variables are independent; in fact, the following relation holds

\[ s + t + u = 4\mu^2 \quad . \]

(130)
For the reaction (I), $s$ represents the squared energy in the c.m. system ($s > 0$), and $t$ the squared momentum transfer ($t < 0$). The invariant scattering amplitude is defined by a function $A(s, t)$, which is related to the usual scattering amplitude $f(k, \theta)$ by

$$A(s, t) = \frac{\sqrt{s}}{2} f(k, \theta) .$$  

(131)

The process

$$A + A' \rightarrow E + B'$$  

(II)

where $A', B$ stand for the antiparticles of $A', B$, is represented by the same graph of Fig. 9, if one considers $k_1, \bar{k}_1 = -k_1$ as ingoing, and $k_2, \bar{k}_2 = -k_2$ as outgoing four-momenta. In this case one has:

$$t = (p_1 + \bar{p}_1)^2 = 4(q^2 + \mu^2)$$
$$s = (p_1 - \bar{p}_2)^2 = -2q^2(1 - \cos \bar{\theta}) .$$  

(132)

The variable $t$ has now the role of the squared energy ($t > 0$), and $s$ of the squared momentum transfer ($s < 0$) in the c.m. system for the reaction (II)

$q$ and $\bar{\theta}$ are the corresponding momentum and scattering angle.

The invariant scattering amplitude for the process (II) is again given by the function $A(s, t)$ in which the roles of the variables $s, t$ have been interchanged with respect to the reaction (I). We can say that the same amplitude $A(s, t)$ represents the two reactions (I) and (II) when the variables $s, t$ are defined in the two different physical regions Eqs. (129) and (132).

We assume now that the scattering amplitude has a Regge pole behaviour in the physical region for the reaction (II). We write, in analogy with Eq. (123):

$$A(s, t) = \beta(t) \sin \pi \lambda(t) P_{\lambda}(t) (-\cos \bar{\theta})$$  

(133)

with

$$\cos \bar{\theta} = 1 - \frac{2s}{4q^2 - t} .$$  

(134)
and including in $\beta(t)$ terms which it is not necessary to specify here (for more details and other references see: Drell, Proc. of the 1962 Int.Conf. on High-Energy Physics at CERN, page 397).

The same amplitude describes the reaction (I) for $s > 0$, $t < 0$. For large values of $s$ (high-energy for the reaction (I)), we can use the asymptotic expansion for the Legendre polynomials ($P_n(z) \sim z^n$ for large $z$) and write:

$$P_{\lambda(t)}(-\cos \theta) \sim P_{\lambda(t)} \left( \frac{2s}{4\mu^2 - s} \right) \simeq g(t) s^{\lambda(t)}.$$  \hspace{1cm} (135)

The scattering amplitude becomes

$$A(s, t) = \frac{\beta(t) \overline{e}(t)}{\sin \pi \lambda(t)} s^{\lambda(t)} \simeq a(t) s^{\lambda(t)}$$ \hspace{1cm} (136)

and the differential cross-section for the reaction (I) can be written as:

$$\frac{d\sigma}{d\Omega} \simeq 4 |a(t)|^2 s^{2\lambda(t) - 1}.$$ \hspace{1cm} (137)

By use of

$$\cos \theta = 1 - \frac{2t}{4\mu^2 - s} \simeq 1 + \frac{2t}{s},$$ \hspace{1cm} (138)

one can also write

$$\frac{d\sigma}{dt} \simeq 16\pi |a(t)|^2 s^{2(\lambda(t) - 1)}.$$ \hspace{1cm} (139)

The optical theorem allows us to evaluate the total cross-section ($t = 0$ in the forward direction):

$$\sigma_{\text{tot}} \simeq \frac{16\pi}{s} \Im A(s, c) \simeq 16\pi \Im a(c) s^{\lambda(c) - 1}.$$ \hspace{1cm} (140)

The experimental indication that the total cross-sections become constant at very high-energy implies:

$$\lambda(c) = 1.$$ \hspace{1cm} (14.1)
This is equivalent to assuming the existence of a Regge pole which dominates all the amplitudes at very high energies; this pole corresponds to the higher trajectory (Pomeranchuk trajectory) considered in the previous section. Near $t = 0$ we can expand $\lambda(t)$ and, assuming that Eq. (141) holds, we get

$$\lambda(t) \approx 1 + \lambda'(0)t.$$  \hspace{1cm} (142)

The differential cross-section Eq. (139) for small $t$ ($t \ll 0$) can be written:

$$\frac{d\sigma}{dt} \approx 16\pi |a(t)|^2 e^{2\lambda'(0)t} \log s.$$  \hspace{1cm} (143)

Theoretical conjectures lead to $\lambda'(0) > 0$. Then, since $t$ is negative, one sees that the differential cross-section presents a forward peak which decreases exponentially with $t$ and shrinks with increasing energy.

It is interesting to compare this behaviour with the diffraction peak obtained in the case of scattering by a black sphere. From Eq. (62) we get:

$$\frac{d\sigma}{d\Omega} \approx \left| \frac{R J_1(kR \sin \theta)}{\sin \delta} \right|^2.$$  \hspace{1cm} (144)

For small angles one gets from Eq. (138):

$$2k \sin \theta/2 \approx k\theta \approx \sqrt{-t}$$

and Eq. (144) can be re-written for large $s$, in the form

$$\frac{d\sigma}{dt} \approx \pi \left| \frac{J_1(R/\sqrt{t})}{\sqrt{-t}} \right|^2.$$  \hspace{1cm} (145)

For reasonable values of $R$ ($R \approx 1$ fermi) and small values of $t$, this formula can be approximated by

$$\frac{d\sigma}{dt} \approx \pi R^2 \exp \left[ -\left( \frac{R}{2} \right)^2 t \right].$$  \hspace{1cm} (146)
We see that in this case the forward peak decreases exponentially, but it is independent of the energy.

The behaviour given by Eq. (143) is a particular feature of the Regge pole hypothesis which, however, does not seem to be exhibited by the experimental results, at least at the available energies ($E \approx 20$ GeV). Maybe the asymptotic behaviour given by the leading Regge pole will appear in the experiments only at higher energies.
II. INTRODUCTION TO UNITARY SYMMETRIES

I. INTRODUCTION

Elementary particles are characterized by a set of quantum numbers like mass, electric charge, spin, intrinsic parity, hypercharge, etc. Different classifications are clearly possible, by collecting particles in subsets with similar properties. Only experiments can decide if a classification is good, in the sense that the underlying scheme has a physical content and its predictions are not contradicted.

We shall consider here some classification according to the so-called unitary symmetries. These are related to the unitary transformations which leave the free Lagrangian of the particles invariant. We remember that the invariance of the Lagrangian under a given transformation corresponds to a conservation law (see e.g.: Roman, Theory of Elementary Particles, North Holland, 1960).

Let us consider, for instance, the baryons. The concept of isospin is based on the idea that there are sets of particles (e.g. proton and neutron) which have identical properties (spin, parity, hypercharge), similar mass, and only different electric charge. If one is concerned with only strong interactions and disregards electromagnetic effects, one can consider on the same footing (extending the equality to the masses) the particles of the same multiplet. One can go further and suppose that all the stable baryons (which decay only via weak or electromagnetic interactions) can be considered as the eight components of a super-multiplet. All these components have the same properties; including the mass, if one differentiates the strong interactions in very strong and medium strong interactions and disregards the latter as giving the mass differences among the isospin multiplets.

In general, if one describes by $\psi_i$ ($i = 1, 2, \ldots, 8$) the i-th component of this baryon field, the free Lagrangian can be written as:
\( Z_0 = \sum_{i,m} \bar{\psi}_i \left( \gamma_\mu \sigma_\nu \right)_i^m \psi_i \). \hspace{1cm} (1)

Suppose that \( Z_0 \) is invariant under a set of linear transformations acting on \( \psi_i \):

\[
\begin{align*}
\psi_i &\rightarrow \psi_i' = \sum_j U_{ij} \psi_j \quad (U\psi)_i \\
\bar{\psi}_i &\rightarrow \bar{\psi}_i' = \psi_i^{\dagger} \gamma_\mu = \left( U\psi_i \right)^{\dagger} \gamma_\mu = \sum_j \bar{\psi}_j U_{j,i} \psi_i.
\end{align*}
\hspace{1cm} (2)
\]

Clearly, the invariance requires the condition

\[
U^{\ast \dagger}_{ik} U_{ij} = U^{\dagger}_{kl} U_{lj} = \delta_{ij} \hspace{1cm} (3)
\]

or in matrix notation

\[
U^{\ast \dagger} = U U^{\ast} = 1. \hspace{1cm} (4)
\]

The matrix \( U \) is then unitary (the symbol \( \ast \) stands for complex conjugate, and \( \dagger \) for Hermitian conjugate).

We shall consider in the following two particular sets of unitary transformations: the group \( SU_3 \) which is related to the isotopic spin, and the group \( SU \), which is related to the so-called "eightfold way".

A brief survey of group theoretical concepts is given in the Appendix, to which we refer for the nomenclature and some results used in the following Chapters.

II. GROUP \( SU_3 \)

We consider here in some detail, the group \( SU_3 \), as a simple example of unitary group, since it is connected with familiar physical ideas.

We start from the unitary transformations acting on a two-component complex vector (spinor) \( \varphi \) and on its complex conjugate \( \varphi^\ast \):

\[
\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \varphi^\ast = \begin{pmatrix} \varphi_1^\ast \\ \varphi_2^\ast \end{pmatrix}. \hspace{1cm} (5)
\]
The transformations

$$\xi' \to U \xi$$

are represented by $2 \times 2$ matrices

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \text{ with } \begin{cases} \alpha \alpha^* + \beta \beta^* = 1 \\ D = \det U, |D| = 1 \end{cases}$$

A unitary transformation can be written, in general, as in Eq. (A1), in terms of a set of linearly independent Hermitian matrices. There are four $2 \times 2$ Hermitian matrices which are linearly independent; for instance, the Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

plus the unit matrix $1$.

Any $2 \times 2$ Hermitian matrix can be written as a linear combination of these four matrices, so that a unitary transformation is given, in general, by

$$U = e^{i \xi_1 \tau_1 + i \xi_2 \tau_2 + i \xi_3 \tau_3}$$

where we have written the $\tau_i$ formally as components of a three-vector

$$\vec{\tau} = (\tau_1, \tau_2, \tau_3)$$

and $\vec{\xi}$ is a three-vector with real components $\xi_1, \xi_2, \xi_3$.

The factor $e^{i \xi_0}$ gives a phase transformation (also $\xi_0$ is real); the invariance of the Lagrangian of the elementary particles under this transformation is connected with the conservation of the baryon number.

We consider here only unimodular matrices (with determinant equal to unity), so that Eq. (9) reduces to

$$U = e^{i \vec{\xi} \cdot \vec{\tau}}.$$ 

A corresponding infinitesimal transformation is then given by

$$U \simeq 1 + i \vec{\xi} \cdot \vec{\tau}.$$
We see that these transformations are completely defined by three real parameters. This can also be seen directly from the matrix Eq. (6), adding the condition $D = 1$.

These transformations define the group $SU_2$; in fact, they correspond to its irreducible representation of dimension 2. The order of the group is 3, since there are three independent generators $(\tau_1, \tau_2, \tau_3)$. The commutation relations of the generators are:

$$[\tau_i, \tau_j] = i\delta_{ij}\tau_k$$  \hspace{1cm} (13)

where $(i,j,k)$ is a cyclic permutation of $(1,2,3)$. Since there is no commuting pair of generators, the rank of the group is one.

The basis vector of the two-dimensional representation can be taken as:

$$\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  \hspace{1cm} (14)

Since $\tau_3$ is in diagonal form, $\xi_+$ and $\xi_-$ represent the eigenstates of $\tau_3$ with eigenvalues +1 and -1, respectively.

One can see easily how $\tau_1, \tau_2$ operate on $\xi_+, \xi_-$. by taking the linear combinations

$$\tau_\pm = \frac{1}{2} (\tau_1 \pm \tau_2)$$  \hspace{1cm} (15)

that is:

$$\tau_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tau_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (16)

One gets

$$\tau_+ \xi_+ = 0, \quad \tau_- \xi_- = \xi_+.$$  \hspace{1cm} (17)

If we define the quantity

$$\hat{I} = \frac{1}{2} \tilde{\tau}$$  \hspace{1cm} (18)

we see that we can write the equation

$$[I_3, I_\pm] = \mp I_\pm.$$  \hspace{1cm} (19)
which corresponds to Eq. (A18) where \( H_i, E_{+}, H_{-} = E_{+}^{+} \) (in this case \( i = 1, \alpha = \pm 1 \)) are replaced by \( I_3, I_1, \) respectively. Clearly, there are only two (one-component) roots: \( \hat{P}(1) \) and \( \hat{P}(-1) = -\hat{P}(1) \). The one-component weights are the eigenvalues of \( I_3 \) [see Eq. (A21)].

One realizes that the operator \( I_3 \), which is called isotropic spin, has the formal properties of the spin \( S \) of a particle. One can diagonalize simultaneously \( I^2 \) and \( I_3 \) (which are analogous to \( S^2 \) and \( S_3 \)), so that \( \xi_+, \xi_- \) can be considered the corresponding eigenstates.

So far we have considered the properties of the group \( SU_2 \) in terms of the two-dimensional irreducible representation. It is easy to show that this is the fundamental representation, which corresponds to the fundamental weight \( 1/2 \). Starting from this representation, one can build all the other irreducible representations of the group.

Each irreducible representation is characterized by the dominant weight \( I = \lambda \lambda (\lambda = 0, 1, 2, \ldots) \), which is the maximum eigenvalue of \( I_3 \); it is denoted by \( D(\lambda \lambda /2) \) and its dimension is given by \( (2I+1) \). The basis vectors are characterized by both \( I \) and \( I_3 \); we denote them in general by \( \xi(I, I_3) \).

The two-dimensional representation is then denoted by \( D(1/2) \); for the two basis vectors we have used the notation: \( \xi_+ = \xi(1/2, 1/2), \xi_- = \xi(1/2, -1/2) \).

The representations \( D(0), D(1) \) can be obtained by direct product of two representations \( D(1/2) \):

\[
D(1/2) \times D(1/2) = D(0) + D(1)
\]

The dimensions of \( D(0) \) and \( D(1) \) are 1 and 3, respectively. The basis for the \( D(1/2) \times D(1/2) \) representation can be written as: \( \xi_+^{(1)}, \xi_-^{(1)}, \xi_+^{(2)}, \xi_-^{(2)} \), where \( \xi_+^{(1)}, \xi_-^{(1)} \) are the basis for the two representations \( D(1/2) \) which appear in Eq. (20). Each term of the type \( \xi_+^{(1)} \xi_-^{(2)} \) represents a four-component object and it can be considered as a vector in the four-dimensional direct product space. The basis for \( D(0) \) is obtained by the antisymmetric combination

\[
\xi(0,0) = \sqrt{\frac{1}{2}} \left[ \xi_+^{(1)} \xi_-^{(2)} - \xi_-^{(1)} \xi_+^{(2)} \right]
\]
and the basis for \( D(1) \) by the symmetric combinations:

\[
\xi(1,1) = \xi_{+}^{(1)} \xi_{-}^{(1)}
\]

\[
\xi(1,0) = \sqrt{\frac{1}{2}} \left[ \xi_{+}^{(1)} \xi_{+}^{(0)} - \xi_{-}^{(1)} \xi_{+}^{(0)} \right]
\]

\[
\xi(1,-1) = \sqrt{\frac{1}{2}} \left[ \xi_{+}^{(1)} \xi_{-}^{(0)} \right]
\]

By applying a general transformation to \( \xi_{+}^{(1)}, \xi_{-}^{(1)} \) (for instance using Eq. (12)) one realizes that \( \xi(0,0) \) transforms into itself, and \( \xi(1,1), \xi(1,0), \xi(1,-1) \) transform into linear combinations of themselves. In other words \( \xi(0,0) \) can be considered as a scalar, and \( \xi(1,i) \) \( (i = 1, 0, -1) \) as a three-vector in a three-dimensional space (isospin space).

From Eq. (11) it is clear that the product of two transformations corresponds to the sum of the generators. Then, if \( I(1), I(2) \) are the isospins corresponding to two representations \( D(1/2) \), the total isospin corresponding to the product representation is given by:

\[
\mathbf{I} = \mathbf{I}(1) + \mathbf{I}(2).
\]

(25)

It is easy to check that the eigenvalues of \( I^2, I_3 \) for the combinations Eq. (21) and (23) are indeed \( (0, 0) \) and \( (1, 1) \) or \( 0 \) respectively. One has to apply Eq. (23) to Eq. (21) and (23) taking into account that \( I(1) \) operates only on \( \xi_{+}^{(1)} \) and \( I(2) \) only on \( \xi_{-}^{(1)} \).

The above results are well known from the elementary quantum formalization of angular momenta: they are expressed here using simple group theoretical language. These results can be generalized to the direct product of irreducible representations of arbitrary dimensions:

\[
D[I(1)] \times D[I(2)] = D[I(1) + I(2)] + \ldots + D[I(1) + I(2) - 1] + \ldots
\]

(24)

The basis for the representation \( D[I(1)] \times D[I(2)] \) can be obtained by the products: \( \xi[I(1), I_3(1)], \xi[I(2), I_3(2)] \). These basis vectors are, however, eigenstates of \( I^2(1), I_3(1), I^2(2), I_3(2) \). It is more useful to obtain a
set of eigenstates which are eigenstates of the operators: \( I^2 = [\hat{I}(1) + \hat{I}(2)]^2 \), 
\( I_3 = I_3(1) + I_3(2) \). They are obtained, in general, by the relation

\[
\xi[I_1, I_3; I(1), I(2)] = \sum_{I_3(1), I_3(2)} \xi[I(1), I(2); I_1, I_3(1), I_3(2)] \xi[I(1), I_3(1)] \cdot \\
\xi[I(2), I_3(2)]
\]

(25)

where the \( C[I(1), I(2)] \)'s, called Clebsch-Gordan coefficients, are the elements of a unitary (in particular, orthogonal) matrix.

The set of eigenstates given by Eq. (25), for a given value of \( I^2 (I_3 = I, I - 1, \ldots, -I) \) form the basis of one of the irreducible representations in the r.h.s. of Eq. (24). The relation Eq. (25) can be inverted in the following way:

\[
\xi[I(1), I_3(1)], \xi[I(2), I_3(2)] = \sum_{I=|I(1) - I(2)|} C[I(1), I(2); I, I_3(1), I_3(2)] \cdot \\
\xi[I, I_3; I(1), I(2)].
\]

(26)

The most commonly used Clebsch-Gordan coefficients are given in Table I.

III. APPLICATION OF THE ISOTOPIC SPIN TO ELEMENTARY PARTICLES

The classification of the elementary particles into isospin multiplets is now easily carried out by assigning each multiplet to an irreducible representation of \( SU_2 \).

We can identify the proton \( \pi \) and the neutron \( n \) with the two eigenvectors \( \xi_+, \xi_- \) which form the basis of the I.R. \( D(1/2) \), by defining the electric charge by

\[
Q = I_3 + \frac{1}{2} Y
\]

(27)

where \( Y \) is the hypercharge.

A linear combination of \( \xi_+, \xi_- \) represents the general state of a nucleon, which can be denoted by
\[ \xi \rightarrow \left( \begin{array}{c} \xi \\ \bar{\eta} \end{array} \right) \]  
\[ \text{and which transforms as the vector } \xi \text{ according to Eq. (5)}. \]

The state of an antinucleon can be identified with \( \xi^* \):
\[ \xi^* \rightarrow \left( \begin{array}{c} \xi^* \\ \bar{\eta} \end{array} \right). \]  
\[ \text{Its transformation properties are given by} \]
\[ \xi'^* \rightarrow U^* \xi^*. \]  
\[ \text{A comparison of Eq. (30) with Eq. (6) shows that} \]
\[ \left( \begin{array}{c} \bar{\eta} \\ \bar{\nu} \end{array} \right) \]  
\[ \text{transforms in the same way as Eq. (28). In the case of antinucleons, the} \]
\[ \text{formula Eq. (27) is to be applied to Eq. (31).} \]

In order to clarify the situation for the case of direct product representations, we give a few specific examples.

Consider first the system of two nucleons: there are four possible states: \( pp, pn, np, nn \). We can classify these states by taking symmetric and antisymmetric combinations of them. The antisymmetric combination
\[ \sqrt{2}(pn - np) \]  
\[ \text{corresponds to the eigenstate Eq. (21) of isospin } I = 0 \text{ (isospin singlet).} \]

The symmetric combinations
\[ \begin{cases} \sqrt{2}(pn + np) \\ \text{nn} \end{cases} \]  
\[ \text{correspond to the eigenstates Eq. (22) of isospin } I = 1 \text{ (isospin triplet).} \]

The charges of the different eigenstates are given by \( K_0 \), (27) with \( Y = 2 \), since the system is composed of two baryons with hypercharge \( Y = 1 \).
Similarly, for a system of a nucleon and an antinucleon, taking
into account Eq. (31), one can write down the eigenstates corresponding to
the singlet \( I = 0 \):

\[
\text{\sqrt{2}} (p^+ n^- + p^- n^+) \tag{34}
\]

and to the triplet \( I = 1 \) \((I_3 \neq 0,0)\):

\[
\begin{aligned}
\text{\sqrt{2}} (p^+ n^- - p^- n^+) \\
\text{\sqrt{2}} (p^0 n^+ - p^+ n^0)
\end{aligned}
\tag{35}
\]

The electric charge is obtained by means of Eq. (27) with \( Y = 0 \).

These states have the same quantum numbers as the \( \eta \) meson and of the
pion; one can then classify the \( \eta \) as an isospin singlet, and the pion as an
isospin triplet: \((\pi^+, \pi^0, \pi^-)\).

By combination of a pion with a nucleon, one obtains six different
states. According to the rule given in Eq. (25), we can classify these
states in a doublet \( I = \frac{1}{2} \) \((I_3 = \pm \frac{1}{2})\):

\[
\begin{aligned}
\text{\sqrt{2}} \, p^0 n^+ - \sqrt{3} \, n^0 p^+ \\
\text{\sqrt{2}} \, p^- n^- - \sqrt{3} \, n^+ p^-
\end{aligned}
\tag{36}
\]

and a quartet \( I = \frac{3}{2} \) \((I_3 = \pm \frac{3}{2}, \pm \frac{1}{2})\):

\[
\begin{aligned}
\text{\sqrt{2}} \, p^0 n^+ + \sqrt{3} \, n^0 p^+ \\
\text{\sqrt{2}} \, p^- n^- + \sqrt{3} \, n^+ p^-
\end{aligned}
\tag{37}
\]

They correspond to the representations \( D(\frac{1}{2}) \), \( D(\frac{3}{2}) \) which are given, according to Eq. (24), by

\[
D(1) \times D(\frac{1}{2}) = D(\frac{1}{2}) + D(\frac{3}{2}) \tag{38}
\]
Following the same procedure, one can classify all the strongly interacting particles into isospin multiplets. Each multiplet is characterized by the value of the isotopic spin \( I \); each component in a multiplet by the third component \( I_3 \), which is connected to the electric charge by Eq. (27).

At first sight all this appears like a formal game. Physics enter when one assumes that the strong interactions in a system depend on the total isospin \( I \) and not on the third component \( I_3 \); in other words, that the strong interactions are charge independent. This invariance property is equivalent to the conservation of the total isotopic spin; the third component is also conserved, since it is connected with the electric charge, through Eq. (27).

Charge independence means, for instance, that in a system of two nucleons, instead of four possible interactions corresponding to the four charge states, there are only two: the isospin singlet interaction and the isospin triplet interaction. Similarly, in the system of a nucleon and a pion, the strong interactions distinguish only between the \( I = \frac{1}{2} \) and the \( I = \frac{3}{2} \) states.

As a physical consequence, the reaction between two particles of isospin \( I(1) \) and \( I(2) \) is completely described in terms of a number of amplitudes equal to the number of irreducible representations into which the direct product \( D[I(1)] \times D[I(2)] \) can be reduced. This number is, in general, much smaller than the number of charge states.

For instance, in the case of the \( \pi - N \) system, one needs only two amplitudes \( A_1, A_2 \) corresponding to the transitions between states of isospin \( I = \frac{1}{2}, I = \frac{3}{2} \) respectively. Each given state of charge (e.g., \( \pi^+, \pi^- \), etc.) can be expressed in terms of the eigenstates of the total isospin, by inverting Eqs. (36) and (37). One gets, for instance:

\[ p\pi^+ \rightarrow \frac{\sqrt{6}}{\sqrt{3}} \xi_1^{\frac{1}{2}} \]

\[ p\pi^- \rightarrow \sqrt{3} \xi_2^{\frac{1}{2}} - \sqrt{3} \xi_3^{\frac{1}{2}} \xi_2^{\frac{1}{2}} \]

\[ n\pi^0 \rightarrow \sqrt{3} \xi_2^{\frac{1}{2}} + \sqrt{3} \xi_3^{\frac{1}{2}} \xi_2^{\frac{1}{2}} \]

(39)
We will now write down the amplitudes and cross-sections for some reactions of experimental interest:

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Amplitude</th>
<th>Cross-section</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^+ p \rightarrow \pi^+ p$</td>
<td>$A_3$</td>
<td>$</td>
</tr>
<tr>
<td>$\pi^- p \rightarrow \pi^- p$</td>
<td>$\frac{1}{5}A_3 + \frac{3}{5}A_1$</td>
<td>$\frac{1}{5}</td>
</tr>
<tr>
<td>$\pi^- p \rightarrow \pi^0 n$</td>
<td>$\frac{\sqrt{5}}{5}A_3 - \frac{\sqrt{2}}{5}A_1$</td>
<td>$\frac{2}{5}</td>
</tr>
</tbody>
</table>

These are simply obtained by expressing the initial and final states in terms of the isospin eigenstates, taking into account that they are orthogonal.

IV. GROUP SU$_3$

The group SU$_3$ can be defined as the group of the unitary unimodular transformations acting on the complex vectors in a linear three-dimensional space. The considerations of the previous Chapter can be easily extended to the present case, by considering here the representation which consists of $3 \times 3$ (unitary unimodular) matrices. It is clear that the number of real independent parameters is 8, so that the order of the group is 8. An infinitesimal transformation can be written as:

$$U = 1 + i\varepsilon \sum_{a} L_a \quad (a=1,2, \ldots, 8)$$

(40)

in terms of 8 real parameters $\varepsilon_a$, and the 8 generators $L_a$, which can be considered either as Hermitian operators, or, in a given representation, as traceless Hermitian matrices.

It can be proved that there are only two commuting generators, which we denote by $H_\sigma, H_\tau$. The other 6 generators will be denoted by $E_{\alpha}(\alpha = \pm 1, \pm 2, \pm 3)$.

It is now important to determine the irreducible representations of the group. We must remember that the representations of the generators are sufficient to give the representations of the group.
We consider first the roots of the group. According to the definition Eq. (A16), there are six roots, which are two-component vectors. Starting with two roots \( \vec{r}(\alpha), \vec{r}(\beta) \) one can build the others, according to Eq. (A21), which can be expressed in the form:

\[
\vec{r}(\gamma) = \vec{r}(\beta) - 2 \frac{\vec{r}(\beta)}{|\vec{r}(\alpha)|} \vec{r}(\alpha) \cos \varphi .
\] (4.1)

As one can see from Fig. 1, the root \( \vec{r}(\gamma) \) can be obtained geometrically by reflection of \( \vec{r}(\beta) \) through a plane perpendicular to \( \vec{r}(\alpha) \).

Since Eq. (4.1) holds for any pair of roots, it is easy to see that the angle between two adjacent roots is 60°.

Using the normalization condition Eq. (A19), one can write the roots as:

\[
\begin{align*}
\vec{r}(1) &= \sqrt{\frac{1}{3}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\vec{r}(2) &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
\vec{r}(3) &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} 
\end{align*}
\] (4.2)

plus the other three obtained by

\[
\vec{r}(-\alpha) = -\vec{r}(\alpha) \quad (\alpha=1,2,3) .
\] (4.3)

These roots can be drawn in a plane, giving the root diagram, as shown in Fig. 2.
Let us now evaluate the weights corresponding to a general representation. A weight for $SU_3$ is a two-component vector

$$\vec{m} = \left(\begin{array}{c} m_1 \\ m_2 \end{array}\right).$$

(44)

Taking into account that the quantity

$$2 \frac{\lambda_{\alpha'} \lambda^*_\alpha}{|\lambda_{\alpha'}|^2}$$

(45)

is an integer (see Appendix), one gets, taking $\alpha = 2, 3$, the two conditions:

$$\sqrt{3}(m_1 + \sqrt{3}m_2) = \lambda_1$$

$$\sqrt{3}(m_1 - \sqrt{3}m_2) = \lambda_2$$

(46)

where $\lambda_1, \lambda_2$ are integer numbers. The general weight Eq. (44) can then be written as

$$\vec{m} = \frac{1}{6} \lambda_1 \left(\begin{array}{c} \sqrt{3} \\ 1 \end{array}\right) + \frac{1}{6} \lambda_2 \left(\begin{array}{c} \sqrt{3} \\ -1 \end{array}\right)$$

(47)

from which it is clear that the two vectors

$$\vec{m}(1,0) = \frac{1}{6} \left(\begin{array}{c} \sqrt{3} \\ 1 \end{array}\right); \quad \vec{m}(0,1) = \frac{1}{6} \left(\begin{array}{c} \sqrt{3} \\ -1 \end{array}\right)$$

(48)

are the two fundamental weights of the group. The vector, given by Eq. (47), and which we shall denote by $\vec{m}(\lambda_1, \lambda_2)$, is the dominant weight which characterizes the irreducible representation $D(\lambda_1, \lambda_2)$. The dimension of the I.R. $D(\lambda_1, \lambda_2)$ is given by

$$N = \left[1 + \frac{1}{6} (\lambda_1 + \lambda_2)\right]\left(1 + \lambda_1\right)\left(1 + \lambda_2\right).$$

(49)

The weight $\vec{m}(1,0)$ is orthogonal to the root $\vec{r}(3)$; we can obtain from it, another two weights by reflection through planes perpendicular to the other roots $\vec{r}(1), \vec{r}(2)$, [refer to formula Eq. (A26) which is similar to Eq. (41)].
These weights form a class of vectors, which are drawn in Fig. 3 (weight diagram); they are the weight of the representation $D(1,0)$ of dimension 3. In the same way one can obtain, starting from the fundamental weight $\pi(0,1)$, the weight diagram for the representation $D(0,1)$, which is shown in Fig. 4. The two representations $D(1,0)$ and $D(0,1)$ have the same dimension 3, but they are inequivalent; they are the fundamental representations of the group.

Starting from these representations, all the other I. R. of the group can be obtained by reduction of direct products. One can prove that the direct product of the two fundamental representations gives:

$$D(1,0) \times D(0,1) = D(0,0) + D(1,1)$$  \hspace{1cm} (50)

i.e. one irreducible representation of dimension 1, and one of dimension 8. Similarly, the direct product of two 8-dimensional representations gives

$$D(1,1) \times D(1,1) = D(0,0) + D(1,1) + D(1,1) + D(5,0) +$$
$$+ D(6,3) + D(2,2)$$ \hspace{1cm} (51)

i.e. one I. R. of dimension 1, two of dimension 8, two (inequivalent) of dimension 10 and one of dimension 27.

In the following we shall denote, for the sake of simplicity, an irreducible representation by $[N]$, where $N$ is given by Eq. (49); we shall distinguish inequivalent representations with the same $N$ with an asterisk, for example, $[3], [3^*]$. In this notation the relation Eq. (51) can be rewritten as:

$$[8] \times [8] = [1] + [8] + [8] + [10] + [10^*] + [27].$$ \hspace{1cm} (51')

The weight diagrams for these I. R. can be obtained, starting from the dominant weight given by Eq. (47), and assigning the relevant values to the parameters $\lambda_1, \lambda_2$. We give in Fig. 5 the weight diagram for the representation $[8]$; we remember that the number of weights is equal to the dimension of the representation: we then have 8 weights, six of which
are simple and are those represented in Fig. 5; moreover there is a weight of multiplicity two, and is zero (represented by a dot and a small circle in the figure). We also give the weight diagram for the representation $[10]$ in Fig. 6.

Knowing the weight diagram, one can easily find the representations of the generators. Consider, for example, the representation $D(1,0)$. The components $m_x, m_y$ of the three weights are the eigenvalues of $H_x, H_y$. These two generators can then be represented by diagonal matrices

$$H_x = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_y = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (52)$$

The corresponding eigenvectors can be taken as the basis of the representation. In the present case, there is an eigenvector for each weight $\vec{m}$, so that we can denote it by $\xi(\vec{m})$ and write

$$H_i \xi(\vec{m}) = m_i \xi(\vec{m}) \quad . \quad (53)$$

From Eq. (53) and Eq. (A18) one immediately gets:

$$H_i E_{-\alpha} \xi(\vec{m}) = [m_i - r_i(\alpha)] E_{-\alpha} \xi(\vec{m}) \quad . \quad (54)$$

This means that $E_{-\alpha} \xi(\vec{m})$ is also an eigenvector of $H_i$, belonging to the eigenvalue $m_i - r_i(\alpha)$: since this is different from $m_i$, the eigenvector $E_{-\alpha} \xi(\vec{m})$ is also orthogonal to $\xi(\vec{m})$. In this way, by normalizing the eigenvectors, one gets a set of orthonormal vectors which form the basis for the representation. Of course, if $m_i - r_i(\alpha)$ is not a weight, one has $E_{-\alpha} \xi(\vec{m}) = 0$. 


Fig. 3: Weight diagram for $D(1,0) = \{3\}$.

Fig. 4: Weight diagram for $D(0,1) = \{3^*\}$. 

$2m_2 = Y$

$\frac{2}{3}$

$\frac{1}{3}$

0

$-\frac{1}{2}$

$\frac{1}{2}$

$\sqrt{3}m_4 = 1_3$

Reflection plane 1
**Fig. 5**: Weight diagram for \{8\}.

**Fig. 6**: Weight diagram for \{10\}.
Using the possible combinations of weight and roots given in Eq. (42) and in Fig. 2 and 3, one gets the following relations:

\[ E_{-1}, \xi[\hat{m}(1)] = \frac{1}{\sqrt{6}} \xi[\hat{m}(2)] \]

\[ E_{-2}, \xi[\hat{m}(1)] = \frac{1}{\sqrt{6}} \xi[\hat{m}(3)] \]  

\[ E_{-3}, \xi[\hat{m}(2)] = \frac{1}{\sqrt{6}} \xi[\hat{m}(3)] \]

which corresponds to the following representation for \( E_{-\alpha} \)

\[ E_{-1} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ E_{-2} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \]  

\[ E_{-3} = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

One can also prove that in this case, as for \( SU_2 \), one has: \( E_\alpha = E_{-\alpha}^+ \), so that the representation of the other three generators \( E_\alpha \) is immediately obtained from Eq. (56).

In the general case the basis of an irreducible representation is denoted by \( \xi([N], \hat{m}, g) \), where \([N]\) characterizes the representation, \( \hat{m} \) is the eigenvector in a given basis, and \( g \) distinguishes among degenerate eigenvectors. The basis \( \xi([N], \hat{m}, g) \) consists of \( N \) orthogonal vectors in the \( N \) dimensional linear space \( V_N \). The representation obtained as a direct product of two irreducible representations of dimensions \( N, N' \) has for a basis the set of \( NN' \) vectors which span the \( NN' \) dimensional space \( V_{NN'} \).
It is clear that in this case the generators of the group \( L_a \), considered as operators acting on a vector space \( V_N \), can be written as the sum of the generators \( L_a(1), L_a(2) \) operating in the two independent vector spaces \( V_N, V_{N'} \):

\[
L_a = L_a(1) + L_a(2).
\]  

Each irreducible representation can be characterized, instead of \([N]\) or in other terms of the parameters \( \lambda_1, \lambda_2 \) by the eigenvalues of two operators which are non-linear combinations of the generators \( L_a \), and which commute with all the \( L_a \). One can show that in this case they have to be a multiple of the unit operator, and they have a fixed constant value for each representation. We shall not consider them explicitly here. Instead, we need an operator which distinguishes, in a given representation, the basis vectors which belong to multiple weights. One can show that the operators \( H, E_+, E_- \) can be taken as generators of a group \( SU_3 \), subgroup of \( SU_3 \); they are equivalent to \( I_3, I_+, I_- \) of formula Eq. (19). The quantity

\[
I^2 = H^2 + \frac{1}{2} E_+ E_- + \frac{1}{2} E_-, E_+ \]  

is an invariant of \( SU_3 \). Moreover one has:

\[
[I^2, H_3] = 0.
\]  

The eigenvalues of \( I^2 \) can be taken as the required labels to distinguish the degenerate eigenvectors.

We shall then use a simpler notation for denoting the basis vectors of a representation, namely by

\[
\xi([N], I^2, I_3, Y).
\]  

The symbols \( I_3, Y \) stand for the eigenvalues of the operators

\[
I_3 = \sqrt{3} H_3, \quad Y = 2 H_2.
\]
The basis vectors for the direct product of two representations is given by

\[
\xi[[N(1)],\{N(2)\},I^2(1),I^2(2),I_3(1),I_3(2),Y(1),Y(2)] = \\
= \xi[[N(1)],I^2(1),I_3(1),Y(1)] \xi[[N(2)],I^2(2),I_3(2),Y(2)].
\]

[62]

It is useful, however, to choose as basis the eigenvectors of another set of commuting operators, namely

\[
\xi[[N],\{N(1)\},\{N(2)\},I^2,I^2(1),I^2(2),I_3,Y]
\]

[63]

where \{N\} characterizes one of the L.R. into which is reduced the direct product, and:

\[
I^2 = [I(1) + I(2)]^2; \quad I_3 = I_3(1) + I_3(2); \quad Y = Y(1) + Y(2).
\]

[64]

The connection between the set of eigenvectors Eq. (63) and the set, Eq. (62), is given by a unitary (in particular, orthogonal) transformation

\[
\xi[[N],\{N(1)\},\{N(2)\},I^2,I_3,Y] = \\
\sum_{I_3(1),I_3(2),Y(1),Y(2)} C_{I(1)I(2)} [I,I_3;I_3(1),I_3(2)] \left( \begin{array}{ccc} [N(1)] & [N(2)] & [N] \\ (I(1)Y(1)) & (I(2)Y(2)) & IY \end{array} \right) \times \\
\times \xi[[N(1)],I^2(1),I_3(1),Y(1)] \xi[[N(2)],I^2(2),I_3(2),Y(2)]
\]

[65]

where the coefficients \(C_{I(1)I(2)}\) are the Clebsch-Gordon coefficients used in Eq. (25) for the similar problem in the case of the group \(SU_2\), and the other terms

\[
\left( \begin{array}{ccc} [N(1)] & [N(2)] & [N] \\ (I(1)Y(1)) & (I(2)Y(2)) & IY \end{array} \right)
\]

[66]

are called isoscalar factors. We give in Table II the isoscalar factors for the case of the direct product \{8\} \times \{8\}, as given by de Swart ("The octet model and its Clebsch-Gordon coefficients", CERN, 1963). We refer to this paper for the other direct products of \(SU_3\).
The formula Eq. (65) can be inverted in the following way

\[
\mathcal{E}[[N(1)], I^2(1), I_3(1), Y(1)], \mathcal{E}[[N(2)], I^2(2), I_3(2), Y(2)] = \\
\sum_{N, I, Y} \mathcal{C}_{I(1)I(2)}[I, I_3(1), I_3(2)] \begin{pmatrix} \{N(1)\} & \{N(2)\} \\ \{I(1)\} & \{I(2)\} \end{pmatrix} Y(2) \quad \mathcal{E}[N, \{N(1)\}, \{N(2)\}, I^2, I_3, Y].
\]

(67)

The formulae Eqs. (65) and (67) are the analogue for SU_3 of Eqs. (25) and (26).

V. APPLICATION OF SU_3 TO THE ELEMENTARY PARTICLES

By generalizing to SU_3, the procedure used for the group SU_2, it is possible to classify the elementary particles and resonances into multiplets. In this case one also assigns each multiplet to an irreducible representation, each element of the multiplet being one of the eigenvectors of the corresponding basis.

In a multiplet of given baryon number, spin and parity, the elementary particles can be characterized by the third component of the isotopic spin and by the hypercharge. We identify these observables with the two operators \( I_3 \) and \( Y \) given by Eq. (61).

Since there are eight stable baryons (not decaying via strong interactions) with spin \( \frac{1}{2} \) and the same parity, we can try to classify them in the representation \( \{8\} \). The assignment of the different baryons to the basis vector is now easy, since we know already the weight diagram. We see, in fact, that the representation \( \{8\} \) contains two isospin doublets with \( Y = \pm 1 \) (\( \pi^+ \pi^- \) and \( \pi^0 \pi^0 \)), an isospin triplet and a singlet with \( Y = 0 \) (\( 2 \pi^0 \pi^0 \) and \( \Lambda^0 \)), so that the eight baryons can be fitted in the weight diagram, as shown in Fig. 7. The same can be done for the eight known pseudoscalar mesons \( (K^+K^-, \pi^+\pi^0\pi^-; \eta; \rho^0\rho^-; \omega; K^0, K^{*0}) \), and the eight vector mesons \( (K^{*+}, K^{*0}, \rho^+, \rho^0, \rho^-; \omega, K^0, K^*) \), as shown in the same figure. The baryon resonances of spin \( \frac{1}{2} \) and parity -, can be fitted into an octet; the baryon resonances of spin \( \frac{1}{2} \) and parity + into a decuplet (representation \( D(3,0) \)). We shall not, however, go into details for this classification, but refer to the copious literature on the
subject (see e.g. Dalitz: The systematics of baryons and meson states, EPJINS 63-53).

What are the physical consequences of this classification? Clearly, if one considers complete symmetry under the transformations of $SU_3$, all the masses of the particles in a multiplet have to be considered identical. One can assume that the mass differences among the isospin sub-multiplets in a multiplet, are given by a symmetry-breaking interaction which is considered "medium strong" with respect to the "very strong" symmetry-conserving interaction. Using the simplest form for the medium strong interaction, which is still invariant under the sub-group $SU_3$ of the isospin, one can obtain a formula for the mass splitting:

$$m = a + bY + c \left[ I(I+1) - \frac{1}{4}Y^2 \right]$$  \hspace{1cm} (68)

where $a, b$ and $c$ are constant for a given representation. By means of Eq. (68), one can then obtain linear relations for the masses of the particles in a given multiplet. (See: Dalitz, quoted reference).

We consider now a problem similar to the one mentioned for the isospin case; namely the problem of expressing a scattering process in terms of the smallest number of independent amplitudes, obtaining in this way, relations among the cross-sections of different particles of the same multiplet. In this case, however, we are far from the good experimental confirmation of the predictions in the analogous problem for $SU_3$. In fact one has to take into account that the higher symmetries at low energies are severely violated by the mass splittings; so that experimental test of these predictions should probably be applied at much higher energies than those at present available.

Let us consider, as an example, the scattering of a meson $M$ by a baryon $B$

$$M + B \rightarrow M' + B'$$  \hspace{1cm} (69)

where the $B, B', M, M'$ belong to the eight-fold representation.
The direct product of the two representations is reduced, according to Eq. (51) into six representation; \([1], [8_s], [8_a], [10], [10^*], [27] \). The fact that the eight dimensional representation occurs twice in the direct product, means that there are two possible couplings between a baryon octet and a meson octet. The two representations \([8_s], [8_a] \) correspond to symmetric and antisymmetric combinations of \(B \) and \(M \), respectively. We have then seven scattering amplitudes, corresponding to the transitions:

\[
A_1 : [1] \to [1] ; \quad A_{1e} : [10] \to [10] ; \quad A_{1o^*} : [10^*] \to [10^*] ; \quad A_{27} : [27] \to [27];
\]

\[
A_{sas} : [8_s] \to [8_s] ; \quad A_{saa} : [8_a] \to [8_a] ; \quad A_{sas} : [8_s] \to [8_s].
\]

(70)

The amplitude \(A_{sas} \) is equal to \(A_{saa} \) for time reversal invariance. In terms of these amplitudes, one can then express all the possible reactions similar to Eq. (69). We give just one example. Let us consider the process

\[
K^- + P \to K^- + p.
\]

By use of the formula Eq. (67) we can write

\[
K^- p = \sqrt{2} \xi([1], 0) + \sqrt{2} \sqrt{3} \xi([8_s], 1) + 1/2 \sqrt{3} \xi([8_a], 1) + 1/2 \xi([8_a], 0) - \\
1/2 \sqrt{5} \xi([8_s], 0) + 3 \sqrt{20} \xi([27], 0) - 1/10 \xi([27], 1) + 1/12 \xi([10], 1) - \\
1/12 \xi([10^*], 1),
\]

(71)

where we have omitted in the eigen states of the different representations the labels corresponding to the eigenvalues of \( I_3 = 0, \ Y = 0 \) which are, of course, the same for all the terms. The total amplitude for the process \(K^- p \to K^- + p \) can then be written as

---

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\[ A(K^+ p \rightarrow K^- p) = \frac{1}{4} A_{\alpha \alpha \alpha} + \frac{1}{3} A_{\alpha \alpha \alpha} + \frac{1}{12} (A_{I,0} + A_{I,0}) + \frac{1}{4} A_{\tau \tau} + \frac{1}{6} A_{\tau} \]  

(72)

This is only given as an example; the list of all the reactions is too long and we refer for it to the tables published by Freund et al. (Nuovo Cimento, 25, 307 1962).

A simpler way of obtaining relations between the cross-sections of different reactions consists in the application of particular transformations of the group SU3, under which the strong interactions are invariant. For instance, by convenient choice of the parameters \( \epsilon_a \) in Eq. (A1) one realizes that particular operations of the group SU3 are the rotations of an angle \( \pi \) around the axes \( 1, 2, 3 \) drawn in Fig. 7. Invariance under these rotations implies invariance under the following exchanges:

\[
\begin{align*}
    p & \leftrightarrow n, \quad \Sigma^+ \leftrightarrow \Sigma^-, \quad \Xi^0 \leftrightarrow \Xi^-; \quad K^+ \leftrightarrow K^0, \quad \pi^+ \leftrightarrow \pi^-, \quad \bar{K}^0 \leftrightarrow K^- \\
    p & \leftrightarrow \Sigma^+, \quad n \leftrightarrow \Sigma^0, \quad \Sigma^- \leftrightarrow \Xi^+; \quad K^+ \leftrightarrow \pi^+, \quad K^0 \leftrightarrow \bar{K}^0, \quad \pi^- \leftrightarrow K^- \\
    n & \leftrightarrow \Sigma^-, \quad p \leftrightarrow \Xi^-, \quad \Sigma^+ \leftrightarrow \Xi^0; \quad K^0 \leftrightarrow \pi^-, \quad K^+ \leftrightarrow K^-, \quad \pi^+ \leftrightarrow \bar{K}^0
\end{align*}
\]

(1) 2) 3)

Combining these operations, one obtains, for instance, relations among cross-sections, as for instance:

\[ \sigma(K^- p \rightarrow \Sigma^- \pi^+) = \sigma(K^- p \rightarrow \Xi^0 K^0) \]  

(73)
ELEMENTS OF GROUP THEORY

We give here a survey on some important properties of the transformation groups, which have been applied to the symmetries of the elementary particles. Mostly definitions and concepts will be presented here; for a more complete survey and derivation of properties we refer to the paper by Berends et al. (Rev. Mod. Phys. 34, 1 1962) and to the literature quoted in it.

Definition of group

A group is a set of elements for which a composition rule, called product, exists, such that:

1) the product of two elements \( e_i, e_j \) of the set is an element of the set:
   \[ e_k = e_i e_j \]

2) there exists a unit element \( e_0 \) such that:
   \[ e_i e_0 = e_0 e_i = e_i \]

3) there exists, for each element in the set, an element \( e_i' \) such that:
   \[ e_i e_i' = e_i' e_i = e_0 \]

   \( e_i' \) is called inverse of \( e_i \) and is usually denoted by \( e_i^{-1} \).

The product is associative:

\[ e_i (e_j e_k) = (e_i e_j) e_k \]

but in general not commutative:

\[ e_i e_j \neq e_j e_i \]

Sub-group and simple group

A sub-set of a group is called sub-group if the above properties hold for all its elements.

A sub-group is invariant if each of its elements commute with each element of the complete group.
A group is **simple** if it contains no invariant sub-groups, except the unit element.

**Lie group**

A group of transformations, which are characterized by a set of continuous parameters, is called **Lie group**.

We consider here the **unitary** transformations (see Chapter I):

\[
U = e^{i \epsilon_a L_a}
\]

which are represented by unitary operators (or equivalently by unitary matrices):

\[U^* U = U U^* = 1.\]

\(L_a\) is a set of **Hermitian** operators (or equivalently Hermitian matrices):

\[L_a^* = L_a,\]

and \(\epsilon_a\) is a set of **real** parameters.

Each transformation given by Eq. (A1) is an element of a Lie group.

An infinitesimal transformation can be written as:

\[U = 1 + i \epsilon_a L_a\]

or for a matrix element:

\[U_{ij} = \delta_{ij} + i \epsilon_a (L_a)_{ij}\]

We consider here unitary transformations which are also **unimodular** (\(\det U = 1\)), so that from Eq. (A3) one gets

\[\Sigma_i (L_a)_{ii} = 0\]

that is, to the operators \(L_a\) there correspond **traceless** matrices.

**Generators of infinitesimal transformation**

The operators \(L_a\) are called **generators** of the infinitesimal transformations. These operators in general do not commute; their **commutation relations** characterize the group, and can be written as:

\[[L_a, L_b] = C_{ab}^d L_d\]

The coefficients \(C_{ab}^d\) are called **structure constants**; they satisfy the antisymmetry condition

\[C_{ab}^d = -C_{ba}^d\]
and the so-called **Jacobi identity**:  

$$C_{ab}^e C_{ef}^e + C_{bf}^e C_{ea}^e + C_{fa}^e C_{eb}^e = 0.$$  

(A7)

**Order and rank of a group**

The number $$r$$ of linearly independent generators $$L_a$$ which characterize all the elements of the group, is called the **order** of the group.

Suppose that $$l$$ of these independent operators $$L_a$$ commute; then the number $$l$$ is called **rank** of the group. If there is no commuting pair of operators, the rank of the group is **one**.

**Representation of the group**

As assumed explicitly above, the quantities $$L_a$$ can be considered as either operators or matrices. In fact, many sets of different matrices can be found, which satisfy the commutation relations Eq. (A5) with the same structure constants. Such sets can be considered as explicit realizations of abstract operators; they are a **representation** of the generators $$L_a$$. By means of Eq. (A2), one can associate to the elements of the group a set of matrices. This set of matrices is a particular **representation** of the abstract group (group of abstract operators).

The dimension of the matrices in a representation is called the **dimension** of the representation.

We consider only **faithful** representations, such that there is a one-to-one correspondence between the elements of the group and the matrices of the representation. In this case, the group can be defined also by one of its representations.

The set of $$n \times n$$ matrices corresponding to a given set of operators is obtained by means of the scalar products:

$$U_{ik} = <\xi(i), U\xi(k)>$$  

(A8)

where the $$\xi$$ form a set of $$n$$ linearly independent orthogonal vectors which are called **basis vectors** in a $$n$$ dimensional linear space.
A change of the basis vectors, obtained by means of a non-singular matrix $S$

$$\xi'(i) = S\xi(i)$$  \hspace{1cm} (A9)

is equivalent to a change of the matrix $U$:

$$U' = S^{-1}US.$$  \hspace{1cm} (A10)

Such transformation is called similarity transformation.

Two representations of a group which are related by a similarity transformation are equivalent.

Irreducible representations

If, by a similarity transformation, all the matrices in a representation reduce to block diagonal form

$$U = \begin{pmatrix} U(1) & & & 0 \\ & \ddots & & \\ & & U(2) & \\ & & & \ddots \end{pmatrix}$$  \hspace{1cm} (A11)

the representation is said to be completely reducible. In this case it can be split into different representations $U(1), U(2), \ldots, U(k)$ (usually of different dimensions). This can be written symbolically:

$$U = U(1) + U(2) + \ldots + U(k).$$  \hspace{1cm} (A12)

If such a reduction is not possible, the representation is called irreducible.

Direct product representation

Given two representations of a group $U^{(m)}(1), U^{(n)}(2)$ of dimensions $m,n$ one can obtain an $mn$ dimensional representation of the group, by taking the direct (or Kronecker) product of the corresponding matrices:

$$u^{(mn)} = U^{(m)}(1) \times U^{(n)}(2).$$  \hspace{1cm} (A13)

One can show that the direct product representation $U^{(mn)}$ is completely reducible, and can then be split into a number of irreducible representations of dimensions $r_1, r_2, \ldots, r_s$: 
\[ U^{(mn)} = U^{(r_1)} + U^{(r_2)} + \ldots + U^{(r_s)} \]  \hspace{1cm} (A14)

where: \( r_1 + r_2 + \ldots + r_s = mn \).

The basis vector of the \( U^{(mn)} \) representation is obtained by the product of the basis vectors \( \xi^{(m)}(i), \xi^{(n)}(k) \) of \( U^{(m)} \) and \( U^{(n)} \):

\[ \xi^{(mn)}(ik) = \xi^{(m)}(i) \xi^{(n)}(k) \]  \hspace{1cm} (A15)

By \( \xi^{(mn)}(ik) \) we denote the \( mn \) terms obtained by multiplying each of the \( m \) components of \( \xi^{(m)}(i) \) by each of the \( n \) components of \( \xi^{(n)}(k) \); they represent a vector in a \( mn \) dimensional linear space.

**Roots**

We denote here by \( H_1 \) (\( i = 1, 2, \ldots, \ell \)) the \( \ell \) operators amongst the \( r \) generators of the group which commute:

\[ [H_1, H_j] = 0 \]  \hspace{1cm} (A16)

We denote by \( E_a \) the remaining \( r - \ell \) operators, for which

\[ [E_a, E_b] = C_{ab} E_y \]  \hspace{1cm} (A17)

These operators \( E_a \) can be chosen in such a way that the remaining commutation relations (between the \( E_a \) and the \( H_1 \)) can be written as:

\[ [H_1, E_a] = r_1(a) \ E_a \]  \hspace{1cm} (A18)

This expression can be considered as an eigenvalue equation for \( E_a \). The quantities \( r_1(a) \), for a given \( a \), can be considered as components of a vector \( \vec{r}(a) \). Since there are \( r - \ell \) operators \( E_a \), there are \( r - \ell \) of such vectors; they are called roots. They can be normalized in such a way that:

\[ \sum_a r_i(a) r_j(a) = \delta_{ij} \]  \hspace{1cm} (A19)

One can show that:

1) If \( \vec{r}(a) \) is a root, \( \vec{r}(-a) = -\vec{r}(a) \) is also a root; the corresponding operator is denoted by \( E_{-a} \). All the set of operators can be denoted by \( E_a \) with: \( a = \pm 1, \pm 2, \ldots, \pm \ell (r - \ell) \).
2) If \( \vec{r}(\lambda) \) and \( \vec{r}(\mu) \) are two roots, then

\[
K = 2 \frac{\vec{r}(\lambda) \cdot \vec{r}(\mu)}{|\vec{r}(\lambda)|^2}
\]

is an integer, and

\[
\vec{r}(\gamma) = \vec{r}(\mu) - K \vec{r}(\lambda)
\]

is also a root. It then follows that:

\[
\vec{r}(\lambda) \cdot \vec{r}(\mu) = \frac{1}{2} m |\vec{r}(\lambda)|^2 = \frac{1}{2} n |\vec{r}(\mu)|^2
\]

where \( m \) and \( n \) are integers. If \( \phi \) is the angle between \( \vec{r}(\lambda) \) and \( \vec{r}(\mu) \), one gets:

\[
\cos \phi = \frac{1}{4} mn
\]

This means that \( \phi \) can have only the values: \( 0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ \).

**Weights**

We consider a representation of dimension \( n \) of the group: the generators \( H_i \) have to be considered here as \( n \times n \) matrices. One can choose the representation in which these matrices are diagonalized simultaneously. The eigenvalues and eigenvectors are defined by

\[
H_i \xi^i = m_i \xi^i
\]

The \( n \) component vector \( \vec{m} = (m_1, m_2, \ldots, m_n) \) is called weight. If a weight belongs to only one eigenvalue it is called simple. Since the eigenvectors corresponding to different weights are linearly independent, there are, at most, \( n \) different weights.

One can prove that: for any weight \( \vec{m} \) and a root \( \vec{r}(\lambda) \), the quantity

\[
K = \frac{2m \cdot \vec{r}(\lambda)}{|\vec{r}(\lambda)|^2}
\]

is an integer, and

\[
\vec{m}' = \vec{m} - K \vec{r}(\lambda)
\]

is also a weight. This corresponds to a reflection of the vector \( \vec{m} \) through
a hyperplane perpendicular to $\tilde{f}(x)$. The weights which are related by reflection and product of reflection are called equivalent.

A weight $\tilde{m} = (m_1, m_2, \ldots, m_n)$ is said to be higher than $\tilde{m}' = (m'_1, m'_2, \ldots, m'_n)$ if the first non-vanishing difference $(m_i - m'_i)$ is greater than zero. The dominant weight is the highest in a set of equivalent weights.

It has been proved that for every simple group of rank $\ell$, every dominant weight can be written as a linear combination of $\ell$ fundamental dominant weights $\tilde{m}^{(1)}, \tilde{m}^{(2)}, \ldots, \tilde{m}^{(\ell)}$

$$\tilde{m} = \sum_{i=1}^{\ell} \lambda_i \tilde{m}^{(i)} \quad (A27)$$

with $\lambda_i$ non-negative integers. Moreover, there exists $\ell$ irreducible representations for which the dominant weight is one of the $\tilde{m}^{(i)}$. These representations are called fundamental (only one of the $\lambda_i$ is different from zero and equal to 1). In general, each irreducible representation is characterized by a given set of the coefficients $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.
TABLE I

Clebsch-Gordan coefficients for \( D(\frac{1}{2}) \times D(\frac{1}{2}) \); \( D(\frac{1}{2}) \times D(1) \) and \( D(1) \times I(1) \):

\[
C_{I(1)I(2)}[I_1I_3;I_1',I_2',I_3']
\]

\( I(1) = I(2) = \frac{1}{2} \)

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<tr>
<th>( I )</th>
<th>( I_3 )</th>
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\( I(1) = \frac{1}{2}, I(2) = 1 \)

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### Table II

Isoscalar factors for $\{8\} \times \{8\}$

\[
\begin{pmatrix}
\{8\} & \{8\} & \{N\} \\
I_1 Y_1 & I_2 Y_2 & I Y
\end{pmatrix}
\]

#### $Y = 2$, $I = 1$

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<th>${N}$</th>
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#### $Y = 2$, $I = 0$

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#### $Y = -2$, $I = 0$

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#### $Y = 1$, $I = \frac{3}{2}$

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\( Y = 0 \) \( \quad I = 1 \)

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<tr>
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<td>( \sqrt{\gamma} )</td>
<td>( \sqrt{\gamma} )</td>
<td>( 0 )</td>
<td>( -\frac{\gamma}{2} )</td>
<td>( -\frac{\gamma}{2} )</td>
<td></td>
</tr>
</tbody>
</table>

7863/π/cm
TABLE II (cont.)

\[ Y = 0 \quad I = 0 \]

<table>
<thead>
<tr>
<th>( I_1, Y_1; \quad I_2, Y_2 )</th>
<th>{27}</th>
<th>{8,}</th>
<th>{1}</th>
<th>{8_2}</th>
<th>{N}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2}, 1; \quad \frac{1}{2}, -1 )</td>
<td>( \sqrt{\frac{3}{7}} )</td>
<td>( \sqrt{\frac{7}{10}} )</td>
<td>( \frac{1}{2} )</td>
<td>( \sqrt{\frac{5}{2}} )</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{2}, -1; \quad \frac{1}{2}, 1 )</td>
<td>( -\sqrt{\frac{3}{7}} )</td>
<td>( -\sqrt{\frac{7}{10}} )</td>
<td>( -\frac{1}{2} )</td>
<td>( \sqrt{\frac{5}{2}} )</td>
<td></td>
</tr>
<tr>
<td>( 1, 0; \quad 1, 0 )</td>
<td>( \sqrt{\frac{5}{2}} )</td>
<td>( -\sqrt{\frac{7}{10}} )</td>
<td>( \sqrt{\frac{6}{5}} )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0, 0; \quad 0, 0 \</td>
<td>( 3\sqrt{\frac{7}{2}} )</td>
<td>( -\sqrt{\frac{7}{5}} )</td>
<td>( -\sqrt{\frac{6}{5}} )</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

\[ Y = -1 \quad I = \frac{\gamma}{2} \]

<table>
<thead>
<tr>
<th>( I_1, Y_1; \quad I_2, Y_2 )</th>
<th>{27}</th>
<th>{10}</th>
<th>{N}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2}, -1; \quad 1, 0 )</td>
<td>( \sqrt{\frac{6}{5}} )</td>
<td>( -\sqrt{\frac{7}{2}} )</td>
<td></td>
</tr>
<tr>
<td>1, 0; \quad \frac{1}{2}, -1 \</td>
<td>( \sqrt{\frac{7}{2}} )</td>
<td>( \sqrt{\frac{6}{5}} )</td>
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\[ Y = -1 \quad I = \frac{\gamma}{2} \]

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</tr>
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<tbody>
<tr>
<td>( \frac{1}{2}, -1; \quad 1, 0 )</td>
<td>( -\sqrt{\frac{3}{7}} )</td>
<td>( 3\sqrt{\frac{7}{10}} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td></td>
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<tr>
<td>1, 0; \quad \frac{1}{2}, -1 \</td>
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<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td></td>
</tr>
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<td>( 3\sqrt{\frac{7}{10}} )</td>
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<td>( -\frac{1}{2} )</td>
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