I. INTRODUCTION

Let us consider the set $S$ of all density matrices $\rho$ of a bipartite system with assigned Hilbert space $\mathbb{C}^n_A \otimes \mathbb{C}^n_B$. Let $\mathcal{M}(pq)$ be the linear space (over the complex field $\mathbb{C}$) of all $pq \times pq$ complex matrices equipped with the inner product $\langle A|B \rangle := \text{Tr}(A^\dagger B)$ for any $A, B \in \mathcal{M}(pq)$. Let us consider the metric $D(A, B) := ((A - B)(A - B))^\dagger$ on $\mathcal{M}(pq)$, for any $A, B \in \mathcal{M}(pq)$. With respect to this metric, the set $S$ forms a compact (which is also convex) subset of $\mathcal{M}(pq)$ generated by $(p^2q^2 - 1)$ real parameters. Deciding whether a given element from this compact set $S$ is separable or entangled (the separability problem) is known to be NP-hard $[3]$. This problem is an instance of the weak membership problem as defined by Gr"{o}tschel et al. $[11]$ (see also $[10]$). Recently the separability problem has been considered and discussed in $[2, 8, 12, 15]$. There are few cases where the separability problem is known to be efficiently solvable. The best known situation is $(p, q) = (2, 3)$ or $(3, 2)$. In this case, the positivity of $\rho^{1/2}$ (the partial transposition of $\rho$ with respect to the system $B$) is equivalent to the separability of $\rho$ $[14, 16]$. Also, the set of all density matrices “very near” (in the sense of some useful metric) to the maximally mixed state is known to be separable $[5]$. Other examples are given in $[3]$. For some discrete family of density matrices (that is, for which no continuity argument can be applied), no such efficient criterion is known to us. Here we consider the family of the density matrices of graphs as defined in [quant-ph/0406165]. The density matrix $\rho_{pq}$ of $G_{pq}$, with $\rho_{pq} = (1/(2n^2))(G_{pq} + G_{pq}^\dagger)$, is an $n \times n$ complex matrix, denoted by $G_{pq}$. The degree matrix of $G$ is the combinatorial laplacian of the graph normalized to have unit trace. We describe a $[\Delta(G)]_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E(G); \\ 0, & \text{if } \{v_i, v_j\} \notin E(G). \end{cases}$

The $[\Delta(G)]_{ij} = \begin{cases} 1, & i = j; \\ 0, & \text{if } i \neq j. \end{cases}$

The laplacian $[L(G)]_{ij} = \begin{cases} |\{v_j : \{v_i, v_j\} \in E\}|, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$

The laplacian $L(G)$ of $G$ is the symmetric positive semidefinite matrix $L(G) := \Delta(G) - M(G)$. Other than this combinatorial laplacian, there are several other types of laplacians associated to graphs $[8]$. The matrix $\rho(G) := \frac{1}{2|E|}L(G)$.
is a density matrix. This is called the density matrix of $G$.  

It should be noted here that the notion of density matrix $\rho(G)$ of a graph $G = (V(G), E(G))$, as defined above, is completely different from the notion of ‘graph states’, introduced by Briegel and Raussendorf. A graph state $|G\rangle$, corresponding to a (simple) graph $G = (V(G), E(G))$, is a common eigenstate (corresponding to the eigen value 1) of the $n = |V(G)|$ no. of $n$-qubit operators $\sigma_{11} \otimes \sigma_{12} \otimes \cdots \otimes \sigma_{1n}, \sigma_{21} \otimes \sigma_{22} \otimes \cdots \otimes \sigma_{2n}, \ldots, \sigma_{n1} \otimes \sigma_{n2} \otimes \cdots \otimes \sigma_{nn}$, where (i) $\sigma_{ij} = \sigma_x$ for all $i \in \{1, 2, \ldots, n\}$, (ii) for $j \neq i$, $\sigma_{ij} = \sigma_z$ if the vertices $v_i$ and $v_j$ of $G$ are connected by an edge, and (iii) for $j \neq i$, $\sigma_{ij} = I$ if the vertices $v_i$ and $v_j$ of $G$ are not connected by an edge. Thus, in this formalism, a two-level system is attached with each vertex of the graph and each edge of the graph represents an interaction (ising type) between the two two-level subsystems attached to the two vertices of the edge.

Let $G$ be a graph on $n = p.q$ vertices $v_1, v_2, \ldots, v_n$. These vertices are represented here as ordered pairs in the following way: $v_1 = (u_1, w_1) \equiv u_1w_1, v_2 = (u_1, w_2) \equiv u_1w_2, \ldots, v_q = (u_1, w_q) \equiv u_1w_q, v_{q+1} = (u_2, w_1) \equiv u_2w_1, v_{q+2} = (w_2, w_2) \equiv u_2w_2, \ldots, v_{2q} = (u_2, w_q) \equiv u_2w_q, \ldots, v_{(p-1)q+1} = (u_p, w_1) \equiv u_pw_1, v_{(p-1)q+2} = (u_p, w_2) \equiv u_pw_2, \ldots, v_{pq} = (u_p, w_q) \equiv u_pw_q$. We associate to this graph $G$ on $n$ labeled vertices (described above) the orthonormal basis $\{|u_i\rangle : i = 1, 2, \ldots, n\} = \{|u_j\rangle \otimes |w_k\rangle : j = 1, 2, \ldots, p; k = 1, 2, \ldots, q\}$, where $\{|u_j\rangle : j = 1, 2, \ldots, p\}$ and $\{|w_k\rangle : k = 1, 2, \ldots, q\}$ are orthonormal bases of the Hilbert spaces $\mathcal{H}_A \cong \mathbb{C}^p$ and $\mathcal{H}_B \cong \mathbb{C}^q$, respectively. The partial transpose of a graph $G = (V, E)$ (with respect to $H_B$), denoted by $G^{T_B} = (V, E')$, is the graph such that $\{u_iw_j, u_kw_j\} \in E'$ if and only if only if $\{u_iw_j, u_kw_j\} \in E$. We propose the following conjecture:

**Conjecture 1** Let $\rho(G)$ be the density matrix of a graph on $n = pq$ vertices. Then $\rho(G)$ is separable in $\mathbb{C}^p_A \otimes \mathbb{C}^q_B$ if and only if $\Delta(G) = \Delta (G^{T_B})$.

A proof of this conjecture would give a simple method for testing the separability of density matrices of graphs, as we would only need to check whether the $n \times n$ diagonal matrices $\Delta(G)$ and $\Delta (G^{T_B})$ are equal. This fact is in some sense analogous to the fact that the separability of all two-mode Gaussian states (which form a continuous family) is equivalent to the Peres–Horodecki partial transposition criterion \[17\]. We prove one side of our conjecture:

**Theorem 2** Let $\rho(G)$ be the density matrix of a graph on $n = pq$ vertices. If $\rho(G)$ is separable in $\mathbb{C}^p_A \otimes \mathbb{C}^q_B$ then $\Delta(G) = \Delta (G^{T_B})$.

We prove the other side of the conjecture for the following two families of graphs:

- Consider a rectangular lattice with $pq$ points arranged in $p$ rows and $q$ columns, such that the distance between two neighboring points on the same row or in the same column is $1$. A nearest point graph is a graph whose vertices are identified with the points of the lattice and the edges have length $1$ or $\sqrt{2}$.
- A perfect matching is a graph $G = (V, E)$ such that for every $v_i$ there is a unique vertex $v_j$ such that $\{v_i, v_j\} \in E$.

Namely, we prove the following two theorems:

**Theorem 3** Let $G$ be a nearest point graph on $n = pq$ vertices. Then the density matrix $\rho(G)$ is separable in $\mathbb{C}^p_A \otimes \mathbb{C}^q_B$ iff $\Delta(G) = \Delta (G^{T_B})$.

**Theorem 4** Let $G$ be a perfect matching on $n = 2k$ vertices. Then the density matrix $\rho(G)$ is separable in $\mathbb{C}^p_A \otimes \mathbb{C}^q_B$ iff $\Delta(G) = \Delta (G^{T_B})$.

See Figure 1 below as examples of perfect matching $H$, the partial transpose graph $H^{T_B}$, nearest point graph $G$, and the partial transpose graph $G^{T_B}$.

The degree condition expressed in the conjecture appears to have value beyond density matrices of graphs. In general, given a density matrix $\rho$ in $\mathbb{C}^p_A \otimes \mathbb{C}^q_B$, let $\Delta(\rho)$ be the matrix defined as follows:

$$\Delta(\rho)_{i,j} = \begin{cases} 
\sum_{k=1}^{pq} \rho_{ik}, & \text{if } i = j; \\
0, & \text{if } i \neq j.
\end{cases}$$

In a circulant matrix each row is a cyclic shift of the row above to the right. This means that a circulant matrix is then defined by its first row. Let $G$ be a finite group of order $n$ and let $\sigma$ be the regular permutation representation of $G$. Then $\sigma$ is an homomorphism from $G$ to the set of permutation matrices of dimension $n$. The fourier transform (evaluated at $\sigma$) of a complex-valued function $f$ on $G$ is defined as the matrix $\hat{f} = \sum_{g} f(g) \sigma(g)$. According to this definition, a complex circulant matrix $M$ of dimension $n$ has the form $M = \sum_{g \in \mathbb{Z}_n} f(g) \sigma(g)$. We prove the following result:
Theorem 5 Let \( \rho \) be a circulant density matrix of dimension \( n = pq \). Then \( \Delta(\rho) = \Delta\left(\rho^{T_B}\right) \) and \( \rho \) is separable in \( \mathbb{C}_A^p \otimes \mathbb{C}_B^q \). Let \( \rho = \sum_{g \in \mathbb{Z}_2} f(g) \sigma(g) \) be a density matrix of dimension \( 2^n \). Then \( \Delta(\rho) = \Delta\left(\rho^{T_B}\right) \) and \( \rho \) is separable in \( \mathbb{C}_A^{2k} \otimes \mathbb{C}_B^{2l} \), where \( k + l = n \).

II. PROOFS

A. Proof of Theorem 2

Let \( L(G) \) be the laplacian of a graph \( G = (V, E) \) on \( n \) vertices \( v_1, \ldots, v_n \). Let \( D \) be any \( n \times n \) real diagonal matrix in the orthonormal basis \( \{|v_1\}, \ldots, |v_n\rangle \) such that \( D \neq 0 \) and \( \text{tr}(D) = 0 \). It follows that there is at least one negative entry in the diagonal of \( D \). Let this entry be \( D_{i,i} = b_i \). Let \( |\psi_0\rangle = \sum_{j=1}^{n} |v_j\rangle \) and \( |\phi\rangle = \sum_{j=1}^{n} \chi_j |v_j\rangle \), where

\[
\chi_j = \begin{cases} 0, & \text{if } j \neq i; \\ k & \text{if } j = i, \end{cases} \quad \text{with } k \in \mathbb{R}.
\]

Let \( |\chi\rangle = |\psi_0\rangle + |\phi\rangle = \sum_{j=1}^{n} (1 + \chi_j) |v_j\rangle \). Then

\[
\langle \chi | \left( L(G) + D \right) | \chi \rangle = \langle \chi | L(G) | \chi \rangle + \langle \chi | D | \chi \rangle = \langle \psi_0 | L(G) | \psi_0 \rangle + \langle \psi_0 | L(G) | \phi \rangle + \langle \phi | L(G) | \psi_0 \rangle + \langle \phi | L(G) | \phi \rangle + \langle \psi_0 | D | \psi_0 \rangle + \langle \phi | D | \psi_0 \rangle + \langle \phi | D | \phi \rangle.
\]

The state \( |\psi_0\rangle \) is an eigenvector (unnormalized) of \( L(G) \), corresponding to the eigenvalue \( 0 \): \( L(G) |\psi_0\rangle = 0 \). Also \( \langle \psi_0 | D | \psi_0 \rangle = \text{tr}(D) = 0 \). Then

\[
\langle \chi | \left( L(G) + D \right) | \chi \rangle = \langle \psi_0 | L(G) | \phi \rangle + \langle \phi | L(G) | \phi \rangle + \langle \psi_0 | D | \phi \rangle + \langle \phi | D | \psi_0 \rangle + \langle \phi | D | \phi \rangle.
\]

Now \( \langle \psi_0 | L(G) | \phi \rangle = \langle \phi | L(G)^T | \psi_0 \rangle = \langle \phi | L(G) | \psi_0 \rangle = 0 \). In fact, \( L(G) = L(G)^T \). Let \( [L(G)]_{j,l} \) be the \( jl \)-th entry of \( L(G) \) with respect to the basis \( \{|v_1\}, \ldots, |v_n\rangle \). Let \( d_i = |\{v_j : \{v_i, v_j\} \in E}\| \). We have

\[
\langle \phi | L(G) | \phi \rangle = k^2 \langle (L(G))_{i,i} \rangle = k^2 d_i;
\]
\[
\langle \phi | D | \phi \rangle = b_i k^2;
\]
\[
\langle \psi_0 | D | \phi \rangle = b_i k;
\]
\[
\langle \phi | D | \psi_0 \rangle = b_i k.
\]

Thus

\[
\langle \chi | \left( L(G) + D \right) | \chi \rangle = k^2 (d_i + b_i) + 2b_i k, \quad \text{with } d_i \geq 0 \text{ and } b_i < 0.
\]
So we can then always choose a positive \( k \), small enough, such that
\[
2b_i k + k^2 (d_i + b_i) < 0.
\]

It follows that
\[
L(G) + D \not\preceq 0.
\]

For any graph \( G \) on \( n = pq \) vertices
\[
v_1 = (u_1, w_1), v_2 = (u_1, w_2), \ldots, v_q = (u_1, w_q), v_{q+1} = (u_2, w_1), v_{q+2} = (u_2, w_2), \ldots, v_{2q} = (u_2, w_q), \ldots, v_{pq} = (u_p, w_q),
\]
consider now the degree condition \( \Delta(G) \). Now
\[
(L(G))^{\Gamma_B} = (\Delta(G) - \Delta(G)) + L(G).\]
Let
\[
D = \Delta(G) - \Delta(G).
\]
Then \( D \) is an \( n \times n \) real diagonal matrix with respect to the orthonormal basis
\[
|v_1\rangle = |u_1\rangle \otimes |w_1\rangle, \ldots, |v_{pq}\rangle = |u_p\rangle \otimes |w_q\rangle.
\]
Also
\[
\text{tr}(D) = \text{tr}(\Delta(G)) - \text{tr}(\Delta(G)) = 0.
\]

As \( G \) is a graph on \( n \) vertices \( v_1, v_2, \ldots, v_n \), as here \( D = \Delta(G) - \Delta(G) \) is a diagonal matrix with respect to the orthonormal basis \( \{|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle\} \), and as here \( \text{tr}(D) = 0 \), therefore, by the above-mentioned reasoning, \( D + L(G) \) is separable if \( D \neq 0 \). Now if \( \rho(G) \) is separable then we must have \( L(G)^{\Gamma_B} = D + L(G) \geq 0 \). Therefore separability of \( L(G) \) implies that \( D = \Delta(G) - \Delta(G) = 0 \).

**B. Proof of Theorem 3**

Let \( G \) be a nearest point graph on \( n = pq \) vertices and \( m \) edges. We associate to \( G \) the orthonormal basis \( \{|v_i\rangle : i = 1, 2, \ldots, n\} = \{|u_j\rangle \otimes |w_k\rangle : j = 1, 2, \ldots, p; k = 1, 2, \ldots, q\} \), where \( \{|u_j\rangle : j = 1, 2, \ldots, p\} \) is an orthonormal basis of \( \mathbb{C}_q^p \) and \( \{|w_k\rangle : k = 1, 2, \ldots, q\} \) is an orthonormal basis of \( \mathbb{C}_p^q \). Let \( j, j' \in \{1, 2, \ldots, p\} \) and \( k, k' \in \{1, 2, \ldots, q\} \). Let \( \lambda_{jk,j'k'} \in \{0, 1\} \) be defined as follows:
\[
\lambda_{jk,j'k'} = \begin{cases} 
1, & \text{if } \{u_j w_k, u_{j'} w_{k'}\} \in E; \\
0, & \text{if } \{u_j w_k, u_{j'} w_{k'}\} \notin E. 
\end{cases} \tag{2.1}
\]
Thus, for the above-mentioned nearest point graph \( G \), \( \lambda_{jk,j'k'} \) can have non-zero values only in the following cases: either (i) \( j' = j \) and \( k' = k + 1 \), or (ii) \( j' = j + 1 \) and \( k' = k \), or (iii) \( j' = j + 1 \) and \( k' = k + 1 \), or (iv) a combination of some or all of the three cases (i) - (iii). Let \( \rho(G) \) and \( \rho(G^{\Gamma_B}) \) be the density matrices corresponding to the graphs \( G \) and \( G^{\Gamma_B} \), respectively. Thus
\[
\rho(G) = \frac{1}{2m} (\Delta(G) - M(G)) \quad \text{and} \quad \rho(G^{\Gamma_B}) = \frac{1}{2m} (\Delta(G^{\Gamma_B}) - M(G^{\Gamma_B}))
\]
Let \( G_1 \) be the subgraph of \( G \) whose edges are all the entangled edges of \( G \). An edge \( \{ij, kl\} \) is **entangled** if \( i \neq k \) and \( j \neq l \). Also let \( G'_1 \) be the subgraph of \( G^{\Gamma_B} \) corresponding to all the “entangled edges” of \( G^{\Gamma_B} \). Obviously \( G_1^{\Gamma_B} = (G_1)^{\Gamma_B} \). Using the above-mentioned notations, we have
\[
\rho(G_1) = \frac{1}{m} \sum_{j=2}^{p} \left( \lambda_{j-1,j} P \left[ \frac{1}{\sqrt{2}} (|u_{j-1} w_1\rangle - |u_j w_2\rangle) \right] \right)
\]
\[
+ \frac{1}{m} \sum_{j=2}^{p} \sum_{k=3}^{q} \left( \lambda_{j-1,k-1,j,k-2} P \left[ \frac{1}{\sqrt{2}} (|u_{j-1} w_{k-1}\rangle - |u_j w_{k-2}\rangle) \right] \right)
\]
\[
+ \lambda_{j-1,k-1,j,k} P \left[ \frac{1}{\sqrt{2}} (|u_{j-1} w_k\rangle - |u_j w_k\rangle) \right]
\]
\[
+ \frac{1}{m} \sum_{j=2}^{p} \left( \lambda_{j-1,q-1,j} P \left[ \frac{1}{\sqrt{2}} (|u_{j-1} w_{q-1}\rangle - |u_j w_{q-1}\rangle) \right] \right), \tag{2.2}
\]
where, for any normalized pure state $|\psi\rangle$, $P[|\psi\rangle]$ denotes the one-dimensional projector onto the vector $|\psi\rangle$. Also we have

$$\rho(G'_{1}) = \frac{1}{m} \sum_{j=2}^{p} \left( \lambda_{(j-1)1,j2} P \left[ \frac{1}{\sqrt{2}} (|u_{(j-1)w_{2}}\rangle - |u_{jw_{1}}\rangle) \right] \right)$$

$$+ \frac{1}{m} \sum_{j=2}^{p} \sum_{k=3}^{q} \left( \lambda_{(j-1)(k-1),j(k-2)} P \left[ \frac{1}{\sqrt{2}} (|u_{jw_{k-2}}\rangle - |u_{jw_{k-1}}\rangle) \right] \right)$$

$$+ \lambda_{(j-1)(k-1),jk} P \left[ \frac{1}{\sqrt{2}} (|u_{jw_{k}}\rangle - |u_{jw_{k-1}}\rangle) \right]$$

$$+ \frac{1}{m} \sum_{j=2}^{p} \left( \lambda_{(j-1)q,j(q-1)} P \left[ \frac{1}{\sqrt{2}} (|u_{jw_{q-1}}\rangle - |u_{jw_{q}}\rangle) \right] \right)$$

(2.3)

One can check that

$$\Delta (G_{1}) = \frac{1}{2m} \left( \lambda_{11,22} P [u_{1w_{1}}] + \sum_{k=3}^{q} \left( \lambda_{1(k-1),2(k-2)} + \lambda_{1(k-1),2k} \right) P [u_{1w_{k}}] + \lambda_{1q,2(q-1)} P [u_{1w_{q}}] \right)$$

$$+ \frac{1}{2m} \sum_{j=3}^{p} \left( \lambda_{(j-2)2,(j-1)1} + \lambda_{(j-1)1,j2} \right) P [u_{jw_{1}}]$$

$$+ \frac{1}{2m} \sum_{j=3}^{p} \sum_{k=3}^{q} \left( \lambda_{(j-2)(k-2),(j-1)(k-1)} + \lambda_{(j-2)(k,j-1)(k-1)} + \lambda_{(j-1)(k-1),j(k-2)} + \lambda_{(j-1)(k-1),j(k)} \right)$$

$$\times P [u_{jw_{k-1}}]$$

$$+ \frac{1}{2m} \sum_{j=3}^{p} \left( \lambda_{(j-2)(q-1),(j-1)q} + \lambda_{(j-1)q,j(q-1)} \right) P [u_{jw_{q}}]$$

$$+ \frac{1}{2m} \lambda_{(p-1)2,p1} P [u_{pw_{1}}] + \frac{1}{2m} \sum_{k=3}^{q} \left( \lambda_{(p-1)(k-2),p(k-1)} + \lambda_{(p-1)k,p(k-1)} \right) P [u_{pw_{k-1}}]$$

$$+ \frac{1}{2m} \lambda_{(p-1)q,pq} P [u_{pw_{q}}]$$.

(2.4)

And

$$\Delta (G'_{1}) = \frac{1}{2m} \left( \lambda_{12,21} P [u_{1w_{1}}] + \sum_{k=3}^{q} \left( \lambda_{1(k-2),2(k-1)} + \lambda_{1k,2(k-1)} \right) P [u_{1w_{k}}] + \lambda_{1(q-1),2q} P [u_{1w_{q}}] \right)$$

$$+ \frac{1}{2m} \sum_{j=3}^{p} \left( \lambda_{(j-2)1,(j-1)2} + \lambda_{(j-1)2,j1} \right) P [u_{jw_{1}}]$$

$$+ \frac{1}{2m} \sum_{j=3}^{p} \sum_{k=3}^{q} \left( \lambda_{(j-2)(k-1),(j-1)(k-2)} + \lambda_{(j-2)(k,j-1)(k-1)} + \lambda_{(j-1)(k-2),j(k-1)} + \lambda_{(j-1)k,j(k-1)} \right)$$

$$\times P [u_{jw_{k-1}}]$$

$$+ \frac{1}{2m} \sum_{j=3}^{p} \left( \lambda_{(j-2)q,(j-1)q-1} + \lambda_{(j-1)q,j(q-1)} \right) P [u_{jw_{q}}]$$

$$+ \frac{1}{2m} \lambda_{(p-1)1,p2} P [u_{pw_{1}}] + \frac{1}{2m} \sum_{k=3}^{q} \left( \lambda_{(p-1)(k-1),p(k-2)} + \lambda_{(p-1)(k-1),pk} \right) P [u_{pw_{k-1}}]$$

$$+ \frac{1}{2m} \lambda_{(p-1)q,pq} P [u_{pw_{q}}]$$.

(2.5)

Let $G_{2}$ and $G'_{2}$ respectively be the subgraphs of $G$ and $G^u$ each containing all the edges of the forms $\{u_{i}w_{j}, u_{i}'w_{j}'\}$ (where $j \neq j'$) as well as $\{u_{i}w_{j}, u_{i}w_{j}\}$ (where $i \neq i'$). Then it is obvious that $\Delta(G_{2}) = \Delta(G'_{2})$, due to the fact that
Using Equation (2.11), it is easy to see from Equation (2.10) that \( \Delta(G) = \Delta(G^{T_B}) \) if and only if \( \Delta(G_1) = \Delta(G'_1) \). Using equations (2.4) and (2.5), we see that the equality of \( \Delta(G_1) \) and \( \Delta(G'_1) \) implies that:

\[
\begin{align*}
\lambda_{11,22} &= \lambda_{1(k-1),2(k-2)} + \lambda_{1(k-1),2k} = \lambda_{1(k-2),2(k-1)} + \lambda_{1k,2(k-1)}, & \text{for } k = 3, 4, \ldots, q, \\
\lambda_{1q,2(q-1)} &= \lambda_{1(k-1),2(q-2)} + \lambda_{1(k-1),2q} = \lambda_{1(k-2),2(q-1)} + \lambda_{1(k-1),2q}.
\end{align*}
\]

(2.6)

for each \( j \in \{3, 4, \ldots, p\} : \\
\lambda_{(j-2)(j-1),1} + \lambda_{(j-1),j2} &= \lambda_{(j-2),2} + \lambda_{(j-1),j1}, \\
\lambda_{(j-2)(j-1),1} + \lambda_{(j-2),j1} &= \lambda_{(j-1),j2} + \lambda_{(j-1),j1}, \\
\lambda_{(j-2)(j-1),1} + \lambda_{(j-2),j1} &= \lambda_{(j-1),j2} + \lambda_{(j-1),j1}, \text{ for } k = 3, 4, \ldots, q, \\
\lambda_{(j-2)(j-1),1} + \lambda_{(j-2),j1} &= \lambda_{(j-1),j2} + \lambda_{(j-1),j1}.
\]

(2.7)

\[
\begin{align*}
\lambda_{(p-1),p2} &= \lambda_{(p-1),2p1}, \\
\lambda_{(p-1)(p-2),p(p-1)} + \lambda_{(p-1)(p-2),p(p-1)} &= \lambda_{(p-1)(p-1),p(p-1)}, \text{ for } k = 3, 4, \ldots, q, \\
\lambda_{(p-1)(p-1),pq} &= \lambda_{(p-1),p(p-1)}. 
\end{align*}
\]

(2.8)

The solution of equations (2.6) - (2.8) is of the form:

\[
\lambda_{ij,i'j'} = \lambda_{ij,j'}, \quad \text{for all } i, i' \in \{1, 2, \ldots, p\} \text{ and all } j, j' \in \{1, 2, \ldots, q\},
\]

(2.9)

and where ever \( \lambda_{ij,i'j'} \) and \( \lambda_{ij',i'j} \) are defined. Equation (2.9) shows that whenever there is an entangled edge \( \{u_iw_j, u_{i'}w_{j'}\} \) in \( G \) (so we must have \( i \neq i' \) and \( j \neq j' \)), there must be the entangled edge \( \{u_iw_j, w_{i'}w_{j'}\} \) in \( G \). The two entangled edges \( \{u_iw_j, u_{i'}w_{j'}\} \) and \( \{u_iw_j, w_{i'}w_{j'}\} \) in \( G \) together give rise to the following contribution (which is again a density matrix) in the density matrix \( \rho(G) \), with the multiplicative factor \( \frac{1}{2} \).

\[
\rho(i, i'; j, j') = \frac{1}{2} \left( P \left( \frac{1}{\sqrt{2}} (|u_iw_j\rangle - |u_{i'}w_{j'}\rangle) \right) + P \left( \frac{1}{\sqrt{2}} (|w_{i}w_{j'}\rangle - |u_{i'}w_{j}\rangle) \right) \right).
\]

(2.10)

Let us write

\[
\frac{1}{\sqrt{2}} (|u_i\rangle \pm |u_{i'}\rangle) = |V (i, i'; \pm)\rangle \quad \text{and} \quad \frac{1}{\sqrt{2}} (|w_j\rangle \pm |w_{j'}\rangle) = |X (j, j'; \pm)\rangle.
\]

(2.11)

Using Equation (2.11), it is easy to see from Equation (2.10) that

\[
\rho(i, i'; j, j') = \frac{1}{2} P [V (i, i'; +) X (j, j'; -)] + \frac{1}{2} P [V (i, i'; -) X (j, j'; +)],
\]

(2.12)

which is a separable state in \( C_X^p \otimes C_B^q \). This shows that, under the constraint \( \Delta(G_1) = \Delta(G'_1) \), \( \rho(G_1) \) is nothing but equal mixture of separable states of the form \( \rho(i, i'; j, j') \), and so, \( \rho(G_1) \) must be separable, which, in turn, shows that \( \rho(G) \) has to be separable. This shows that a nearest point graph \( G \) is separable in \( C_X^p \otimes C_B^q \) if and only if \( \Delta(G) = \Delta(G^{T_B}) \). - We invite the reader to give a shorter proof! -

C. Perfect matchings

(C.1) Proof of Theorem 4

Definition (degree condition): For any graph \( G \) on \( n = pq \) vertices \( v_1 = (u_1, w_1), v_2 = (u_1, w_2), \ldots, v_q = (u_1, w_2), v_{(q+1)} = (u_2, w_1), v_{(q+2)} = (u_2, w_2), \ldots, v_{2q} = (u_2, w_2), \ldots, v_{(p-1)q+1} = (u_p, w_1), v_{(p-1)q+2} = (u_p, w_2), \ldots, v_{pq} = (u_p, w_q) \), the equation \( \Delta(G) = \Delta(G^{T_B}) \) is called the degree condition, where \( G^{T_B} \) is the graph with \( V (G^{T_B}) = V (G) \) and \( \{(u_i, w_j), (u_{i'}, w_{j'})\} \in E (G^{T_B}) \) if and only if \( \{(u_i, w_j), (u_{i'}, w_{j'})\} \in E (G) \).

We consider here only those graphs \( G \) on \( n = pq \) vertices, where \( n \) is even and \( E(G) \) consists of edges of the forms \( \{(i_k, j_k), (i'_k, j'_k)\} \), where (i) \( k \) runs from 1 up to \( n/2 \), (ii) \( (i_k, j_k) \neq (i'_k, j'_k) \) for all \( k \), (iii) \( (i_k, j_k) \neq (i_l, j_l) \) whenever
Therefore, \( p \) matchings on the same set of \( k \) by \( q \) matching on \( q \) where \( \rho \equiv (i_k, j_k, \cdots, i_f, j_f) \) and \( \rho \equiv (i_k', j_k', \cdots, i_f', j_f') \) whenever \( k \neq l \). Thus \( G \) is nothing but a perfect matching on \( n = pq \) vertices. In addition to above-mentioned conditions, if we have \( i_k \neq i_k' \) and \( j_k \neq j_k' \) for each \( k \in \{1, 2, \ldots, n/2\} \), then \( G \) is called as a perfect entangling matching. We denote the set of all such perfect entangling matchings on the same set of \( n = pq \) vertices as \( \mathcal{P}_{p,q} \). The density matrix \( \rho(G) \) of the graph \( G \) is given by

\[
\rho(G) = \frac{1}{n} \sum_{k=1}^{n/2} P \left[ \frac{1}{\sqrt{2}} (|i_k j_k \rangle - |i_k' j_k' \rangle) \right].
\]

Let \( G \in \mathcal{P}_{p,q} \). Let \( G^p \) be the graph with vertex set as \( V(G) \) and \{\( (i_k, j_k), (i_k', j_k') \)\} \( \in E(G^p) \) if and only if \{\( (i_k, j_k), (i_k', j_k') \)\} \( \in E(G) \). Let \( \mathcal{P}_{p,q}^S = \{G \in \mathcal{P}_{p,q} : G^p \in \mathcal{P}_{p,q}\} \). It can be easily shown that for any perfect matching \( G \) on \( n = pq \) vertices, \( G \in \mathcal{P}_{p,q}^S \) if and only if \( \Delta(G) = \Delta(G^p) \). Following are the two examples of ‘canonical’ perfect entangling matchings:

1. **Cris-cross:** A cris-cross \( C \) is given by \( C = (V(C) = \{(i_1, 1), (i_1, 2), (i_2, 1), (i_2, 2)\}, E(C) = \{(i_1, 1), (i_2, 2)\}, \{(i_2, 1), (i_1, 2)\}\} \) (where \( i_1 \neq i_2 \)).

2. **Tally mark:** A tally mark \( T \) is given by \( T = (V(T), E(T)) \) where \( V(T) = \{(i_k, 1): k = 1, 2, \ldots, r\} \cup \{(i_k, 2): k = 1, 2, \ldots, r\} \), and \( E(T) = \{(i_1, 1), (i_2, 2)\}, \{(i_2, 1), (i_3, 2)\}, \ldots, \{(i_{r-1}, 1), (i_r, 2)\}\} \) (where \( 1 \leq i_1 < i_2 < \cdots < i_r \leq p \) and \( r' \leq p' \)). We are now ready to give a proof of Theorem 4.

**Proof of Theorem 4:** Let \( G \) be a perfect matching on \( n = 2p \) vertices \( v_1 \equiv (1, 1), v_2 \equiv (1, 2), v_3 \equiv (2, 1), v_4 \equiv (2, 2), \ldots, v_{(2p-1)} \equiv (p, 1), v_{2p} \equiv (p, 2) \).

Let us first assume that \( \rho(G) \) is separable in \( C_A^p \otimes C_B^p \). Then by Theorem 2, we have \( \Delta(G) = \Delta(G^p) \).

Next we assume that \( \Delta(G) = \Delta(G^p) \). Let us denote the subgraph of \( G \), consisting of all its unentangled edges, as \( G_2 \) and the subgraph of \( G \), consisting of all its entangled edges, as \( G_3 \). As \( G \) is a perfect matching, therefore \( G \) is the disjoint union of \( G_1 \) and \( G_2 \): \( G = G_1 \cup G_2 \). Thus \( V(G) \) is the setwise disjoint union of \( V(G_1) \) and \( V(G_2) \), while \( E(G) \) is the setwise disjoint union of \( E(G_1) \) and \( E(G_2) \). Let us take \( E(G_2) = \{(i_k, 1), (j_k, 2)\}: k = 1, 2, \ldots, q\} \), where \( q \) is a non-negative integer with \( q \leq p \), \( 1 \leq i_1 < i_2 < \cdots < i_q \leq p \). We will prove that \( G_2 \) is a perfect entangling matching on \( n = p'q' \) vertices. So, we must have, \( G^p = G_2 \cup G_4 \), and hence, \( V(G_4) = V(G_1) \) (this is true for arbitrary values of \( p' \) and \( q' \) provided \( n = p'q' \) is even). Thus we see that both \( G_2 \) as well as \( G_4 \) are perfect entangling matchings on the same subset of vertices of \( G \) (this is also true for arbitrary values of \( p' \) and \( q' \) provided \( n = p'q' \)). It then follows that the two subsets \( \{i_k: k = 1, 2, \ldots, q\} \) and \( \{j_k: k = 1, 2, \ldots, q\} \) of \( \{1, 2, \ldots, p\} \) must be same. This is so because if some \( j_k \notin \{i_k: k = 1, 2, \ldots, q\} \), then vertex \( (j_k, 1) \) of the (entangled) edge \( \{(i_k, 2), (j_k, 1)\} \) in \( G_4 \) will belong to \( V(G_1) \) (and hence, to \( V(G_3) \)) - a contradiction.

Therefore, \( G_2 \) (and hence, \( G_4 \)) is a perfect entangling matching on the set of \( 2q \) vertices \( (i, j), \) where \( i \in \{1, i_2, \ldots, i_q\} \) and \( j \in \{1, 2\} \). Note that this fact is true not only for \( n = 2p \) but for any general \( n = p'q' \), provided \( n \) is even (and so, for any \( G \in \mathcal{P}_{p,q} \), \( G \in \mathcal{P}_{p,2q} \) if and only if \( \Delta(G) = \Delta(G^p) \)).

Now, it is known that (see Lemma 4.4 in [1]) any perfect entangling matching \( G' \) on \( n = 2p' \) vertices \( v'_1 \equiv (1, 1), v'_2 \equiv (1, 2), v'_3 \equiv (2, 1), v'_4 \equiv (2, 2), \ldots, v'_{(2p'-1)} \equiv (p', 1), v'_{2p'} \equiv (p', 2) \) can be transformed in to a ‘canonical’ perfect entangling matching \( G_0 \) on the same set of vertices by applying a suitable permutation on the first label of the vertices \( v'_1, v'_2, \ldots, v'_{2p'} \), where, by ‘canonical’ perfect entangling matching, we mean either (i) a cris-cross, or (ii) a tally mark, or (iii) a disjoint union of some tally-markers and/or some cris-crosses (this kind of result is still lacking for a general \( G \in \mathcal{P}_{p,2q} \) and we don’t know what should be the ‘canonical’ form of such a \( G \)). As \( \rho(G_0) \) is known to be separable in \( C_A^p \otimes C_B^p \) (and hence, in \( C_A^p \otimes C_B^p \), as the orthonormal basis \( \{|i_k\}: k = 1, 2, \ldots, q\} \) of \( C_A^p \) is contained inside the orthonormal basis \( \{|l\}: l = 1, 2, \ldots, p\} \) of \( C_A^p \)). Also \( \rho(G_1) \) is separable in \( C_A^p \otimes C_B^p \), as \( G_1 \) consists of only unentangled edges of \( G \). Now \( \rho(G) = \frac{1}{p}[(p-q)\rho(G_1) + q\rho(G_2)] \).

Hence \( \rho(G) \) is separable in \( C_A^p \otimes C_B^p \). \( \square \)
During the proof of Theorem 4, we have proved the following result:

**Corollary 1:** Let $G$ be a perfect matching on $n = p.q$ vertices $v_1 = (u_1, w_1), v_2 = (u_2, w_2), \ldots, v_n = (u_p, w_q)$ for which $\Delta(G) = \Delta(G^T_A)$. Then $G$ is a disjoint union of $N$ no. of perfect matchings $G_1, G_2, \ldots, G_N$, where (i) $V(G_i) = \{(u_{ij}, w_{ik}) : j = 1, 2, \ldots, p_i; k = 1, 2, \ldots, q_i\}$, (ii) $\bigcup_{i=1}^N u_{ij} = \{u_1, u_2, \ldots, u_p\}$ and $\bigcup_{i=1}^N w_{ik} = \{w_1, w_2, \ldots, w_q\}$, (iii) for any two different $i, i' \in \{1, 2, \ldots, N\}$, either $\{u_{ij} | j = 1, 2, \ldots, p_i\} \bigcap \{u_{ij'} | j = 1, 2, \ldots, p_{i'}\} = \emptyset$ or $\{w_{ik} | k = 1, 2, \ldots, q_i\} \bigcap \{w_{ik'} | k = 1, 2, \ldots, q_{i'}\} = \emptyset$ or both, and (iv) for each $i \in \{1, 2, \ldots, N\}$, either $G_i$ consists of only entangled edges or only unentangled edges, but not both.

Thus we see that, for any perfect matching $G$, when the degree condition is satisfied, it is enough to study the separability of the density matrices of its pairwise disjoint entangled subgraphs (i.e., subgraphs each of whose edge is entangled), each of which is a perfect entangling matching on its own right (i.e., it is a perfect entangling matching on a set $S$ of vertices taken from $V(G)$ such that all the elements of $S$ can be labelled by two labels). See Figure 2 for an illustration.

![Figure 2](image_url)

**FIG. 2:** A perfect matching $G$ on 16 vertices $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, $(2,3)$, $(2,4)$, $(3,1)$, $(3,2)$, $(3,3)$, $(3,4)$, $(4,1)$, $(4,2)$, $(4,3)$, $(4,4)$, for which the degree condition is satisfied. $G$ is the disjoint union of of the following four perfect matchings: (i) the unentangled graph: $G_1 = (V(G_1), E(G_1))$ with $V(G_1) = \{(1,1), (1,2), (2,1), (2,2)\}$ and $E(G_1) = \{\{(1,1), (2,1)\}, \{(1,2), (2,2)\}\}$, (ii) the cris-cross: $G_2 = (V(G_2), E(G_2))$ with $V(G_2) = \{(3,1), (3,2), (4,1), (4,2)\}$ and $E(G_2) = \{\{(3,1), (4,2)\}, \{(3,2), (4,1)\}\}$, (iii) the perfect entangling matching: $G_3 = (V(G_3), E(G_3))$ with $V(G_3) = \{(1,3), (1,4), (3,3), (3,4), (4,3), (4,4)\}$ and $E(G_3) = \{\{(1,3), (3,4)\}, \{(1,4), (4,3)\}, \{(3,3), (4,4)\}\}$, and (iv) the unentangled graph: $G_4 = (V(G_4), E(G_4))$ with $V(G_4) = \{(2,3), (2,4)\}$ and $E(G_4) = \{\{(2,3), (2,4)\}\}$. A speciality of the case $n = p.q$ is also reflected in the following Lemma.

**Lemma 1:** $P_{p,q} = P_{p,2}^S$.

**Proof:** By definition, $P_{p,2}^S \subset P_{p,2}$. Let $G \in P_{p,2}$, where $V(G) = \{(k,1) : k = 1, 2, \ldots, p\} \cup \{(k,2) : k = 1, 2, \ldots, p\}$ and $E(G) = \{\{(k,1), (ik,2)\} : k = 1, 2, \ldots, p\}$ where (i) for each $k \in \{1, 2, \ldots, p\}$, the particular element in $\{1, 2, \ldots, p\}$ is $k$ and (ii) $i_k \neq i_l$ whenever $k \neq l$. Thus we see that $G^T_A$ is a graph on $2p$ vertices such that $V(G^T_A) = V(G)$ and $E(G^T_A) = \{(k, 1) : (k, 1) \in E(G)\}$ for all $k = 1, 2, \ldots, p$ with the properties that (i) for each $k \in \{1, 2, \ldots, p\}$, $i_k$ is a particular element in $\{1, 2, \ldots, p\}$ and (ii) $i_k \neq i_l$ whenever $k \neq l$. So $G^T_A$ must be a perfect entangling matching vertex set as $V(G)$. Therefore, $P_{p,2} \subset P_{p,2}^S$. □

The result in Lemma 1 can not, in general, be extended for the case of $P_{p,q}$ if $q > 2$ (see, for example, figures 2 and 3 in [2]).

**(C.2) Properties of general perfect entangling matchings**

If $\rho(H)$ is separable in $\mathbb{C}^p_A \otimes \mathbb{C}^q_B$, where $H$ is the subgraph of a perfect entangling matching $G$ on $pq$ vertices such that $H$ consists of all the entangled edges in $G$, then $\rho(G)$ will be automatically separable in $\mathbb{C}^p_A \otimes \mathbb{C}^q_B$. So the relevant question is: what can we say about separability of $\rho(G)$ whenever $G \in P_{p,q}^S$, with $q > 2$? Note that it is irrelevant to consider separability of $\rho(G)$ for an arbitrary $G \in P_{p,q}$, as $\rho(G)$ is inseparable if $G \in P_{p,q} \backslash P_{p,q}^S$ (because, in that case, the degree condition is not satisfied). As we have mentioned during the proof of Theorem 4, we still don’t have a
‘canonical’ set of perfect entangling matchings on $n = p, q$ vertices, to one (or a disjoint mixture of some of) which, any element of $\mathcal{P}_{p,q}$ can be transformed via local permutation(s) on one or both the labels the vertices. Moreover, even if we have that canonical set, we still don’t have any proof of separability of the corresponding density matrices. But for a particular class of perfect entangling matchings $G$ on $n = p, 2r$ vertices, for each of which the degree condition is satisfied, one can show that $\rho(G)$ is separable in $\mathbb{C}_A^p \otimes \mathbb{C}_B^p$.

Let $G \in \mathcal{P}_{p,2r}$, where $G = \bigcup_{k=1}^r G_{jk,l_k}$, with $V(G_{jk,l_k}) = \{(a, j_k) : a = 1, 2, \ldots, p\}$ and $E(G_{jk,l_k}) = \{(a, j_k), (a', j_k') \} : a = 1, 2, \ldots, p\}$ such that for each $k \in \{1, 2, \ldots, r\}$, (i) $i_a \in \{1, 2, \ldots, p\}$ \{a\} for each $a \in \{1, 2, \ldots, p\}$ and (ii) $j_k, j_k' \in \{1, 2, \ldots, 2r\}$ with the properties that $j_k \neq l_k, j_k \neq j_k'$ (if $k \neq k'$). Thus we see that for each $k \in \{1, 2, \ldots, r\}$, $G_{jk,l_k}$ is a perfect entangling matching on $2p$ vertices $(1, j_k), (1, l_k), (2, j_k), (2, l_k), \ldots, (p, j_k), (p, l_k)$ and with $p$ edges $(1, j_k), (1, l_k), (2, j_k), (2, l_k), \ldots, (p, j_k), (p, l_k)$. So, by Lemma 4.4 of [2], $G_{jk,l_k}$ can be transformed into a canonical perfect entangling matching on same set $V(G_{jk,l_k})$ of vertices. And so, $\rho(G_{jk,l_k})$ is separable in $\mathbb{C}_A^p \otimes \mathbb{C}_B^p$ (and so, by Theorem 2, $\Delta(G_{jk,l_k}) = \Delta(G_{jk,l_k}^{\Gamma_{1k}})$). Therefore, $\rho(G) = \bigoplus_{k=1}^r \rho(G_{jk,l_k})$ is separable in $\mathbb{C}_A^p \otimes \mathbb{C}_B^p$ and $\Delta(G) = \bigoplus_{k=1}^r \Delta(G_{jk,l_k}) = \bigoplus_{k=1}^r \Delta(G_{jk,l_k}^{\Gamma_{1k}}) = \Delta(G_{jk,l_k}^{\Gamma_{1k}})$.

The set of all elements in $\mathcal{P}_{p,2r}$ of which is a disjoint union of exactly $r$ number of elements of $\mathcal{P}_{p,2}$, is denoted here by $\mathcal{E}_{p,2r}$. Let $G \in \mathcal{E}_{p,2r}$. Then, as described above, $G$ is a disjoint union of $r$ elements $G_{j_1,l_1}, G_{j_2,l_2}, \ldots, G_{j_r,l_r}$ of $\mathcal{P}_{p,2}$. Note that each element $G_{j_k,l_k}$ of $\mathcal{P}_{p,2}$ is a disjoint union of $N_k$ number of elements $G_{l_1, (L_{ik}) \in \mathcal{P}_{L_{ik}}} \in \mathcal{P}_{L_{ik}}$, $G_{j_k,l_k} = (L_{ik}) \in \mathcal{P}_{L_{ik}}, \ldots, G_{N_k,l_k} (L_{ik}) \in \mathcal{P}_{L_{ik}}$, such that we can have a further splitting of any $G_{L_{ik}}$ (as a disjoint union of perfect entangling matchings) is possible (see Figure 3 for an illustration). For each $i \in \{1, 2, \ldots, N_k\}$, we must have $V(G_{j_k,l_k} (L_{ik})) = \{a_{m}^{(ik)} : m = 1, 2, \ldots, L_{ik}\}$ and $L_{ik} = \{a_{m}^{(ik)} : m = 1, 2, \ldots, L_{ik}\}$, where $\{a_{m}^{(ik)} : m = 1, 2, \ldots, L_{ik}\} = \emptyset$ if $i \neq i'$ and $\bigcup_{i=1}^{N_k} \{a_{m}^{(ik)} : m = 1, 2, \ldots, L_{ik}\} = \{1, 2, \ldots, p\}$. Now by using Lemma 4.4 of [2], we have

$$
\rho(G_{j_k,l_k} (L_{ik})) = \frac{1}{L_{ik}} \sum_{i=0}^{L_{ik} - 1} P \left[ \frac{1}{\sqrt{L_{ik}}} \sum_{m=1}^{L_{ik}} \right. \exp \left( \frac{2\pi i (m-1)l}{L_{ik}} \right) U_{ik} \left| a_{m}^{(ik)} \right\rangle \right] \otimes P \left[ \frac{1}{\sqrt{2}} \left( |j_k\rangle - \exp \left( -\frac{2\pi il}{L_{ik}} \right) |l_k\rangle \right) \right],
$$

(2.13)

where $U_{ik}$ is the permutation matrix corresponding to a permutation on the labels $a_{1}^{(ik)}, a_{2}^{(ik)}, \ldots, a_{L_{ik}}^{(ik)}$ for $i' \in \{1, 2, \ldots, N_k\}$ and $k = 1, 2, \ldots, r$. So, we have

$$
\rho(G_{j_k,l_k}) = \frac{1}{N_k} \sum_{i'=1}^{N_k} \rho(G_{j_k,l_k} (L_{ik})),
$$

and finally

$$
\rho(G) = \frac{1}{r} \sum_{k=1}^{r} \rho(G_{j_k,l_k}).
$$

(2.14)

Note that the range of $\rho(G)$ (where $G \in \mathcal{E}_{p,2r}$) will always contain at least $pr$ no. of pairwise orthogonal product states, namely the states $\frac{1}{\sqrt{L_{ik}}} \sum_{m=1}^{L_{ik}} \exp \left( \frac{2\pi i (m-1)l}{L_{ik}} \right) U_{ik} \left| a_{m}^{(ik)} \right\rangle \otimes \frac{1}{\sqrt{2}} \left( |j_k\rangle - \exp \left( -\frac{2\pi il}{L_{ik}} \right) |l_k\rangle \right)$ where $\sum_{i=1}^{N_k} L_{ik} = p$ and $k = 1, 2, \ldots, r$. The range of $\rho(G)$ can also contain some other (possibly infinite in number) product states if either (i) $L_{ik} = L_{ik'}$ for different $i, i' \in \{1, 2, \ldots, N_k\}$, or (ii) $\{a_{1}^{(ik)}, a_{2}^{(ik)}, \ldots, a_{L_{ik}}^{(ik)}\}$ for different $k, k' \in \{1, 2, \ldots, r\}$ (but for same $i$). All the above-mentioned $pr$ no. of pairwise orthogonal product states are reliably distinguishable by using local operations and classical communication (LOCC).

Is there any $\mathcal{P}_{p,2r}$ such that $G \notin \mathcal{E}_{p,2r}$? Yes, there are such perfect entangling matchings: for $p = 3, r = 2$, there is (up to local permutations on the labels of right hand and/or left hand sides) one such $G$ which contains
FIG. 3: A perfect entangling matching $G \in P_{6,2}^S$ on the 24 vertices $(1,1)$, $(1,2)$, ..., $(6,4)$. $G$ is a disjoint union of two perfect entangling matchings $G_{13}$ and $G_{24}$ in $P_{6,2}$, where $V(G_{13}) = \{(i,1) | i = 1,2,\ldots,6\}$ and $V(G_{24}) = \{(i,2) | i = 1,2,\ldots,6\} \cup \{(i,4) | i = 1,2,\ldots,6\}$. Thus $G \in P_{6,2}^S$. $G_{13}$ itself is a disjoint union of the cris-cross $G_{113}(2)$ (with $V(G_{113}(2)) = \{(1,1),(3,1)\} \cup \{(1,3),(3,3)\}$ and $E(G(2)) = \{((1,1),(3,3)),((3,1),(1,3))\}$) and the perfect entangling matching $G_{213}(4)$ (with $V(G_{213}(4)) = \{(2,1),(4,1),(5,1),(6,1)\} \cup \{(2,3),(4,3),(5,3),(6,3)\}$ and $E(G_{213}(4)) = \{((2,1),(5,3)),((4,1),(6,3)),(5,1),(4,3)),((6,1),(2,3))\}$). $G_{213}(4)$ can be transformed (via the local permutation $4 \leftrightarrow 5$ on the first label) to a tally mark. And $G_{24}$ is a perfect entangling matching on the vertices $(1,2)$, $(2,2)$, ..., $(6,2)$, $(1,4)$, $(2,4)$, ..., $(6,4)$ which can be transformed (via first applying the local permutation $2 \leftrightarrow 3$, $4 \leftrightarrow 5$ and then applying the local permutation $5 \leftrightarrow 6$, both on the first label) to a tally mark.

neither any cris-cross nor tally mark (see Figure 4). Higher the values of $p$ and/or $r$, higher will be the number of such different $G$’s (not containing cris-crosses or tally marks). From now on, we shall only consider those perfect entangling matchings, none of which contains a cris-cross or tally mark. Let $H \in P_{p,2r}^S \setminus E_{p,2r}$. Is $\rho(H)$ a separable state in $C_A^p \otimes C_B^r$? In order to answer this question, we need to see whether there is any product state (of $C_A^p \otimes C_B^r$) within the range $(R_H)$ of $\rho(H)$. Also we consider here the density matrix $\rho(H_+) = \frac{1}{pr}(I - \rho(H))$, where $I$ is the $2pr \times 2pr$ identity matrix. Let $R_{H_+}$ be the range of $\rho(H_+)$. We have the following conjecture:

**Conjecture 2:** Let $H \in P_{p,2r}^S \setminus E_{p,2r}$ such that $H$ neither contains any cris-cross nor any tally mark. Then the range $R_H$ of $\rho(H)$ (the range $R_{H+}$ of $\rho(H_+)$) contains exactly $pr$ number of product states $|\psi_1\rangle \otimes |\phi_1\rangle, |\psi_2\rangle \otimes |\phi_2\rangle, \ldots, |\psi_{pr}\rangle \otimes |\phi_{pr}\rangle$ of $C_A^p \otimes C_B^r$. Moreover, (i) all these product states are pairwise orthogonal, (ii) all the states $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_{pr}\rangle$ are different but one can always have at least one class of exactly $p$ of them all of which are pairwise orthogonal, (iii) all the states $|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_{pr}\rangle$ are different but one can always have at least one class of exactly $2r$ of them all of which are pairwise orthogonal, and (iv) all the states $|\psi_1\rangle \otimes |\phi_1\rangle, |\psi_2\rangle \otimes |\phi_2\rangle, \ldots, |\psi_{pr}\rangle \otimes |\phi_{pr}\rangle$ are reliably distinguishable by LOCC.

Validity of Conjecture 2 would directly show that for any $H \in P_{p,2r}^S \setminus E_{p,2r}$, which neither contains any cris-cross nor any tally mark, $\rho(H) = \frac{1}{pr} \sum_{j=1}^{pr} P[|\psi_j\rangle \otimes |\phi_j\rangle]$, and hence, $\rho(H)$ is separable in $C_A^p \otimes C_B^r$. If an $H \in P_{p,2r}^S \setminus E_{p,2r}$ contains some cris-crosses and/or tally marks, the rest part of $H$ (eliminating out all these cris-crosses, tally marks) will be again an element of $P_{p,2r}^S \setminus E_{p,2r}$, for some $p' \leq p$ and $r' \leq r$, such that this new graph does not contain any cris-cross or tally mark. As cris-crosses or tally marks always form separable density matrices, therefore we see that for any $H \in P_{p,2r}^S \setminus E_{p,2r}$, $\rho(H)$ is separable in $C_A^p \otimes C_B^r$, provided the above-mentioned conjecture is true. Note that the validity of Conjecture 1 automatically implies that for any $G \in P_{p,2r}$, $\rho(G)$ is separable if $G \in P_{p,2r}^S \setminus E_{p,2r}$. However, the statement in Conjecture 2 is much stronger than just saying that $\rho(G)$ is separable if $G \in P_{p,2r}^S \setminus E_{p,2r}$. 

D. Proof of Theorem 5

Let $\rho$ be a circulant density matrix of dimension $n = pq$. As we already mentioned in the introduction, we can write $\rho = \sum_{g \in \mathbb{Z}_n} f(g)\sigma(g)$ where $f$ is a complex-valued function. Obviously, there will be some constraints imposed by the fact that $\rho$ is positive semidefinite and hermitian. It is well-known that $\rho$ is diagonalized by the Fourier transform $FT(\mathbb{Z}_n)$ over $\mathbb{Z}_n$ [7]: $[FT(\mathbb{Z}_n)]_{j,k} = \exp(2\pi i j k/n)$. The eigenvectors of $\rho$ are then the columns of $(FT(\mathbb{Z}_n))^\dagger$. We prove the theorem in two steps: (1) we prove that if $|\lambda\rangle$ is an eigenvector of $\rho$ then $|\lambda\rangle = |a\rangle \otimes |b\rangle$, where $|a\rangle \in \mathbb{C}^p$ and $|b\rangle \in \mathbb{C}^q$, for any $p$ and $q$ such that $n = pq$. (2) Then, for any chosen $p$ and $q$, we prove that $\Delta(\rho) = \Delta(\rho_{FB})$, where the partial transpose is taken with respect to the standard orthonormal basis $\{|ij\rangle : i = 1,\ldots,q; j = 1,\ldots,p\}$ of $\mathbb{C}^p \otimes \mathbb{C}^q$.

(1) Let $A$ be an $n \times n$ matrix which is diagonalized by a unitary matrix $U$. We take $n = pq$. So $UAU^\dagger = D$, where $D = \text{diag}(\lambda_1,\ldots,\lambda_n)$, with respect to the standard orthonormal basis $\{|\psi_1\rangle,\ldots,|\psi_n\rangle\}$. Thus $A(U^\dagger|\psi_i\rangle) = \lambda_i(U^\dagger|\psi_i\rangle)$ for $i = 1,2,\ldots,n$, and $UAU^\dagger = \sum_{i=1}^n \lambda_i|\psi_i\rangle\langle\psi_i|$. Thus $U^\dagger|\psi_i\rangle$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda_i$. Also, $\langle\psi_i|UU^\dagger|\psi_j\rangle = \langle\psi_i| |\psi_j\rangle = \delta_{ij}$. Let $U$ be any $n \times n$ unitary matrix with its $(j,k)$-th entry as $u_{jk}$ with respect to the orthonormal basis $\{|\psi_1\rangle, |\psi_2\rangle,\ldots,|\psi_n\rangle\}$. Then $U^\dagger = (\omega_{jk})_{j,k=1}^n$ where $\omega_{jk} = u_{kj}^*$ for all $j,k$. Now $U^\dagger|\psi_i\rangle = \sum_{j=1}^n \omega_{ji}|\psi_j\rangle = \sum_{j=1}^n u_{ji}^*|\psi_j\rangle$, which is the $i$-th column of $U^\dagger$. Assuming that $\rho$ is a circulant matrix, with respect to $\{|\psi_1\rangle,\ldots,|\psi_n\rangle\}$, we have

$$U = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{pmatrix}.$$  

The unitary matrix $U$ is the Fourier transform over $\mathbb{Z}_n$ [7]. We can see that

$$\rho = \sum_{j=1}^n \lambda_j P[U^\dagger|\psi_j\rangle]$$

$$= \sum_{j=1}^n \lambda_j \left[ \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \exp \left( -\frac{2\pi i (j - 1) l}{n} \right) |\psi_{l+1}\rangle \right]$$

$$= \sum_{j=1}^n \lambda_j \left[ \frac{1}{\sqrt{n}} \sum_{a=0}^{p-1} \sum_{b=0}^{q-1} \exp \left( -\frac{2\pi i (j - 1) (aq + b)}{n} \right) |a\rangle \otimes \frac{1}{\sqrt{pq}} \sum_{b=0}^{q-1} \exp \left( -\frac{2\pi i (j - 1) b}{pq} \right) |b\rangle \right].$$

It follows that $\rho$ is a separable density matrix provided that $\lambda_j \geq 0$ for $j = 1,\ldots,n$.  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{G is the representative perfect entangling matching on 12 vertices (1, 1), (1, 2), \ldots (3, 4) such that $G \in \mathcal{P}_3(2,2) \setminus \mathcal{E}_3(2,2)$.}
\end{figure}
(2) The general form a circulant density matrix is

\[
\rho = \begin{bmatrix}
a_1 & a_2 & \cdots & a_n \\
a_n & a_1 & \cdots & a_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_2 & a_3 & \cdots & a_1 \\
\end{bmatrix}.
\]

Since the matrix is symmetric, we have

\[ a_1 = a_1^*, a_2 = a_n^*, a_3 = a_{n-1}^*, \ldots, a_l = a_{n-l+2}^*. \tag{2.15} \]

Consider \( \rho \) as a block-matrix with \( p^2 \) blocks, each block being a \( q \times q \) matrix:

\[
\rho = \begin{bmatrix}
A_{1,1} & A_{1,2} & \cdots & A_{1,p} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,p} \\
\vdots & \vdots & \ddots & \vdots \\
A_{p,1} & A_{p,2} & \cdots & A_{p,p}
\end{bmatrix}.
\]

Consider the block \( A_{1,m+1} \). Then \([A_{1,m+1}]_{1,1} = a_{mq+1} \). Let \( l = mq + 1 \). Then

\[
A_{1,m+1} = \begin{bmatrix}
a_l & a_{l+1} & \cdots & a_{l+q-1} \\
a_{l-1} & a_l & \cdots & a_{l+q-2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{l-q+1} & \cdots & \cdots & \cdots
\end{bmatrix}.
\]

Now, consider the block \( A_{1,p-m+1} \). Then \([A_{1,p-m+1}]_{1,1} = a_{n-l+2} = a_{n+1-mq} \). Then

\[
A_{1,n-m+1} = \begin{bmatrix}
a_{n-l+2} & a_{n-l+3} & \cdots & a_{n-l+q+1} \\
a_{n-l+1} & a_{n-l+2} & \cdots & a_{n-l+q} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n-l-q+3} & \cdots & \cdots & \cdots
\end{bmatrix}.
\]

Applying the condition expressed in Equation (2.15), one can verify that

\[ A_{1,m+1} = A_{1,n-m+1}^\dagger. \]

This argument extends to all blocks of the \( i \)-th block-row of \( \rho \). For example, the first block-row of \( \rho \) is then of the form

\[
A_{1,1} \left( = A_{1,1}^\dagger \right) A_{1,2} \ A_{1,3} \ \cdots \ A_{1,p/2+1} \ A_{1,p/2+1}^\dagger \ \cdots \ A_{1,3}^\dagger \ A_{1,2}^\dagger
\]

if \( p \) is even, and

\[
A_{1,1} \left( = A_{1,1}^\dagger \right) A_{1,2} \ A_{1,3} \ \cdots \ A_{1,(p+1)/2} \left( = A_{1,(p+1)/2}^\dagger \right) \ \cdots \ A_{1,3}^\dagger \ A_{1,2}^\dagger
\]

if \( p \) is odd. It is then clear that \( \Delta(\rho) = \Delta(\rho^\Gamma) \), that is the the row sums of \( \rho \) are invariant under the partial transpose. It should be noted here that each element of \( \Delta(\rho) \) (as well as of \( \Delta(\rho^\Gamma) \)) is real due to equation (2.15).

The same reasoning applies to the second part of the theorem. The only difference is that \( \rho = \sum_{g \in \mathbb{Z}_2^2} f(g) \sigma(g) \) is diagonalized by the Hadamard matrices of Sylvester type, \( H^n = H^{n-1} \otimes H \), where \( H \) is the \( 2 \times 2 \) Hadamard matrix [18].

III. OPEN PROBLEMS

In this paper we have studied the separability of a class of states associated with the combinatorial laplacians of graphs. The graphs for these states compactly encodes information about their bipartite entanglement. We have
shown that invariance of the degree matrices under partial transposition gives, in many cases, significant information about the separability of the states. Now the Peres-Horodecki partial transposition condition (known as the PPT criterion) is only a necessary condition (in general) for separability of any bipartite density matrices \[ \{1, 2\} \]. In fact, all the practical separability conditions, available so far, are either necessary or sufficient for general bipartite density matrices (see, for example, \[ \{4\] ). The degree condition, described in this paper, is of course weaker than the PPT criterion, as not all bipartite density matrices (not even the separable ones) can be described as density matrices generated from graphs. Nevertheless the validity of Conjecture 1 (together with Theorem 2) would imply the PPT criterion, as not all bipartite density matrices (no t even the separable ones) can be described as density matrices (see, for example, \[ \{4\] ).

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Acknowledgement SLB currently holds a Royal Society — Wolfson Research Merit Award. SG, SS and RCW acknowledge the support received from the EPSRC. Part of of this work has been carried out while SS was at The Caesarea Rothschild Institute, Israel.

A. (Partial transposition as a local permutation) It is not difficult to see that for any graph on \( n = pq \) vertices \( v_1 = u_1 w_1, \ldots, v_{pq} = u_p w_q \), if \( \Delta(G) = \Delta(G^{T_B}) \) then there is a permutation matrix \( P \) on the labels \( w_1, \ldots, w_q \) such that

\[
\Delta(G^{T_B}) = (I \otimes P) \Delta(G) (I \otimes P^{-1}) \quad \text{and} \quad M(G^{T_B}) = (I \otimes P) M(G) (I \otimes P^{-1}).
\]

This says that if the degree condition is satisfied then the operation of partial transposition is nothing but a local permutation. Note that this is, in general, false for the case of any given bipartite separable density matrix. The relation between separability of density matrices of graphs and isomorphism remains to be studied.

B. (Structure of bipartite Hilbert spaces) The validity of of Conjecture 6 can be traced back to basic problems in the structure of any bipartite Hilbert space. Given a subspace \( S \) of dimension \( d \) of \( \mathcal{H} \), what are the necessary and sufficient conditions under which at least one of the following situations hold good?

1. \( S \) contains at least one linearly independent product state;
2. \( S \) contains only \( d' \) linearly independent product states, where \( d' < d \);
3. \( S \) contains more than \( d \) product states (in which case they must be linearly dependent);
4. \( S \) contains exactly \( d \) linearly independent product states and these are pairwise orthogonal;
5. \( S \) contains only \( d' \), where \( d' \leq d \), pairwise orthogonal product states that one can extend to a full orthogonal product basis of \( \mathcal{H} \), etc.

C. (Multiparty entanglement) As a generalization of our result to density matrices of graphs having multiple labels on their vertices, we expect that if \( G \) is a graph on \( n = p_1 p_2 \ldots p_m \) vertices

\[
v_i = u_{1s_i(1)} u_{2s_i(2)} \ldots u_{ms_i(m)}, \quad \text{where} \quad s_i^{(j)} \in \{1, \ldots, p_j\} \quad \text{for} \quad j = 1, \ldots, m \quad \text{and} \quad i = 1, \ldots, n,
\]

then \( \rho(G) \) is a separable density matrix in \( C_{p_1 p_2 \ldots p_{j-1} p_{j+1} \ldots p_m} \otimes C_{A_j} \) if and only if \( \Delta(G) = \Delta(G^{T_{A_j}}) \). Moreover, we expect that \( \rho(G) \) is a completely separable density matrix in \( C_{A_1} \otimes C_{A_2} \otimes \cdots \otimes C_{A_m} \) if and only if \( \Delta(G) = \Delta(G^{T_{A_j}}) \) for \( j = 1, \ldots, m \).

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