AN ELEMENTARY INTRODUCTION TO YANG-MILLS THEORIES AND TO THEIR APPLICATIONS TO THE WEAK AND ELECTROMAGNETIC INTERACTIONS
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Introduction

Gauge theories have been first considered by Yang and Mills more than twenty years ago. Their aim was to construct a theory where the global isospin symmetry of strong interactions could be extended into a local symmetry, so that one could be allowed to perform independent isospin transformations in different space-time regions.

Invariance under space time dependent symmetry transformations is, in fact, so intimately related to the very notion of a local field theory, that it was apparent from the outset that, if a local field theory had to be relevant at all in describing the fundamental interactions, the Yang-Mills concept had to play a fundamental role.

This explains why gauge theories have appeared, from time to time, in different parts of particle physics.

In strong interactions, the original Yang-Mills theory led to the concept of universal vector meson couplings, D dominance etc., and played an important role in the discovery of the SU(3) symmetry. Consideration of unified gauge theories for weak and e.m. interactions was started by Schwinger as early as in 1957.

Progress has been slow, however, and difficult, for various reasons. Invariance under local gauge transformations leads to massless gauge fields. One had to reconcile this fact with the remarkable absence of massless vector mesons both in strong and in weak interactions. Furthermore, to get a consistent perturbative treatment of gauge theories proved to be a formidable problem, which took a long while to be solved.

In the middle sixties, the introduction into Yang-Mills theories of the notion of spontaneous symmetry breaking provided an appealing way of giving gauge fields a mass (the Higgs mechanism). This opened the way to the construction of concrete unified theories for the weak and e.m. interactions of the known leptons, accomplished by Weinberg and Salam. Consideration of hadronic weak interactions had to wait a little longer, however, until in 1970 it was realized by Glashow, Iliopoulos and Maiani that a new hadronic quantum number (charm) was needed.

On the more formal side, investigations on the quantization and the perturbative expansion of gauge theories, associated, among others, to the names of De Witt, Feynman, Faddeev and Popov, Wettman, culminated in 1971 in the work of 't Hooft, whereby the complete renormalization program for a spontaneously broken gauge theory was accomplished.

This and the subsequent works in this field, put the Yang-Mills theories of weak and e.m. interactions at the same theoretical level as quantum electrodynamics. The experimental discovery of weak neutral currents and the most recent evidence for charmed particles gave them the concrete support of real facts.

The success of the quark model, and the idea of an unbroken color symmetry for strong interactions has led to further developments. Asymptotically free gauge theories have been proposed, where the observed strong interactions arise from a more fundamental, Yang-Mills, interaction of quarks with the colored gluons. Prompted by the observed scaling in deep inelastic processes, the elaboration of such theories has been made possible by the spectacular progress achieved in field theory in recent years (E. Wilson works on renormalization group, the Callan-Symanzik equation, etc.).

It is conceivable, though by no means proved, that all the fundamental interactions (except gravity) are indeed Yang-Mills interactions. In this connection, it is interesting to point out a change which has taken place, in going from the original Yang-Mills theory to the most recent ones.

In the former, strong interactions were associated to the observed isospin symmetry. In the color gauge theories, strong interactions are associated to the hidden color symmetry, while isospin, SU(3) and all other flavor related broken symmetries are to be associated to the gauge group of the weak and e.m. interactions.

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In these lecture notes, we shall deal exclusively with a gauge theory of weak and e.m. processes. The aim is to give an elementary introduction to the subject by discussing the general underlying ideas, and the way these ideas can be put to work in a concrete theory, based on the gauge group $SU(2) \otimes U(1)$.

All our arguments will be carried on at a classical level, or, equivalently, at a level where only t.e. Feynman diagrams (no internal loops) are considered. This limitation has excluded from the outset any reference to strong interaction gauge theories, where e.g. renormalization effects are essential. Finally, while I tried to indicate as clearly as possible where theory makes contact to experiments, no detailed comparison is carried out of the theoretical predictions with the presently available experimental results. For many topics, this is done in the other courses.

The plan of these lectures is as follows. In Section 1 we review the notion of a global symmetry and introduce the associated conserved currents. In Sections 2 and 3 we discuss the general idea of local gauge symmetry, the construction of the Yang-Mills lagrangian, and work out the elementary properties of the resulting interaction. The connection to observed weak interactions is considered in Section 4, where a mass is given to the gauge fields by adding an ad hoc term to the lagrangian. Spontaneous symmetry breaking is discussed in Section 5, and the Higgs mechanism in Section 6, using as a working example the bosonic sector of the Weinberg-Salam model. The weak interactions of leptons are considered in Section 7, where contact is first made to experimentally testable predictions. In Section 8 we discuss the e.m. interaction of the charged intermede bosons. The way one can give a mass to elementary fermions is discussed in Section 9, with reference to the leptons. The general formalism is worked out in detail, and the possible arising of electron and muon number violation is discussed. Hadronic weak interactions are discussed in Section 10, with reference to the four quark model, and the possibility of still more quark types is considered in Section 11. Finally, Section 12 contains a few conclusive remarks, and some comments on the open problems.

Given the pedagogical character of these lectures, very few references are made to original papers. Reference to the original contributions can be found in the general references listed at the end.

1. Global Symmetries

We shall work in the framework of lagrangian field theory. The dynamics is therefore specified by the lagrangian $\mathcal{L}$, which is a function of the various fields (collectively denoted by $\Psi$) and of their 4-space derivatives ($\partial_\mu \Psi$). The action, $S$, is the integral over 4-space of the lagrangian and the equations of motion are obtained by equating to zero the variation of $S$, resulting from infinitesimal, arbitrary variations of the fields. In formulæ:

\begin{align}
\mathcal{L} &= \mathcal{L}(\Psi, \partial_\mu \Psi) \\
S &= \int d^4 x \mathcal{L} \\
0 &= \delta S = \int \left[ \partial_\mu \delta \Psi + \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \right] \delta \Psi
\end{align}

Partially integrating the term with the derivative in eq. (1.3), discarding integrals of 4-divergences, and setting to zero the coefficient of $\delta \Psi$, we get from (1.3) the equation of motion:

\begin{align}
\frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \Psi &= 0
\end{align}

If we have many fields $\Psi_i$, we get an equation like (1.4) for each field component.

We have a global symmetry whenever it is possible to perform an infinitesimal transformation:

\begin{align}
\Psi 
\rightarrow \Psi' = \Psi + \delta \Psi
\end{align}

such that: 1) it leaves $\mathcal{L}$ invariant: $\mathcal{L}(\Psi') = \mathcal{L}(\Psi)$

2) it is the same at all space-time points.

Furthermore, we speak of an internal symmetry when the transformation (1.5) does not mix fields with different space-time properties. Stated differently, internal symmetries correspond to transformations which commute with the space-time transformations (Lorentz transformations and 4-dimensional translations). We will restrict, in the following, to internal symmetries, choosing as a working example the isospin symmetry.

Suppose we have a doublet of fields, which can have either spin 1/2 (e.g. proton and neutron, or u and d quarks) or spin zero (e.g. $K^+$ and $K^0$). We collect these two fields in a single isospinor:

\begin{align}
\Psi = \begin{pmatrix} P \\ N \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}
\end{align}

Consider now the infinitesimal transformations:

\begin{align}
\Psi' = \Psi + \delta \Psi \\
\delta \Psi = i \xi \frac{\gamma_5}{2} \Psi(\chi)
\end{align}

$a,b,\in \mathbb{R}$
where $\tau_a$ are the 2x2 Pauli matrices and $\xi^a$ are three infinitesimal parameters. Here and in the following we understand sum over repeated indices.

The $\tau_a$ matrices obey the commutation rules:

$$[\frac{\tau_a}{2}, \frac{\tau_b}{2}] = i \xi_{abc} \frac{\tau_c}{2} \quad (1.7)$$

($\xi_{abc}$ is the totally antisymmetric tensor, $\xi_{123} = -1$).

We say that $\mathcal{L}$ is isospin-invariant if:

$$\mathcal{L}(\psi') = \mathcal{L}(\psi) \quad \text{i.e.}$$

$$\delta \mathcal{L} = 0$$

An important consequence of this is the existence of a triplet of conserved currents (Noether's theorem):

$$0 = \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi^a} \delta \psi^a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi^a} \partial_\mu \delta \psi^a = \frac{i}{2} \xi^a \left( \frac{\partial \psi^a}{\partial \psi^b} - \frac{\partial \psi^b}{\partial \psi^a} \right) \mathcal{J}^a$$

$$+ \frac{i}{2} \xi^a \left( \frac{\partial \psi^a}{\partial \partial_\mu \psi^b} \right) \mathcal{J}^a$$

Since the first term in the last formula vanishes by (1.4), we see that the currents:

$$\mathcal{J}^a = \frac{i}{2} \xi^a \left( \frac{\partial \psi^a}{\partial \psi^b} - \frac{\partial \psi^b}{\partial \psi^a} \right) \mathcal{J}^a \quad (1.8)$$

are conserved:

$$\mathcal{D}^\mu \mathcal{J}^a = 0 \quad (1.9)$$

A well known consequence of (1.9) is the existence of conserved "charges". We set:

$$\mathcal{I}^a = \int d^3x \mathcal{J}^a (x, t) \quad (1.10)$$

then:

$$\frac{d}{dt} \mathcal{I}^a = \int d^3x \left[ \mathcal{D}^\mu \mathcal{J}^a + \nabla \cdot \mathcal{J}^a \right] = \int d^3x \nabla \cdot \mathcal{J}^a = 0 \quad (1.11)$$

If we quantize the theory by canonical quantization, $\mathcal{I}^a$ become operators whose equal time commutation relations can be easily computed to be:

$$\left[ I^a(t), I^b(t) \right] = i \xi_{abc} I^c(t) \quad (1.12)$$

The operators $\mathcal{I}^a$ give a realization (or representation) of the commutation rules (1.7) which, in turn, correspond to the algebra of the infinitesimal rotations in a 3-dimensional abstract space, the isospin-space. Furthermore eq. (1.11) becomes:

$$\frac{d}{dt} \mathcal{I}^a = i \left[ H, \mathcal{I}^a \right] = 0 \quad (1.13)$$

The operators $\mathcal{I}^a$ commute with the hamiltonian $H$; therefore they connect states with the same energy. All states will thus appear in isospin-multiplets. For example, $\psi \psi$ bound states will form families of isospin multiplets (with $I = 1$ or 0) mass being equal for members of the same multiplet.

We usually split $\mathcal{L}$ into a free and an interaction part:

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$$

where

$$\mathcal{L}_{\text{free}} = \psi^+ (i \gamma^\mu \gamma^\nu \psi \gamma^\mu \psi) \quad \text{(for spin 1/2 fields)}$$

$$\mathcal{L}_{\text{free}} = \psi^+ (i \gamma^\mu \gamma^\nu \gamma^\rho \psi \gamma^\mu \psi) \quad \text{(for spin 0 complex fields)}$$

(We set $\gamma^\mu = \gamma^\mu$, $\gamma^\nu = \gamma^\nu$, $\gamma^\rho = \gamma^\rho$ is the Dirac matrices). If $\mathcal{L}_{\text{int}}$ does not contain derivative couplings (as is frequently, but not always, the case), the explicit form of $\mathcal{J}^a$ can be directly computed from $\mathcal{L}_{\text{free}}$, since $\nabla \psi = \gamma^\mu \partial_\mu \psi$.

We obtain

$$\mathcal{J}^a = \psi^+ \gamma^\mu \gamma^\nu \gamma^\rho \psi \quad \text{(spin 1/2 case)}$$

$$\mathcal{J}^a = -i [ \psi^+ \gamma^\mu \gamma^\nu \gamma^\rho \psi - \psi^+ \gamma^\mu \gamma^\nu \gamma^\rho \psi ] \quad (1.14)$$

$$\mathcal{J}^a = -i \psi^+ \gamma^\mu \gamma^\nu \gamma^\rho \psi \quad \text{(spin 0 case)}$$

An internal symmetry transformation for Fermi fields could involve, besides matrices acting on "internal" indices (like the Pauli matrices), also the Dirac $\gamma^\mu$ matrix ($\gamma^\mu = i \gamma^\mu \gamma^\nu \gamma^\rho \psi$, $\gamma^\mu \gamma^\nu \gamma^\rho \psi = \gamma^\mu \psi$, $\gamma^\mu \gamma^\nu \gamma^\rho \psi = \gamma^\mu \psi$). This is because $\gamma^\mu$ transforms as a pseudoscalar under Lorentz transformations. Thus we may enlarge the set of transformations (1.6), adding the so-called "chiral isospin" transformations:

$$\delta \gamma^a = i \gamma^a \gamma^\mu \gamma^\nu \gamma^\rho \psi \quad a = 1, 2, 3 \quad (1.15)$$

$\gamma^a$ being again three infinitesimal parameters.

Eqs. (1.6) and (1.15) form a new group of infinitesimal transformations, whose generators obey the commutation rules:

$$[ \frac{\tau_a}{2}, \frac{\tau_b}{2} ] = i \xi_{abc} \frac{\tau_c}{2} \quad (1.16)$$

$$[ \frac{\tau_a}{2}, \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \psi ] = i \xi_{abc} \frac{\tau_c}{2} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \psi$$

The chiral algebra commutators can be set into a simpler form, if we express the infinitesimal generators in terms of the so-called left and righthanded generators:

$$L^a = \frac{i}{2} \gamma^\mu \frac{\tau_a}{2} \gamma^\rho \psi \quad (1.17)$$

$$R^a = \frac{i}{2} \gamma^\rho \frac{\tau_a}{2} \gamma^\mu \psi$$

We get:

$$[ L^a, L^b ] = i \xi_{abc} L^c \quad (1.18)$$

$$[ R^a, R^b ] = i \xi_{abc} R^c$$

$$[ L^a, R^b ] = 0$$

For massless Fermi fields (see Appendix I)
\[ Y_F = 2 \times \text{(helicity)}, \text{so that } L^a \text{ acts only on helicity } = -1/2 \text{ states, while } R^a \text{ acts on helicity } = +1/2 \text{ states. Eq. (1.18) show that chiral transformations are nothing but independent (hence commuting) isospin rotations performed over the two helicity states of the fermion field. (Recall that for massless particles the helicity is a Lorentz invariant quantity). The group structure of eqs. (1.16) or (1.15) is usually referred to a chiral } SU(1/2) \odot SU(2).\]

Chiral transformations can be a symmetry of \( \mathcal{L} \) only in the case of massless fermions. We can in fact compute the variation of \( \mathcal{L}_{\text{free}} \) under chiral transformations. Recalling that \( Y_F \) anticommutes with \( Y_\mu \) , one finds:

\[ \delta \mathcal{L}_{\text{free}} = -2i \eta^a \bar{\psi} \gamma_\mu \tau^a \mathcal{L} \psi \]

so that \( \mathcal{L}_{\text{free}} = 0 \) only when \( m = 0 \).

2. **Gauge Transformations**

The existence of a symmetry expresses the fact that certain choices are purely conventional and have no effect on the dynamics. Isospin symmetry, for example, implies that we can choose as we please the orientation of the axes in isospin space. Consequently, the definition of the fields to be associated to the proton (isospin "up") and to the neutron (isospin "down") is entirely conventional.

A global symmetry implies however that once we have fixed what we define to be isospin "up" at a given point in space-time, we must maintain the same definition at all other points. This seems to be rather unnatural and not in line with the general ideas underlying the concept of a local field theory, where it is meaningful to compare different quantities only at the same point, and not at distant points. It seems therefore desirable and legitimate to investigate theories where the invariance under the global transformations (1.6) is extended to include transformations which can be different at different space-time points, e.g. transformations of the form:

\[ \delta \psi = i \xi^a(x) \tau^a \psi(x) \]

\( \xi^a(x) \) being now infinitesimal, arbitrary functions of the four coordinates, \( x \). In such theories the orientation of the isospin axes, besides being conventional as before, can be chosen at will at any spacetime point \( x \), irrespectively of the orientation we have chosen at any other space-time point \( x' \neq x \).

Requiring invariance under spacetime dependent transformations is by no means a trivial constraint. On the contrary, such a "geometrical" principle will force us to restrict to a particular class of field theories, gauge theories, where a set of vector fields (gauge fields), interact with the other fields in a perfectly prescribed manner. A similar situation is encountered in general relativity, where a "geometrical" invariance principle (namely invariance under general coordinate transformations) leads to a prescribed form of the interaction of matter with the gravitational field.

To understand the problems which arise in enlarging a global symmetry into the symmetry under space-time dependent transformations, let us consider the case of a free, spin 1/2, isodoublet, whose lagrangian is:

\[ \mathcal{L}_{\text{free}} = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi \]

While \( \mathcal{L}_{\text{free}} \) is invariant under (1.6), it is not invariant under (2.1), the reason being that \( \mathcal{L}_{\text{free}} \) transforms quite differently from \( \psi \) itself, so that the term \( \bar{\psi} \gamma^\mu \partial_\mu \psi \) is not invariant. In fact:

\[ \delta (\bar{\psi} \gamma^\mu \partial_\mu \psi) = i \gamma^\mu \tau^a \partial_\mu \psi \]

\[ \delta (\bar{\psi} \gamma^\mu \partial_\mu \psi) = i \gamma^\mu \tau^a \partial_\mu \psi \neq 0 \]

A similar situation is found in classical electrodynamics. There, we describe a charged particle by a complex field \( \psi \), charge conservation being related, by the Noether's theorem, to the invariance under global phase transformations:

\[ \psi \rightarrow \psi' = \psi + \delta \psi \]

\[ \delta \psi = i \xi \psi \]

(\( \xi \) is an infinitesimal constant). As is well known, however, the electrodynamics lagrangian is also invariant under space-time dependent transformations. This is because when we subject \( \psi \) to space-time dependent phase transformations, \( \xi = \alpha(x) \), the terms analogous to the dangerous terms in (2.2), namely the terms proportional to \( \partial^\mu \alpha \), arising from the variation of \( \bar{\psi} \psi \), are exactly compensated if we subject, at the same time, the e.m. field \( A_\mu \) to a gauge transformation:

\[ A_\mu \rightarrow A_\mu' = A_\mu + \delta A_\mu \]

\[ \delta A_\mu = - \partial_\mu \alpha \]

(\( e \) = electric charge). This compensation, in turn, can be traced back to the fact that the e.m. field \( A_\mu \) is introduced in the lagrangian either through the "minimal prescription":

\[ \bar{\psi} \psi \rightarrow (\bar{\psi} + i e A_\mu \psi) \]
or through the gauge invariant quantity:
\[ F^a_{\mu \nu} = \partial^\mu A^a_{\nu} - \partial^\nu A^a_{\mu} \]

Similarly to electrodynamics, to enforce the symmetry under (2.1), we are therefore led to introduce a set of vector fields \( A^a_{\mu} \) (one for each group generator), which will be assumed to transform in such a way that the field combination:
\[ \nabla^\mu \Psi^* \equiv (\partial^\mu + \frac{1}{2} F^a_{\mu \nu} \frac{1}{2} A^a_{\nu} ) \Psi^* \]  
transforms precisely like \( \Psi^* \):
\[ \delta (\nabla \Psi) = i \epsilon^a(x) \frac{1}{2} F^a_{\mu \nu} \nabla^\nu \Psi^* \]

In eq. (2.3), \( g \) is a coupling constant (analogous to the electric charge \( e \)) which describes the interaction of \( A^a_{\mu} \) with the field \( \Psi^* \). The requirement that (2.3) transforms according to (2.4) can then be easily shown (See Appendix II) to lead uniquely to:
\[ \delta A^a_{\mu} = - \epsilon_{abc} \epsilon^b(x) A^c_{\mu} - \frac{1}{2 g} \nabla^\mu \epsilon^a(x) \]  

In the following we will refer to \( A^a_{\mu} \) and \( \nabla \Psi \) with the words "gauge fields" and "covariant derivative", respectively. Also, in analogy to electrodynamics, we will call the transformations (2.1) and (2.5) "gauge transformations".

The usefulness of the covariant derivative lies in the fact that, due to (2.4), any function of \( \Psi \) and \( \nabla \Psi \) which is invariant under the global transformations (1.6) is also invariant under (2.1). This is clear from the fact that neither (2.1) nor (2.4) depend upon \( \partial_{\alpha} \epsilon^a(x) \), and therefore any function of \( \Psi \) and \( \nabla \Psi \) will behave precisely in the same way for constant or space-time dependent transformations.

This observation gives us the clue to the construction of lagrangians invariant under space-time dependent transformations.

Consider a lagrangian \( \mathcal{L}(\Psi, \nabla \Psi) \) describing all the interactions of \( \Psi^* \), except for the interaction with the gauge fields \( A^a_{\mu} \), and assume \( \mathcal{L} \) to be invariant under (1.6). Then, by the above observation, the new lagrangian obtained by the "minimal prescription"
\[ \nabla_{\mu} \Psi^* \rightarrow \nabla_{\mu} \Psi^* \]

namely, the lagrangian:
\[ \mathcal{L}(\Psi, \nabla \Psi) \]  

is invariant if we subject \( \Psi^* \) to the transformations (2.1) and \( A^a_{\mu} \) to (2.5). Note that (2.6) contains a perfectly prescribed way the interaction of \( \Psi^* \) with \( A^a_{\mu} \).

The lagrangian (2.6) cannot still be the total lagrangian. In fact, since \( \mathcal{L}(\Psi, \nabla \Psi) \) is at most quadratic in \( \partial_{\mu} \Psi^* \), \( \mathcal{L}(\Psi, \nabla \Psi) \) will be at most linear in the first derivatives of \( A^a_{\mu} \), while we need terms quadratic in \( \partial_{\mu} A^a_{\mu} \), in order to obtain meaningful (i.e. 2nd order) equations of motion for the fields \( A^a_{\mu} \).

To accomplish this, we have to construct a gauge invariant lagrangian for the fields \( A^a_{\mu} \) alone. Following again electrodynamics, one defines the gauge covariant curl:
\[ F^a_{\mu \nu} = \partial^\mu A^a_{\nu} - \partial^\nu A^a_{\mu} - g \epsilon_{abc} A^b_{\mu} A^c_{\nu} \]  

A straightforward, if not simple, algebra (see Appendix II) shows that \( F^a_{\mu \nu} \) transforms linearly, if we subject \( A^a_{\mu} \) to (2.5); namely:
\[ \delta F^a_{\mu \nu} = - \epsilon_{abc} \epsilon^b(x) F^c_{\mu \nu} \]

Hence:
\[ \mathcal{L}_{YM} = - \frac{i}{4} F^a_{\mu \nu} (F^a)^{\mu \nu} \]

is gauge invariant and it is the required lagrangian for the Yang-Mills gauge fields alone.

We close this section with a number of observations on the properties of the covariant derivatives and on the extension of the Yang-Mills idea to symmetry groups other than the isospin group, \( SU(2) \).

First note that, by eqs. (1.6) and (2.3), the covariant derivative of \( \Psi^* \) can also be expressed as:
\[ \nabla_{\mu} \Psi^* = \partial_{\mu} \Psi^* + g A^a_{\mu} \frac{\delta \Psi^*}{\delta A^a_{\mu}} \]

where \( \delta \Psi^* \) is the variation of \( \Psi^* \) under the global transformation (1.6). Eq. (2.10) allows us to write down the covariant derivative of any field or function of fields, provided we know how do they transform under (1.6). Examples:

1) consider an invariant field or function of fields, \( \mathcal{L}(x) \).
\[ \delta \mathcal{L}(x) = 0 \]

hence
\[ \nabla_{\mu} \mathcal{L}(x) = \partial_{\mu} \mathcal{L}(x) \]

2) consider a field \( \delta \Psi \) with isospin greater than 1/2 (e.g. I = 2). Then
\[ S \bar{\Phi} = i \varepsilon^a \mathbf{T}^a \Phi \]

\( \mathbf{T}^a \) being the appropriate matrices describing infinitesimal rotations over the space of the given isospin. \( \mathbf{T}^a \) obey precisely the same commutation rules as \( \tau_a \). Then

\[ \nabla_{\varepsilon} \Phi = \nabla_\varepsilon \Phi + i \varepsilon^a \mathbf{T}^a \Phi \]

(iii) Suppose \( A(x) = \psi^-(x) \Phi(x) \) and \( \Phi \) transforming in a given way under (1.6). Then:

\[ \nabla_{\varepsilon} A = (\nabla_{\varepsilon} \psi) \Phi + \psi^- (\nabla_{\varepsilon} \Phi) \]

\[ \nabla_{\varepsilon} A = (\nabla_{\varepsilon} \psi) \Phi + \psi^- (\nabla_{\varepsilon} \Phi) \]

As this example shows, the covariant derivative shares many properties of the usual derivative. However, covariant derivatives do not commute:

\[ (\nabla_{\varepsilon} \nabla_{\varepsilon} - \nabla_{\varepsilon} \nabla_{\varepsilon}) \Phi = i \varepsilon^a \mathbf{F}^a \mathbf{T}^a \Phi \]

\( \mathbf{T}^a \) being the matrices appropriate to \( \Phi \) (i.e. \( \mathbf{\tau}^a = \mathbf{\tau}^a/2 \) if \( \Phi \) has isospin 1/2).

(iv) By (2.10), we see that the covariant derivative of any field is determined by the behavior of the field under global transformations. One could therefore try to define a "covariant derivative" of the gauge fields themselves (which transform as \( I = 1 \) fields under global transformations) and try to set:

\[ F_{\mu \nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \]

It turns out that this is wrong. The right hand side of this equation is not equal to (2.7), and does not transform linearly under (2.1). The "covariant derivative" of \( A^a_\mu \) has no meaning. We can, of course, define covariant derivatives of \( F_{\mu \nu}^a \):

\[ (\nabla_\mu F_{\mu \nu}^a) = \partial_\mu F_{\mu \nu}^a - \varepsilon_{abc} A_{\mu}^b F_{\mu \nu}^c \]

The extension of the Yang-Mills formalism to other groups is entirely trivial. For simple groups, i.e. such that we cannot divide the generators into two or more sets of mutually commuting generators, we simply substitute, in the previous formulae, \( \tau_a/2 \) with matrices (call them \( \mathbf{\tau}^a/2 \)) which obey the commutation rules appropriate to the group:

\[ \left[ \frac{\mathbf{\tau}^a}{2}, \frac{\mathbf{\tau}^b}{2} \right] = i \varepsilon_{abc} \frac{\mathbf{\tau}^c}{2} \]

and further replace \( \varepsilon_{abc} \) by \( f_{\mu \nu \lambda} \), the structure constants of the group. Again only one coupling constant appears. If the group is semi-simple (i.e. the generators can be divided in mutually commuting sets, as is the case, for example of chiral \( SU(2) \otimes SU(2) \)), or it contains abelian factors, the same holds true, but we can have different couplings for each simple factor and for each abelian factor (this is why the Weinberg-Salam model, which is based on the group \( SU(2) \otimes U(1) \), has two independent couplings).

3. Properties of the Yang-Mills Interactions

We have seen in the last section that it is possible to obtain a well defined gauge invariant interaction of the fields \( \psi^- \) (which we will call "matter fields") with the gauge fields \( A^a_\mu \), by the so-called "minimal prescription". The rule was:

(i) to consider the lagrangian \( \mathcal{L} (\psi, \bar{\psi}) \) of the matter fields without \( \psi^- A \) interaction;

(ii) to make the substitution \( \bar{\psi} \psi \rightarrow \nabla_\mu \psi \);

(iii) add the Yang-Mills lagrangian (2.9). In this way, one arrives to the total lagrangian:

\[ \mathcal{L}_{\text{tot}} = -\frac{1}{4} F_{\mu \nu}^a (F^a)^{\mu \nu} + \mathcal{L} (\psi, \nabla_\mu \psi) \quad (3.1) \]

In the case where \( \psi^- \) has no other interaction but the Yang-Mills one, \( \mathcal{L} (\psi, \nabla_\mu \psi) \) is the free lagrangian, which is totally determined once we know the kinematic properties of \( \psi^- \) (spin, isospin, mass). In this case, \( \mathcal{L}_{\text{tot}} \) is completely determined by kinematics and by the requirement of gauge symmetry. This uniqueness property is a special feature of Yang-Mills theories, and it makes them very attractive for describing the fundamental interactions.

In fact, many think the e.m., weak and strong interactions are to be described by Yang-Mills interactions. If this were the case, knowing which are the fundamental fields (e.g. lepton and quarks) and what is the gauge group, one would determine, from the gauge principle, the form of all interactions, except gravity. At present, this program is far from being completed. To restrict to weak and e.m. interactions, we will see later that one has to introduce new, yet unseen, scalar particles whose couplings to leptons and quarks are largely undetermined. Even in this circumstance, however, the interaction of gauge fields with the matter fields is determined by the gauge symmetry, and it has many peculiar features which we will now illustrate, restricting to a very simple case.
Let us therefore consider the case of a spinor, isodoublet matter field (e.g. electron and neutrino, degenerate in mass) interacting only with the Yang-Mills (isotriplet) gauge fields.

In this very simple case:
\[
\mathcal{L}_{\text{int}} = -\frac{i}{4} F_{\mu \nu}^a \left( \bar{\psi} \gamma^\mu A^\nu - \bar{\psi} A^\mu \gamma^\nu \right) + \bar{\psi} i \gamma^5 \gamma^\mu \gamma^\nu \psi
\]
\[
\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}
\]  
(3.2)

\[\Delta \psi\] is given by (2.3).

Written explicitly, the lagrangian (3.2) gives rise to terms with different degrees of homogeneity in the fields (e.g. terms bilinear, trilinear etc. in the fields):

\[
\mathcal{L} = -\frac{1}{4} \epsilon_{abc} \epsilon^{def} A^a \left( \partial_\mu A^\nu - \partial_\nu A^\mu \right) + \partial_\mu A^a \gamma^\mu
\]
\[
- \partial_\mu \bar{\psi} \gamma^\mu \psi + g \epsilon_{abc} A^a \bar{\psi} \gamma^\mu A^b \psi
\]
\[
- \frac{1}{4} \epsilon_{abc} \epsilon_{def} A^a A^b A^c A^d A^f
\]
\[
\equiv - \frac{1}{2} \bar{\psi} \gamma^\mu A_\mu \psi + g \bar{\psi} A^a \gamma^\mu A_\mu 
\]
\[
+ \frac{1}{2} \bar{\psi} \gamma^\mu A_\mu \psi
\]
\[
(3.3)
\]

Note that when \( g \rightarrow 0 \), \( \mathcal{L} \) reduces to the quadratic terms, which are therefore indicated as the free part, \( \mathcal{L} \) free, of \( \mathcal{L} \). According to the usual methods in field theory:

i) the bilinear terms, "\( A^2 \)" and "\( \bar{\psi} \psi \)"

describe the free propagation of the particles associated to the \( A \) and \( \psi \) fields; we shall discuss the "\( A^2 \)" term in detail, the "\( \bar{\psi} \psi \)" term leading to the familiar fermion propagator, as e.g. in standard quantum electrodynamics (QED);

ii) Higher than bilinear terms describe interactions of the fields; we shall discuss the "\( A \bar{\psi} \psi \)" term and, later on, the "\( A^3 \)" term;

iii) Unlike QED, the lagrangian (3.2) contains a self interaction of the gauge fields with themselves, represented by the "\( A^4 \)" and "\( A^3 \)" terms.

These terms are uniquely determined by the structure of \( F_{\mu \nu}^a \), i.e. by the principle of gauge covariance. Their presence is a necessary consequence of the fact that the gauge fields are coupled to all fields carrying a non vanishing isospin, and therefore also to themselves (eq. (2.5) indicates \( I = 1 \) for \( A_\mu^a \)).

The self-coupling of gauge fields is a very crucial feature of Y-M theories. It is the main reason why such theories are not a mere transcription of QED, and it is responsible for all the difficulties encountered in getting a consistent treatment to all orders in perturbation theory. In fact, Y-M theories are more similar, in this respect, to quantum gravity than they are to electrodynamics (the graviton is similarly coupled to all forms of energy, including its own). Similarly to gravity, a Y-M theory is not a free theory even in the absence of matter fields.

We restrict now to the "\( A^3 \)" terms, to derive the equations of motion in the free limit, \( g = 0 \). Applying eq. (1.4) to \( \mathcal{L} \) free \( \equiv \mathcal{L} \), and observing that:

\[
\frac{\partial \mathcal{L}}{\partial A^a_{\mu}} = 0 \quad ; \quad \frac{\partial \mathcal{L}}{\partial A^a_{\mu}} = g \epsilon_{abc} A^b \eta^5 \psi - \partial_\mu \bar{\psi} \gamma^\mu A_\mu 
\]

we get:

(\( \Box \equiv \partial_\mu \partial^\mu \) )

\[
\left( \gamma^\mu \partial_\mu - \partial_\mu \gamma^\mu \right) A_\mu^a (\alpha) = 0
\]  
(3.4)

Taking the Fourier transform of (3.4) ( \( i \partial_\mu \rightarrow k_\mu \) ), we get

\[
G^{\beta \lambda} (k) A_\lambda^a (k) \equiv \left( -\delta^{(4)} (k^2 \gamma^5 k \cdot k) \right) A_\lambda^a (k) = 0
\]  
(3.5)

As is well known, the inverse of the matrix \( G^{\beta \lambda} \)

is the propagator for the \( A^- \)-field, i.e. the Fourier transform of the amplitude for finding a gauge particle of given type at the space-time point \( x \), if it has been created at \( x = 0 \). However, the matrix \( G^{\beta \lambda} \)

has no inverse ! In fact, Lorentz covariance restricts \( [G^{\beta \lambda}]^{-1} \) to be of the form:

\[
G^{-1}_{\beta \lambda} = A k^2 \delta_{\beta \lambda} + B k_\beta k_\lambda
\]

\( A \), \( B \) being functions of \( k \).

But:

\[
\left( -\delta_{\beta \lambda} k^2 + k_\beta k_\lambda \right) \left( A k^2 \delta_{\beta \lambda} + B k_\beta k_\lambda \right) =
\]

\[
= A \left( -\delta_{\beta \lambda} k^2 + k_\beta k_\lambda \right)
\]

which cannot be equal to \( \delta_{\beta \lambda} \) for any function \( A \) and \( B \). This is a well-known difficulty, first found in quantizing the photon field. It has to do with the fact that, by gauge invariance, the four components of \( A_\lambda^a \) are not independent dynamical degrees of freedom. For example, by a suitable gauge transformation we may require

\[
\delta^a \partial_\mu A^a \equiv 0
\]  
(3.6)

everywhere. One possible way out is to substitute (3.6) into (3.4). We thus get
\[ \square A^\alpha_\beta(x) = 0 \]

or, in Fourier space:
\[ g^{\alpha\lambda} k^\lambda A^\alpha_\lambda(k) = 0 \]

and we find the so-called Feynman propagator:
\[ D_{\mu \nu}(k) = g^{\mu \nu}(k) = \frac{\varrho_{\mu \nu}}{k^2} \quad (3.7) \]

(To reproduce the correct space-time behavior of propagators, here and in the following one should add a positive imaginary part, \( + i \varepsilon \), to the denominators; furthermore, the correct propagator has an additional \((-1)\) factor.) Alternatively, we can take into account (3.6) by the method of Lagrange multipliers. We add to \( L_{\text{free}} \), a term:
\[ -\frac{1}{2\alpha} \left( \partial^\alpha A^\alpha_\lambda \right) \left( \partial^\lambda A^\alpha_\lambda \right) \]

which vanishes when (3.6) holds, and then take the variation of the lagrangian, with fixed \( \alpha \). We get, in place of (3.4), the equation:
\[ \left[ -\partial^\alpha \square + (\alpha - \frac{1}{2}) \partial^\alpha \partial^\lambda \right] A^\alpha_\lambda(x) = 0 \]

which leads to the propagator (again up to a \((4\alpha)\) factor):
\[ D_{\mu \nu} = \frac{1}{k^2} \left[ \varrho_{\mu \nu} - (1 - \alpha) \frac{k_\mu k_\nu}{k^2} \right] \quad (3.8) \]

(3.8) gives back (3.7) for \( \alpha = 1 \), while \( \alpha = 0 \) leads to the so-called Landau gauge propagator. Physical results will be, of course, independent from \( \alpha \).

Propagators in momentum space have poles at values of \( k^2 \) equal to the (mass)^2 of the propagating particle. The previous equations then show that the gauge fields are to be associated to massless particles.

We turn now to the interaction terms, restricting to the \( A \cdot \bar{\psi} \psi \) interaction. The \( "A\bar{\psi}\psi" \) term in (3.3) can be written as:
\[ g A^\alpha_\mu \bar{\psi}^\alpha \psi = -g A^\alpha_\mu (\Lambda^\alpha)^\mu \quad (3.9) \]

where \( \Lambda^\alpha_\mu \) is the Noether current, associated to the global 1-spin symmetry of the lagrangian before the introduction of the gauge fields (see eq. (1.14)).

Eq. (3.9) would remain valid also in presence of further matter fields, \( J^\alpha_\mu \) being in that case the total Noether current associated to 1-spin conservation, in absence of the gauge fields. Eq. (3.9) expresses the universality of the coupling of gauge fields to 1-spin carrying matter fields.

It is important to observe that \( J^\alpha_\mu \) is not conserved, in the presence of gauge fields. Indeed, applying Noether's theorem to (3.2), one derives that the total, conserved 1-spin current is now:
\[ J^\alpha_\mu = \chi_{\mu \nu \rho} \partial^\nu J^\rho_\mu \quad (3.10) \]

The additional term reflects again the fact that the gauge fields themselves carry a non-vanishing 1-spin.

Matrix elements of \( \mathcal{A}_{\mu \nu} \) give the amplitude for emission and absorption of a gauge particle by an electron or neutrino.

We can apply the previous considerations to study the scattering of two fermions by the exchange of a gauge field. This is illustrated in Fig. 1, which gives a picture of the process in space-time.

The amplitude for such a process is the product of three terms:
- (production amplitude at \( x \))
- (amplitude for propagating from \( x \) to \( x' \))
- (absorption amplitude at \( x' \))

to be integrated over all \( x \) and \( x' \). In momentum space, this is simply the product:
\[ \mathcal{A} = g^2 J^\alpha_\mu \mathcal{D}^{\mu \nu}(k) J^\nu_\mu \quad (3.11) \]

where \( J^\alpha_\mu \) (\( J^\alpha_\mu' \)) is the Fourier transform of the matrix element of \( J^\alpha_\mu (x) (J^\alpha_\mu(x')) \) between ingoing and outgoing fermions and \( \mathcal{D}^{\mu \nu} \) is the 4-momentum transferred by the vector particle. In this particular case, it is easy to see that the currents \( J^\alpha_\mu (x) \) and \( J^\alpha_\mu (x') \) are conserved, so that:
\[ k^\mu J^\alpha_\mu = k^\mu J^\alpha_\mu' = 0 \quad (3.12) \]

This is so because we assume the external particles to be real (on the mass-shell) particles. A real, free particle cannot irradiate gauge mesons, so that there will be no loss of isospin from e.g. the \( \gamma \)-line, except for that which flows into the exchanged gauge field. More formally, the matrix element of the conserved current (3.10) coincides, to the order here considered, with the matrix element of \( J^\alpha_\mu \) alone. Hence the latter is conserved. Eq. (3.12) makes it
irrelevant the choice between the two propagators (3.7) or (3.8). In either case we get (we omit from now on the isospin index, which is irrelevant)

$$A = \frac{i}{k^2} \sum_{\rho} \mathcal{J}_{\rho} \cdot \mathcal{A}_{\rho} \cdot \mathcal{J}_{\rho} = 0$$

$$\left( \mathcal{J}_z \mathcal{J}_o - \mathcal{J}_3 \mathcal{J}_3' + \mathcal{J}_4 \mathcal{J}_4' - \mathcal{J}_5 \mathcal{J}_5' \right)$$

(3.13)

Let us denote by $\mathcal{A}_{2}$ the space part of $\mathcal{A}_{2}$ and by $\omega$ its 4th component, $(\omega = \omega z \cdot \mathcal{A}_{2})$, and choose $\omega$ in the direction of the $z$-axis. We can use again (3.12) to eliminate $\mathcal{J}_3$ and $\mathcal{J}_3'$:

$$\mathcal{J}_3 = \frac{\omega}{q} \mathcal{J}_o' \quad ; \quad \mathcal{J}_3' = \frac{\omega}{q} \mathcal{J}_o$$

We find:

$$A = -\frac{\omega}{q^2} \left( \frac{1}{q^2} \mathcal{J}_o \mathcal{J}_o' + \frac{\omega}{q^2} \left( \mathcal{J}_4 \mathcal{J}_4' + \mathcal{J}_5 \mathcal{J}_5' \right) \right)$$

(3.14)

The two terms in (3.14) have a very simple interpretation. The first one represents an instantaneous, coulombic interaction among the fermions. Indeed, if we Fourier transform back to $x$ space:

$$\mathcal{A}_{2} \rightarrow \frac{\omega}{q} \mathcal{J}_o' \left( t - t' \right) \frac{1}{|x - x'|}$$

which is just the Coulomb interaction; also $\mathcal{J}_o$ and $\mathcal{J}_o'$ give the "charge" densities of the two external particles. The appearance of Coulomb forces confirms that we are exchanging massless particles. The second term in (3.14) has a pole when $q^2 = \omega^2$. At the pole, the amplitude describes the propagation of free waves between $x$ and $x'$. The bracket in (3.14) indicates that we have only two types of waves, those generated by $\mathcal{J}_5$ (and absorbed by $\mathcal{J}_5'$) and those generated by $\mathcal{J}_5$. We can transform the square bracket in (3.14) according to:

$$\mathcal{J}_o \mathcal{J}_o' + \mathcal{J}_5 \mathcal{J}_5' = \left( \frac{\mathcal{J}_4 + \mathcal{J}_4'}{\sqrt{2}} \right) \left( \frac{\mathcal{J}_4' - \mathcal{J}_4}{\sqrt{2}} \right) +$$

$$+ \left( \frac{\mathcal{J}_5 + \mathcal{J}_5'}{\sqrt{2}} \right) \left( \frac{\mathcal{J}_5' - \mathcal{J}_5}{\sqrt{2}} \right)$$

Under a space rotation of an angle $\Theta$ around the $z$-axis:

$$\mathcal{J}_4 \pm i \mathcal{J}_5 \rightarrow e^{\pm i \Theta} \left( \mathcal{J}_4 \pm i \mathcal{J}_5 \right)$$

Such a behavior is typical of the eigenstates of angular momentum in the $z$-direction. An eigenstate with angular momentum along $z$ equal to $\mathcal{J}_z$ picks up, under such a rotation, a factor $e^{\pm i \Theta \mathcal{J}_z}$. Hence we see that the two waves have $\mathcal{J}_z = \pm \mathcal{J}_z$.

In conclusion, we have learned that gauge fields describe particles which are:

1) massless;
2) exist in only two polarization states (namely $\mathcal{J}_z = \pm \mathcal{J}_z$, if they propagate along the $z$-axis);
3) couple to matter fields universally, through the current $\mathcal{J}_\alpha^A$ which is associated by the Noether's theorem to the global symmetry of matter fields.

4. Unbroken Yang-Mills theories and weak interactions

In this section we want to have a first look to the possible applications of the Yang-Mills theory to real weak interactions. Can we describe $\gamma$-electron scattering or $\mu$-decay or the neutron $\beta$-decay by a process similar to that illustrated in Fig. 1? The inspection of the relevant amplitude, (3.13), reveals two features, one very good and one very bad.

The amplitude contains the product of two currents (one for each fermion line), the currents themselves being those currents associated to a global symmetry of the theory without gauge interactions. It is precisely so in weak interactions. The amplitude e.g. for the neutron $\beta$-decay is indeed proportional to the product of two currents, (one changing $M \rightarrow P$, the other creating the lepton pair) which are indeed the currents associated to some global symmetry. Discovering the relations between weak currents and particle symmetries (the so-called CVC hypothesis of Feynman and Gell-Mann, the Cabibbo theory, the relation with chiral symmetry etc.) has been in fact one of the main lines of progress in weak interactions. An underlying gauge theory would give a solid foundation to this fact. This is the good thing.

The bad feature of (3.13) is the factor $\sqrt{\mathcal{A}}/k^2$, related to the masslessness of the gauge particles. There is no trace of massless bosons in weak interactions. If weak interactions are to be mediated by vector bosons, they must be on the contrary very heavy. How can we overcome this trouble? One possibility is to add by brute force a mass term

$$\mathcal{H} = \frac{1}{2} A_{\mathcal{A}} A_{\mathcal{A}} + \frac{1}{2} M^2 A_{\mathcal{A}} A_{\mathcal{A}} + \frac{1}{2} M^2 A_{\mathcal{A}} A_{\mathcal{A}}$$

(3.1) to the lagrangian

Such a term is not gauge invariant, so we are contradicting the philosophical bases of Y-M theory.

Let us see, nonetheless, what happens.

If we add a mass term to the free lagrangian in
(3.3), we get the new equation of motion:
\[
\left( g^\beta_\lambda \square - \partial^\beta \partial_\lambda + m^2 g^\beta_\lambda \right) A^\alpha_\lambda (x) = 0
\]
i.e., in momentum space:
\[
\left( -g^\beta_\lambda k^2 + k^\alpha k^\lambda + m^2 g^\beta_\lambda \right) A^\alpha_\lambda (k) = 0
\]
The operator acting on \( A^\alpha_\lambda \) has now an inverse (we have broken the gauge invariance) and we get the propagator:
\[
D^\alpha_\nu (k, \mu) = \frac{1}{k^2-M^2} \left( -g^\alpha_\nu + k_\nu k_\mu \right) \frac{1}{H^2} (4.1)
\]
If we compute again the amplitude for Fig. 1, we get now:
\[
A = \frac{g^2}{k^2-M^2} J_{\nu} J^{\prime\mu} (4.2)
\]
(since the currents are still conserved, the \( H^2 \) term in (4.1) has no effect). Finally, in the case where \( M^2 >> k^2 \), we get:
\[
A = -\frac{g^2}{M} J_{\nu} J^{\prime\mu} (4.3)
\]
This is precisely the form of the observed weak amplitudes (Fermi interaction) if we identify \( \frac{g^2}{M^2} \) with the Fermi constant. Putting a mass term, we have retained the good feature, and have eliminated the bad one!

The agreement with physics has however been achieved at a very high price. To see this, let us compare the new propagator (4.1), with the old ones, (3.7) or (3.8). For very large \( k \), we see that:
\[
(4.1) \sim 1 \\
(3.7) or (3.8) \sim \frac{1}{k^5} \rightarrow 0
\]
The massive theory is much less convergent in the ultraviolet region. This has the very serious consequence that the higher order corrections will be much more divergent now than they were before. Indeed, the structure of divergences of a massive Y-M theory is so bad that the theory cannot be cast in a sensible (technically : renormalizable) form. To elaborate a little more on this, let us consider in detail eq. (4.2). Putting again \( k = (\vec{q}, \omega) \) and using the conservation equation for \( J_{\nu} \), we have:
\[
A = -3^2 \left\{ \frac{J_\alpha J^\prime_\beta + 1}{4k^2M^2} \left( \frac{H^2}{k^2-M^2} J_3 J^\prime_3 + \right) + J_4 J^\prime_4 + J_5 J^\prime_5 \right\} (4.4)
\]
Comparing (4.4) with (3.14) we see that:
1) the instantaneous interaction (i.e., the first term) is no more coulombic. Fourier transforming to \( x \)-space:
\[
\mathcal{F} \rightarrow \bar{e} \left( \frac{\omega}{k^2-M^2} \right) \frac{1}{|\vec{x}-\vec{x'}|}
\]
and we get a Yukawa interaction, This had to be expected, since we have given a mass to the exchanged particle.

11) The second term, at \( k^2 = M^2 \), represents the propagation of massive waves, but there are now three types of waves! The new wave is generated by \( J_3 \). This is invariant under rotations around the \( z \)-axis and so it has \( J_2 = 0 \). Putting a mass term has given to the theory a new degree of freedom, represented by the longitudinally polarized waves. Before, this degree of freedom was eliminated by gauge invariance.

It is precisely the longitudinal wave which is responsible for the incurable ultraviolet pathologies of the massive Yang-Mills theory.

In conclusion, we are faced with a serious dilemma. Either:

1) we stay with the unbroken (massless) theory: this is a consistent theory, which however can have no application in physics;

or:

11) we introduce a mass term: this gives a theory which is very appealing on phenomenological grounds, but is theoretically impracticable.

We will see a way out to this dilemma in Section 6.

5. Spontaneously broken global symmetries.

Leaving aside Yang-Mills theories for a while, we consider now the problem of symmetry breaking.

Most of the symmetries observed in nature are not exact. Isospin symmetry is broken, as indicated by the proton-neutron mass difference; \( SU(3) \) symmetry is broken, to a larger extent, as indicated by the large proton-\( \Lambda \) mass difference, and so on.

A simple way to describe symmetry breaking would be to add explicit non-invariant terms to the lagrangian. We want to discuss here an alternative way in which a symmetry can be broken, usually referred to as "spontaneous symmetry breaking". The idea is to have a theory where the lagrangian is still exactly symmetric under the group transformations, but it gives rise, for dynamical reasons, to a ground state which is not invariant. The ground state of a field theory represents, in the quantized theory, the vacuum state, i.e. the state with no particles. In
turn, the non-invariance of the vacuum state leads to a well definite pattern of symmetry breaking effects. The application of this idea to particle physics, pioneered by the work of Nambu and Jona-Lasinio, has been forbidden for many years by the discovery that, under quite general conditions, the spontaneous breaking of a continuous symmetry leads to the appearance of massless scalar bosons (Goldstone theorem), about the existence of which we have no evidence whatsoever. We will see in the following, how massless Goldstone bosons appear in a particular example.

It is very remarkable that gauge theories do not satisfy the general conditions I have alluded to above, and indeed if we extend the global symmetry into a gauge symmetry, the unwanted massless Goldstone bosons disappear. At the same time a corresponding number of previously massless gauge mesons acquire a mass. This remarkable phenomenon (called the Higgs phenomenon) cures at the same time the bad features of the spontaneously broken symmetry and of the Yang-Mills theory (all related to the presence of massless particles), and opens the way to the construction of realistic models of weak interactions. We will discuss the Higgs phenomenon in the next section.

Let us consider a theory of a self-interacting scalar field \( \phi \). The interaction will be isospin invariant, \( \phi \) being an isodoublet. We will choose the very simple lagrangian:

\[
\mathcal{L} = (\partial_{\mu}\phi^a)(\partial^{\mu}\phi^a) - \mu^2 \phi^a \phi^a - \lambda (\phi^a \phi^a) \phi^a \\
\equiv (\partial_{\mu}\phi^a)(\partial^{\mu}\phi^a) - \sqrt{2} (\phi^a \phi^a)
\]

(5.1)

where:

\[
\phi = \left( \begin{array}{c} k^+ \\
\kappa 
\end{array} \right) \equiv \left( \begin{array}{c} k_1 + k_2 \\
k_3 + i k_4 
\end{array} \right)
\]

(5.2)

\( k_1 \) being real fields, and

\[
\phi^a \phi^a = (k^+)^* k^+ + (\kappa^*)^* \kappa^a = \frac{1}{2} \left( k_1^2 + k_3^2 \right)
\]

Our aim is to study the mass spectrum of the particles associated to the field \( \phi \).

The standard procedure, which we followed in Section 3 for the gauge fields is to separate, in the lagrangian, the terms bilinear in \( \phi \), from the higher order terms:

\[
\mathcal{L} = \phi^a \phi^a + \phi^a \phi^a \\
\phi^a \phi^a = (\partial_{\mu}\phi^a)(\partial^{\mu}\phi^a) - \mu^2 \phi^a \phi^a
\]

(5.3)

and to study the equations of motion of \( \phi^a \).

In this case, one finds, applying eq. (1.4) the equation of motion:

\[
(\Box + \mu^2) \phi(x) = 0
\]

which describes the propagation of a spin zero, complex isodoublet, with an I-spin invariant mass:

\[
M = \sqrt{\mu^2}
\]

(5.4)

Eq. (5.4) evidently requires \( \mu^2 > 0 \).

In this analysis, however, one is tacitly assuming that the lowest energy state (i.e. the state with no particles, the vacuum) corresponds to the field configuration \( \phi = 0 \). Only in this case, in fact, it is meaningful to expand \( \mathcal{L} \) in powers of \( \phi \), associating the propagation of particles to the small oscillations around \( \phi = 0 \).

For scalar fields, on the contrary, it may happen that the lowest energy configuration (the vacuum) corresponds to:

\[
\phi = \text{const} + \phi_0 \neq 0
\]

(5.5)

(the constancy of \( \phi \) in the ground state is required for the vacuum to be translation invariant).

In that case, particles should be associated to the oscillations of \( \phi \) around \( \phi_0 \), rather than around a vanishing value, and the expansion (5.3) would not make sense. Rather, one has to put:

\[
\phi(x) = \phi_0 + \chi(x)
\]

(5.6)

and expand \( \mathcal{L} \) in powers of \( \chi(x) \):

\[
\mathcal{L} = \chi^1 + \chi^2 + \chi^3
\]

(5.7)

The true particle spectrum is then determined by the equation of motion obtained from the \( \chi^2 \) term.

What determines the ground state field configuration? The answer is that such a configuration must correspond to an absolute minimum of the energy (or hamiltonian) density. The hamiltonian density is given by:

\[
\mathcal{H} = \phi^a \frac{\partial}{\partial \phi^a} \phi^a + \frac{\partial}{\partial \phi^a} \phi^a - \mathcal{L} = \phi^a \phi^a\phi^a + \sqrt{2} \phi^a + \phi^a \phi^a + \phi^a \phi^a
\]

(5.8)

\( \phi^a \) is the space gradient of \( \phi^a \).

To have a sensible theory, \( \mathcal{H} \) must be bounded from below, for any field configuration. Since the first two terms in (5.8) are positive definite, this requires the function \( \sqrt{2} \phi^a \phi^a \) to be bounded.
from below, and this, in turn, is obeyed provided \( \lambda > 0 \). Observe that we have not obtained any constraint on the sign of \( \mu^2 \). Indeed we will see that \( \mu^2 < 0 \) is also possible, and leads precisely to the interesting case of spontaneous symmetry breaking.

The form of the hamiltonian, eq. (5.8) is such that an absolute minimum is obtained for a field \( \Phi(\mathbf{x}) \) such that:

1) \( \Phi(\mathbf{x}) = \text{const} = \Phi_0 \) (so that the derivative terms in \( \mathcal{H} \) vanish);

2) \( V(\Phi^* \Phi_0) = \text{minimum} \)

Fig 2 shows the shape of \( V \) as a function of \( \Phi \)
\( (\Phi^2 = \Phi^* \Phi) \), for the two cases: \( \mu^2 > 0 \) (a) and \( \mu^2 < 0 \) (b). We see that either:

\[ \Phi_0 = 0 \quad \text{(case (a))} \]  

or

\[ \Phi_0 \neq 0 \quad \text{(case (b))} \]  

The sign of \( \mu^2 \), therefore, determines whether the ground state corresponds to a vanishing field or not. The first case, (a), corresponds to the case we have previously discussed, and leads to an I-spin symmetric mass spectrum. Let us consider now the more interesting case (b).

To this aim, we choose our isospin frame, so that the constant isospinor \( \Phi_0 \) is a "down" isospinor:

\[ \Phi_0 = \begin{pmatrix} 0 \\ \eta \end{pmatrix} \]  

whence:

\[ \eta = \sqrt{-\frac{\mu^2}{2 \lambda}} \]  

by (5.10). The ground state situation is illustrated in Fig. 3 (a). Here the 4-dimensional Minkowski space is represented as a two dimensional space. To any point \( x \equiv (x, t) \) there is associated a constant spinor \( \Phi_0 \), represented by an arrow of constant length and orientation (to be able to draw a picture, I have squeezed also the 3-dimensional isospin space into two dimensions). Fig. 3 (b) illustrates the situation for a perturbation of the ground state. At any point \( x \), the spinor \( \Phi(x) \) may differ from \( \Phi_0 \) both in orientation and in length (i.e. \( \Phi^*(x) \Phi(x) \neq \Phi_0^* \Phi_0 \)).

To parametrize conveniently the deviations of \( \Phi(x) \) from \( \Phi_0 \), we observe that any spinor \( \Phi(x) \) can be considered as a "down" spinor with respect to suitable isospin axes. Let us denote by:

\[ U(x) = e_i \tau_i \Theta \]  

(5.13)

the \((2 \times 2)\) isospin rotation needed to bring the "tilted isospin frame" at the point \( x \) (i.e. the frame where \( \Phi(x) \) is "down") into the frame where \( \Phi_0 \) is "down". We also denote by:

\[ (\gamma + \tau_i \sigma(x)) \frac{1}{\sqrt{\lambda}} \]  

(5.14)

the components of \( \Phi(x) \) in the tilted frame. Then the components of \( \Phi(x) \) in the frame where \( \Phi_0 \) is "down" are given by:

\[ \Phi(x) = U(x) \left( \begin{pmatrix} 0 \\ \eta \end{pmatrix} \right) \]  

(5.15)

Following eq. (5.6), we can therefore set:

\[ \chi(\mathbf{x}) = \Phi(\mathbf{x}) - \Phi_0 = U(x) \begin{pmatrix} 0 \\ \eta \end{pmatrix} - \begin{pmatrix} 0 \\ \eta \end{pmatrix} \]  

(5.16)

which, for small perturbations, i.e. for small \( \theta^a(x) \) and \( \bar{\theta}^a(\mathbf{x}) \), reduces to:

\[ \chi(\mathbf{x}) \simeq \begin{pmatrix} \theta^a(x) \\ \bar{\theta}^a(\mathbf{x}) \end{pmatrix} \]  

(5.17)

The equations above show that we can parametrize the deviations from the vacuum configuration by 4 real functions (fields) \( \theta^a(x) \) (\( a = 1, 2, 3 \)) and \( \bar{\theta}^a(\mathbf{x}) \). Our next task will be to express the lagrangian in eq. (5.1) in terms of these fields, and to examine the structure of the quadratic terms (corresponding to the term \( ^* \chi \eta \) in eq. (5.7)). We find:

\[ \bar{\theta}^a \theta^b \chi \equiv \frac{1}{2} \left( \bar{\theta}^a \theta^b \chi \right) \]  

+ higher order terms

\[ -\sqrt{\chi} \cdot \Phi \, \Phi = -\sqrt{\left( \begin{pmatrix} \gamma^a \\ \eta^a \end{pmatrix} \right)} \]  

- \, \chi^a \eta^a \]  

- \, \chi^a \eta^a \]  

\[ \frac{1}{2} \left( 4 \lambda \eta^a \right)^2 + \text{higher order terms} \]

so that:

\[ \chi^a = \frac{1}{2} \left( \bar{\theta}^a \theta^b \right) \]  

\[ \eta^a = \frac{1}{2} \left( \bar{\theta}^a \theta^b \right) \]  

\[ \eta^a = \frac{1}{2} \left( \bar{\theta}^a \theta^b \right) \]  

\[ \eta^a = \frac{1}{2} \left( \bar{\theta}^a \theta^b \right) \]  

(5.18)

Eq. (5.18) describes 4 types of particles, similarly to the \( \mu^2 > 0 \) case, but with a different mass spectrum. We have:

1) one massive scalar boson, with mass

\[ m = \sqrt{-\mu^2} \]  

associated to the field \( \theta \);
Fig. 2  Shape of the potential $V(y)$, $y^2 = \Psi^* \Psi$

a) Field configuration in the ground state. The heavy arrows represent the value of $\Psi_0$ in an isospin frame superimposed to the $(x,t)$ space. The dotted axes are the axes of the isospin frame where $\Psi_0$ is a "down" spinor.

b) Field configuration in a perturbed state. The heavy arrows represent the value of $\Psi$ at various space-time points. At the point $(x,t)$, the standard isospin axes, where $\Psi_0$ is "down", and the isospin axes defined by $\Psi(x,t)$ are shown (dotted and undotted axes, respectively).

Fig. 3
for any value of \( \phi \). It is easy to see that 
\( T_4, T_2, \) and 
\( i T_8 \) do not satisfy (5.20), 
while 
\( T_1 \) does : 
\[
\frac{(1+T_1)}{2} \phi_0 = 0
\]

In conclusion, out of four generators, only one obeys (5.20) (i.e. annihilates the vacuum), the other three do not, and the symmetry generated by them is broken. Thus there are as many broken generators as many Goldstone particles we found. This is precisely the rule we looked for, and it is absolutely general. If we have a continuous symmetry group, which is spontaneously broken by the non vanishing value, \( \phi_0 \), that a scalar field takes in the ground state, there will be one Goldstone boson for each group generator \( T_1 \) such that 
\[ T_1 \phi_0 \neq 0 \]

The residual symmetry associated to those generators which obey eq. (5.20) remains unbroken.

6. The Higgs phenomenon

We transform now the lagrangian (5.1) into a gauge invariant lagrangian, using the minimal prescription given in Sects 2 and 3. The new lagrangian is therefore :
\[
\mathcal{L} = (\nabla^\mu \phi)(\nabla^\nu \phi) - V(\phi^* \phi) - \frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} - \frac{1}{4} G_{\mu \nu} G^{\mu \nu} \tag{6.1}
\]

We have taken into account that the full symmetry of (5.1) is \( SU(2) \otimes \mathbb{R}(1) \), as discussed in the previous section, and, correspondingly, we have introduced 4 gauge fields : an isotriplet \( A_{\mu}^a \) \((a = 1, 2, 3)\) and a singlet \( B_{\mu} \). The corresponding gauge curls are given by (2.7) for \( F_{\mu \nu}^a \), while :
\[
G_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{6.2}
\]

since \( B_{\mu} \) is associated to the abelian group \( \mathbb{R}(1) \). The covariant derivative of \( \phi \) is the generalization of (2.3) to \( SU(2) \otimes \mathbb{R}(1) \) and therefore it contains two coupling constants (see the final comments in Section 2) :
\[
\nabla^\mu \phi = (\partial_\mu + i \frac{1}{2} A_\mu^a \frac{T_a}{2} + i \frac{g}{2} \epsilon_{\mu \lambda \nu} B_\nu \frac{T_3}{2}) \phi \tag{6.3}
\]

\( \epsilon \) denotes the 2x2 unit matrix.

Vacuum state. Again we look for the classical solution of (6.1) which represents the ground (vacuum) state. As before, it is obtained for constant fields \( \phi_0, (A_{\mu}^a)_0, (B_{\mu})_0 \). However, we must keep into
account the fact that the ground state must be
Lorentz invariant. This means that at any space-time
point \( x \) there must not be any preferred direction
in Minkowski space. Non-vanishing values for \( A^a_\mu \)
or \( B_\mu \) would give just that, so we conclude:

\[
(A^a_\mu)_{\alpha} = (B_\mu)_{\alpha} = 0 \quad (6.4)
\]

The Hamiltonian density obtained from (6.1),
with the conditions (6.4) obviously coincides with
(5.8), so we are back to our previous problem, and
we find again the two possible solutions (5.9) and
(5.10).

**Particle spectrum.** We put ourselves in the broken
symmetry case (5.10). Consider a field configuration
\( \Phi(x) \) which differs a bit from the constant
distribution, \( \Phi_0 \). As before, the isospinor \( \Phi(x) \)
is a "down" spinor in an isospin frame which differs,
with respect to the isospin frame defined by \( \Phi_0 \),
by the rotation \( \mathcal{U}(x) \), given by (5.13). The situation
is however different from the one we had before,
in that the lagrangian (6.1) is now invariant under
space-time dependent (gauge) isospin transformations.
As we discussed in Section 2, this means that we can
choose at any space-time point \( x \) any orientation of the
isospin axes we please, independently from what
we do at any other space-time point \( x' \). In particular,
we can choose at \( x \) our isospin axes to coincide with those in which \( \Phi(x) \) is exactly a down spinor,
and at the same time choose the axes at \( x' \) as those in which \( \Phi(x') \) is also a down spinor. If we do
so, then by definition:

\[
\Phi(x) = \begin{pmatrix} \sigma(x) \\ \gamma \cdot \sigma(x) \end{pmatrix} \quad (6.5)
\]

More formally we can get eq. (6.5) by parametrizing
\( \Phi(x) \) as we did in eq. (5.13), and then absorbing
the matrix \( \mathcal{U}(x) \) into the redefinition of the iso-
spin axes. (A side remark. The possibility of doing
so depends upon two facts: gauge invariance and the
fact that (5.13) gives an allowed parametrization of
\( \Phi(x) \). It is possible to show that the second
condition is fulfilled in the broken theory, while it is
not in the case of the exact symmetry (\( \mu > 0, \gamma = 0 \)).
In the unbroken theory the parametrization (6.5) is
illegal and the subsequent analysis does not hold).
Eq. (6.5) may seem absurd at first sight. A complex
isospin field is described by four real fields (see
eqs. (5.2) or (5.17)) (6.5) contains only one
real field, \( \sigma \). We seem to have lost the transverse
degrees of freedom, i.e. those previously associated
to the fields \( \theta \). We will see shortly the solution
to this seemingly paradoxical fact. To determine

the mass spectrum, we have again to substitute (6.5)
into (6.1) and collect all terms which are of 2nd
order in the fields \( \sigma, A^a_\mu \) and \( B_\mu \). The
covariant derivative term reduces according to:

\[
\begin{aligned}
(\nabla_\mu \phi^\dagger)(\nabla_\nu \phi) &= \left( \frac{1}{2} (\partial_\mu \sigma)(\partial_\nu \sigma) + \\
&+ \frac{1}{2} \left( \frac{g^2}{2} \right) \left[ A^a_\mu \partial^\nu A^a_\nu + \frac{1}{2} (\frac{g^2}{2}) A^a_\mu A^a_\nu \right] + \\
&+ \frac{1}{4} \eta^2 \left( \frac{g^2}{2} A^a_\mu A^a_\nu - \frac{g}{2} \theta \right) \left( \frac{g^2}{2} A^a_\mu A^a_\nu - \frac{g}{2} \theta \right) + \\
&+ \text{higher order terms}
\end{aligned}
\]

while \( \nabla^\dagger \phi \) gives the \( \sigma \) mass term as before:

\[
\left( \frac{1}{2} \nabla_\mu \phi \right)^2 = c_\text{const} + \frac{1}{2} (-g^2 m^2) \sigma^2 + h.o.t. \quad (6.7)
\]

Finally, the 2nd order terms in the gauge field lag-
grangian give:

\[
- \frac{1}{2} A^a_\mu \nabla_\nu A^a_\nu - \frac{1}{2} \frac{g^2}{2} A^a_\mu A^a_\nu - \frac{1}{4} G^a_\mu G^a_\nu \quad (6.8)
\]

where we have set \( A^a_\mu \equiv \partial_\mu A^a_\nu - \frac{1}{2} g^2 \theta A^a_\nu \). Adding up
(6.6), (6.7) and (6.8) we get the total free lagrangian.
Actually, eq. (6.6) contains mixed products of
\( A^a_\mu \) and \( B_\mu \), and we have to diagonalize it. To this
end, we define two orthogonal combinations of
\( A^a_\mu \) and \( B_\mu \):

\[
Z_\mu = \cos \theta A^a_\mu - \sin \theta B_\mu \\
A_\mu = \sin \theta A^a_\mu + \cos \theta B_\mu \quad (6.9)
\]

and ask \( \theta \) to be such as to make the free lagrangian
diagonal in \( Z_\mu \) and \( A_\mu \). Evidently, eq.
(6.6) implies that:

\[
\tan \theta = \frac{g}{\theta} \quad (6.10)
\]

whence:

\[
\begin{aligned}
L_{\text{free}} &= \frac{1}{2} (\partial_\mu \sigma)(\partial_\nu \sigma) - \frac{1}{2} (-g^2 m^2) \sigma^2 - \\
&- \frac{1}{4} A^a_\mu A^\dagger \nu A^a_\nu + \frac{1}{2} \frac{(g^2)}{2} A^a_\mu A^a_\nu - \\
&- \frac{1}{4} A^a_\mu A^\dagger \nu A^a_\nu + \frac{1}{2} \frac{(g^2)}{2} A^a_\mu A^a_\nu \\
&- \frac{1}{4} Z_\mu Z^\dagger + \frac{1}{2} \frac{(g^2)}{2} Z^\dagger Z \\
&- \frac{1}{4} A_\mu A^\dagger \nu A_\nu \quad (6.11)
\end{aligned}
\]

Eq. (6.11) shows that in the presence of gauge fields,
a spontaneously broken situation does not lead to massless scalar particles. There is only one, massless scalar field (with \( m = \sqrt{2\mu^2} \) as before). On the other hand, three vector fields have acquired a non-vanishing mass. More precisely we have a common mass:

\[
M^2 = \frac{1}{2} \cos^2 \theta
\]

(6.12)

for \( A^a_\mu \) and \( A^b_\mu \), and a different mass:

\[
M^2_2 = \frac{\frac{1}{2} \cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}
\]

(6.13)

for \( Z_\mu \). Recall, from Section 3, that a massive vector field has one more degree of freedom, with respect to the massless case, namely that associated to longitudinal waves. There are, therefore, three more degrees of freedom for vector fields in the broken theory, with respect to the unbroken situation. This exactly compensates for three degrees of freedom we seemed to have lost in discussing eq. (6.5). These degrees of freedom have been simply transferred from the scalar fields to the longitudinal modes of vector fields. As Sidney Coleman puts it, the gauge fields have eaten the Goldstone bosons and grown heavy.

There is still one massless vector field in (6.11), namely \( A_\mu \). This is so because the full symmetry of (6.1) has not been completely broken by \( Q_\mu \). As we have seen in Section 5, there is still one conserved generator, namely:

\[
\frac{\tau_1 + \tau_3}{2} = Q
\]

(6.14)

Indeed, if we go back to (6.3) and express the terms containing \( A^a_\mu \) and \( B_\mu \) as functions of \( A_\mu \) and \( Z_\mu \), we get:

\[
\begin{align*}
&= \frac{1}{\cos \theta} \left( \frac{\tau_1}{2} - i \tilde{u}^e_\mu Q A^e_\mu \right) Z_\mu + \frac{1}{2} \tilde{u}^e_\mu Q A^e_\mu
\end{align*}
\]

(6.15)

\( Q \) being the matrix (6.14). We see that \( Z_\mu \) is coupled to a broken generator (similarly to \( A^a_\mu \) and \( A^b_\mu \) which are coupled to \( \tau_1/2 \) and \( \tau_3/2 \)) while \( A_\mu \) is coupled precisely to the conserved generator, \( Q \).

What we have seen above is an illustration of the Higgs phenomenon (actually discovered also by Brost and Englert, and by Guralnik, Hagen and Kibble). It enables us to associate massive particles to some of the gauge fields (those corresponding to broken generators) without having to introduce explicitly a mass term in the lagrangian.

The relevance of the Higgs phenomenon is greatly enhanced by the following result (essentially proven by 't Hooft, and which I cannot possibly explain in any detail in these lectures): a gauge theory spontaneously broken by the Higgs mechanism is renormalizable. In such a theory, similarly to QED and unlikely the massive \( \gamma \)-\( \mu \) theory we sketched in Section 3, we can compute physical amplitudes to any given order in the coupling constants, in terms of only a finite number of parameters, namely those contained in the lagrangian we started with.

Stated differently, the Higgs phenomenon allows us to have a physically sensible theory (and no unobserved massless particles) which is also theoretically tractable in high orders of perturbation theory.

The example discussed above is the basis of the models of weak and e.m. interactions we shall discuss in the following sections.

Replacing \( A^a_\mu \) and \( A^b_\mu \) by the complex fields:

\[
\begin{align*}
W_\mu &= \frac{1}{\sqrt{2}} \left( A^a_\mu + i A^b_\mu \right) \\
W^a_\mu &= \sqrt{2} (A^a_\mu - i A^b_\mu)
\end{align*}
\]

(6.16)

we will identify \( W_\mu \) and \( W^a_\mu \) with the fields associated to the charged intermediate boson (the one mediating e.g. the neutrino \( \beta \)-decay) and \( Z_\mu \) with the neutral intermediate boson, responsible for neutral current processes (e.g. \( \nu - \bar{\nu} \) scattering). The leftover massless field, \( A_\mu \), will of course be identified with the electromagnetic field. The mixing angle \( \theta \), defined in eqs. (6.9) and (6.10), is the so-called Weinberg-Salam angle, and, according to eq. (6.15) we will set:

\[
\tan \theta = \frac{e}{Q}
\]

(6.17)

\( e = \) electric charge.

The mixing angle \( \theta \) has been, actually, first introduced by Glashow\(^1\), who was also the first to consider a unified model based on the gauge group \( SU(2) \otimes U(1) \).

7. Weak Interactions of electron-like and muon-like leptons in the Weinberg-Salam model

We are now ready to construct a concrete model for the weak and e.m. interactions of the known leptons (\( \nu, e \) and \( \nu', \mu \)) based on the gauge group \( SU(2) \otimes U(1) \). In doing so, we must keep into account that leptons are coupled in conventional weak interactions (that is in these processes mediated
by W-exchange) through pure V-A currents. To reproduce this feature, we shall associate the action of the charged generators of SU(2) × U(1) on the lepton fields, with the chiral isospin generators we have introduced in Section 1.

Let us, therefore, arrange the four lepton into two independent doublets:

\[ E = \begin{pmatrix} \nu \\ e \end{pmatrix}, \quad M = \begin{pmatrix} \nu' \\ \mu \end{pmatrix} \]

We will then define the action of an infinitesimal SU(2) transformation according to:

\[ \delta E = i \frac{a}{2} \left( 1 - \gamma_5 \right) \begin{pmatrix} \nu \\ e \end{pmatrix} = i \frac{a}{2} E^a \begin{pmatrix} \nu \\ e \end{pmatrix} \quad (7.1) \]

and similarly for SM. As we noticed in Section 1, the transformation (7.1) corresponds, for zero mass particles, to an isospin rotation of the lefthanded states (i.e. states with negative helicity), the righthanded states remaining unaffected. To make this fact more explicit, we define a lefthanded doublet:

\[ E_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \frac{1 + \gamma_5}{2} \begin{pmatrix} \nu \\ e \end{pmatrix} \quad (7.2) \]

and similarly a muonic doublet, \( \eta_L \). Eq. (7.1) can then be interpreted as saying that the lefthanded fields behave as (weak) isodoublets:

\[ \delta E_L = i \frac{a}{2} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad (7.3) \]

(and similarly for \( \eta_L \)) while righthanded fields behave like (weak) isosinglets:

\[ \delta e_R = \delta \begin{pmatrix} \nu_R \\ e_R \end{pmatrix} = 0 \quad (7.4) \]

and similarly for \( \nu_R, \nu'_R, \mu_R \). To go further, we must specify the action of the U(1) transformations.

This is in fact determined by the requirement that the photon field, defined by (6.9) couples precisely to the electric charge. To see this, let us put all the lepton fields into a single column vector \( \chi \):

\[ \chi = \begin{pmatrix} \nu_L \\ e_L \\ \vdots \\ \nu'_R \end{pmatrix} \]

and assume:

\[ \delta \chi = i \frac{a}{2} \Gamma \chi \quad (7.5) \]

\( \Gamma \) being an 8x8 matrix. The coupling of \( A^3_{\mu} \) and \( B_{\mu} \) to the leptons resulting from (7.3), (7.4) and (7.5) will be:

\[ \bar{\chi} \gamma^\mu \begin{pmatrix} a A^3_{\mu} L^3 + \frac{1}{2} g B_{\mu} \gamma \end{pmatrix} \chi = \]

\[ = \bar{\chi} \gamma^\mu \begin{pmatrix} a s \cos \theta L^3 + \frac{1}{2} g' \gamma \end{pmatrix} + \]

\[ + \bar{\chi} \gamma^\mu \begin{pmatrix} a s \sin \theta \gamma \end{pmatrix} \chi = \]

\[ = \bar{\chi} \gamma^\mu \begin{pmatrix} a s \cos \theta L^3 + \frac{1}{2} g' \gamma \end{pmatrix} + \]

\[ + \bar{\chi} \gamma^\mu \begin{pmatrix} a s \sin \theta \gamma \end{pmatrix} \chi \quad (7.6) \]

We have therefore to satisfy the condition:

\[ Q = L^3 + \frac{1}{2} Y \quad (7.7) \]

which determines \( Y \), once \( Q \) and \( L_3 \) are known. The above condition leads to the weak hypercharge assignment shown in Table 1. Note that, since both \( Q \) and \( L_3 \) are diagonal, so is \( Y \), and the U(1) transformations act, in this case, as phase transformations. Note also that \( \gamma_5 \) is coupled to a linear combination of the currents associated to the electric charge and to the 3rd component of weak isospin.

\begin{table}
<table>
<thead>
<tr>
<th>( L_3 )</th>
<th>( \nu_L )</th>
<th>( e_L )</th>
<th>( \nu_R )</th>
<th>( e_R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>+\frac{1}{2}</td>
<td>-\frac{1}{2}</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>-2</td>
<td></td>
</tr>
</tbody>
</table>

We can now write down the covariant derivatives of the lepton fields:

\[ \nabla_\mu E_L = (\partial_\mu + ig \frac{a}{2} A^a_{\mu} - ig' \frac{1}{2} \partial_\mu) E_L \]

\[ \nabla_\mu E_R = (\partial_\mu - ig \frac{1}{2} B_{\mu}) E_R \]

\[ \nabla_\mu \nu_R = \partial_\mu \nu_R \]

and similarly for muon-like fields. To fully determine the lepton lagrangian, one could make the hypothesis that the leptons have no other interaction than with the gauge fields. In this case one could simply apply the minimal prescription, \( \theta \to \theta \), to the free lepton lagrangian. However, to be able to
do so, the free lagrangian itself must be invariant under the global transformations, eqs. (7.3) to (7.5). Since these transformations involve chiral transformations, an invariant free lagrangian can only be achieved in the limit where all leptons have a vanishing mass, as discussed in Section 1.

To obtain a non vanishing mass for electrons and muons, we have to postulate that leptons possess other interactions but the weak and electromagnetic ones. We will see in the next section that a suitable interaction of the leptons with the scalar isodoublet, \( \varphi \), can indeed give rise to the observed lepton masses.

For the time being, however, we will neglect these additional interactions and, restricting to the limit of zero mass for all leptons, we will construct the lepton lagrangian from the free, massless lagrangian. In this limit, the lepton lagrangian is simply:

\[
\mathcal{L}_{\text{Lag}} = i \left( \bar{\psi}_e \not{D} c + \bar{\psi}_\mu \not{D} M_\mu \right) + i \left( \bar{\psi}_e \mathcal{V} \psi_e + \bar{\psi}_\mu \mathcal{V} \psi_\mu \right) + \mathcal{V}^0 \not{V} \psi_e \psi_\mu
\]

(7.8)

while the total lagrangian is the sum of (7.8) and (6.1).

The lagrangian (7.8) describes the weak and e.m. interactions of leptons, which we will now briefly review.

To this aim, we extract from (7.8) the interaction terms, which turn out to be:

\[
\mathcal{L}_{\text{int}} = -\frac{g}{\sqrt{2}} \bar{\psi}_\mu \gamma^\mu \frac{i}{2} \not{G} \psi_e + \frac{g}{\sqrt{2}} \not{G} \not{V} \psi_e + \not{G} \left( \bar{\psi}_e \gamma^\mu \not{G} \psi_\mu \right)
\]

(7.9)

It is of course more convenient to express \( \mathcal{L}_{\text{int}} \) in terms of the physical vector fields \( \omega_\mu \), \( \phi_\mu \), and \( \phi_\mu^* \). Using eqs. (6.9), (6.10), (6.16) and (6.17), one gets finally:

\[
\mathcal{L}_{\text{int}} = -\frac{g}{\sqrt{2}} \bar{\psi}_\mu \gamma^\mu \left( \mathcal{A}_\mu^* - h.c. \right) + \frac{g}{\sqrt{2}} \left( \bar{\psi}_e \gamma^\mu \left( \mathcal{A}_\mu^* - h.c. \right) \right)
\]

(7.10)

The \( \mathcal{A}_\mu \) terms describe the well known e.m. interaction of photons with electrons and muons, which we shall not discuss. The other terms describe the emission and absorption of the heavy intermediate bosons, \( \mathcal{W} \) and \( \mathcal{Z} \). In second order of perturbation theory (as discussed in Sections 3 and 4) they give rise to lepton-lepton scattering, or to crossing related processes, through \( \mathcal{W}^\pm \) and/or \( \mathcal{Z} \) exchange.

\textbf{W-mediated processes (charged currents).} \textbf{W}-exchange gives rise e.g. to the process:

\[
y^\prime + e \rightarrow \mu + \nu
\]

(7.11)

or to the crossing related \( \mu \) decay. Applying eq. (4.2), and using the weak charged currents given in (7.10), we find the low energy amplitude (valid in the limit of very small momentum transfer with respect to the \( \mathcal{W} \) mass):

\[
\mathcal{A} = \frac{g^2}{8 M_W^2} \left[ \bar{\mu} \gamma^\mu (1-\gamma_5) \mathcal{W} \right] \left[ \bar{\nu} \gamma^\nu (1-\gamma_5) e \right]
\]

(7.12)

The amplitude (7.12) coincides with the V-A current \( \times \) current amplitude. The coefficient multiplying the four fermion amplitude is conventionally written as \( G_F^2 \), so that one finds the relation:

\[
\frac{G_F^2}{M_W^2} = \frac{g^2}{8 M_W^2}
\]

(7.13)

(\( G = \) Fermi constant \( \sim 10^{-5} M_p^{-2} \); \( M_p = \) proton mass).

We may combine eq. (7.13) with eq. (6.17), to obtain a lower bound for the \( \mathcal{W} \) mass:

\[
M_W^2 = \frac{g^2}{4 \sqrt{2} G_F^2} = \frac{e^2}{4 \sqrt{2} G_F^2} \sin^2 \theta = \frac{\left( \frac{\pi d}{4 \sqrt{2} G} \right) \sin^2 \theta}{\sin^2 \theta} = \frac{37.5 \text{ GeV}^2}{\sin^2 \theta}
\]

(7.14)

At high energy, the amplitude for the process (7.11) is modified, with respect to (7.12), by the effect of the \( \mathcal{W} \)-propagator (see eq. (4.22)). Eq. (7.14) indicates that appreciable modifications will appear only for \( \nu^\prime \) energies above the present FNAL range. Indeed, the experimental lower limit on the \( \mathcal{W} \) mass that present neutrino experiments have been able to set, is around 10 GeV.

Information coming from semi-leptonic neutral current processes indicates that \( \sin^2 \theta \sim 0.3 \) (with large errors). Correspondingly, one gets

\[
M_W \sim 6 \text{ GeV}
\]

\textbf{Leptonic width of the \( \mathcal{W} \).} Eq. (7.10) gives directly the amplitude for the decay:

\[
\mathcal{W}^+ \rightarrow e^+ \nu
\]

which occurs in lowest order. The corresponding width can be computed by standard methods. It is use-
ful to cast the result in a more general form, considering a coupling of the type:

\[ W_{\nu} \left( g_L \bar{e}_L \gamma^\nu e_L + g_R \bar{e}_R \gamma^\nu e_R \right) \]

One finds in this case:

\[ \Gamma = \left( \frac{g_L^2 + g_R^2}{4\pi} \right) \frac{M_W}{\alpha} \]  

(7.15)

From eq. (7.10), we read:

\[ g_R = 0, \quad g_L = g/\sqrt{2} \]

whence

\[ \Gamma (w \rightarrow e\nu) = \frac{1}{12} \left( \frac{2\pi}{4\pi} \right) \frac{M_W}{\alpha} \approx \frac{23}{\sin^3 \theta} \text{MeV} \]

(7.16)

To get the total width, one must add to \( \Gamma (w \rightarrow e\nu) \) an equal contribution from the \( \mu \nu^\prime \) mode, the hadronic width and possible contributions from heavy leptons. In total, this may give about a factor of 10, leading to a value for \( \Gamma_{\text{total}} (w) \) in the GeV region.

**Z-mediated processes (neutral currents).** The prototype neutral current process is:

\[ \nu^\prime + e \rightarrow \nu + e \]  

(7.16)

whose low energy amplitude, derived as before, is:

\[ A = \frac{g^2}{16 \cos^2 \theta} M_Z^2 \left[ \bar{\nu}^\prime \gamma^\nu (1 - \gamma^\nu) \nu \right] \left[ \bar{e} \gamma^\nu (g_\nu - g_\alpha \gamma^\nu) e \right] \]  

(7.17)

where

\[ g_\nu = 1 - 4 \cos^2 \theta \]

\[ g_\alpha = 1 \]

(7.18)

We may observe that:

1) The electron neutral current is not pure \( \nu^{-} \).

2) Using eq. (6.13) we may eliminate the \( \Theta \) dependence of the effective coupling in (7.17) according to:

\[ \frac{g^2}{16 \alpha^2 \theta} M_Z^2 = \frac{g^2}{16 \alpha^2 \theta} = \frac{\alpha}{\sqrt{2}} \]  

(7.19)

so that the scale of the amplitude (7.17) is determined by the Fermi constants only.

\( Z \) exchange contributes also to the process:

\[ \nu + e \rightarrow \nu + e \]  

(7.20)

in addition to \( W \)-exchange, see Fig. 4. The low energy amplitude is still a four fermion amplitude, of the form:

\[ A (\nu e) = \frac{g^2}{2 \sqrt{2}} \left[ \bar{\nu} \gamma^\nu (1 - \gamma^\nu) \nu \right] \left[ \bar{e} \gamma^\nu (g_\nu - g_\alpha \gamma^\nu) e \right] \]  

(7.21)

but now:

\[ g'_\nu = - \left( 1 + 4 \sin^2 \theta \right) \]

\[ g'_\alpha = -1 \]

(7.22)

**Fig. 4.** Feynman diagrams for \( \nu e \) scattering

Summing up, we have seen that the amplitudes of processes (7.16) and (7.20), and of the crossing related processes, are completely determined, in terms of one single parameter, \( \Theta \). It is obviously of the utmost importance to have experimental checks of these predictions, especially since, in these processes, there are no uncontroled strong interaction effects.

At present, a few events of (7.16) have been observed in Gargamelle and, quite recently, process (7.20) has been unambiguously observed\(^2\). Within the very limited accuracy presently available, data are in both cases consistent with the Weinber-Salam model, with \( \sin^2 \Theta \sim 0.3 \).

Combining eqs. (7.14) and (6.13), we may again obtain a lower bound on the \( Z \) mass. We find:

\[ M_Z^2 = \frac{(75.0 \text{ GeV})^2}{(\sin^2 \Theta)^2} \]

Using again \( \sin^2 \Theta \sim 0.3 \), we estimate

\[ M_Z \sim 82 \text{ GeV} \]

**Leptonic width of the \( Z \).** From eq. (7.10) we get the couplings of \( Z \) to \( e^+ e^- \):

\[ g_L^2 = \frac{g^2}{4 \cos^2 \theta} \left( 1 - 2 \sin^2 \theta \right) \]

\[ g_R^2 = \frac{g^2}{\cos^2 \Theta} \]

and using (7.15), we get:
\[
\Gamma(Z \rightarrow e^+ e^-) = \Gamma(W \rightarrow e v) \frac{1 - 4 \sin^2 \theta + 8 \sin^4 \theta}{2 \cos^3 \theta} \\
\approx \frac{92 \text{ MeV}}{(\sin 2 \theta)^3} (1 - 4 \sin^2 \theta + 8 \sin^4 \theta)
\]

In the same way, one can find the width for \(Z \rightarrow \nu \bar{\nu}\):
\[
\Gamma(Z \rightarrow \nu \bar{\nu}) = \frac{2 \frac{\text{MeV}}{24 \cos^3 \theta}}{2 \cos^3 \theta} = \frac{9 \text{ MeV}}{(\sin 2 \theta)^3}
\]

The total \(Z\) rate, as in the case of the \(W\), can be about one order of magnitude larger than the previous rates.

Other \(Z\)-related processes. \(Z\)-exchange gives rise to observable effects also in processes not involving neutrinos. An important example is the production of \(\mu^+ \mu^-\) pairs in colliding rings:
\[
e^+ e^- \rightarrow \mu^+ \mu^-(7.23)
\]

which can take place through one photon and one \(Z\) exchange. At low energy the effect of \(Z\)-exchange is negligibly small, but it increases linearly with the center of mass energy. Already in the Petra energy range the interference of \(Z\) with \(Y\) exchange may give rise to detectable effects (e.g. the backward-forward asymmetry of \(\mu^+\)). When \(2 E_Z = M_Z\), the weak amplitude dominates, as one is producing real \(Z\)'s.

At still larger energies, \(Y\) and \(Z\) exchange remain comparable, and gradually weak and e.m. interactions merge together.

Another important effect, is related to \(Z\) exchange between electrons and nucleons in atoms. This gives rise to a parity violating potential and therefore to P-violating mixing of the atomic levels. There are, at present, various experimental groups attempting to detect P-violation in heavy atoms. No firm evidence of such effects has been yet achieved.

8. The anomalous magnetic moment of the \(W\)

As a side exercise, we consider the coupling of the photon to the \(W\)-meson.

This coupling arises from the trilinear self coupling of gauge fields, which we have previously called \(A^3\). Since the structure of the \(A^3\) terms is precisely determined by gauge invariance, we will get a well defined \(Y-W\) interaction, which, as we shall see, has a very peculiar structure.

In the \(SU(2) \otimes U(1)\) model, only the isoscalar fields \(A_\mu^a\) have a trilinear interaction, of the form already given in eq. (3.3). There are no trilinear terms involving \(B_\mu\), since \(B\) is associated to an abelian group (see eq. (6.2)) which commutes with \(SU(2)\). If we write down explicitly the \(g A^3\) term in (3.3), we get:
\[
g A^3 = \frac{1}{2} g \varepsilon_{abc} A_\mu^a A_\nu^b A_{\mu \nu}^c =
\]
\[
= g \left( A_\mu^a A_\nu^b A_{\mu \nu}^c + A_\mu^a A_\nu^b A_{\mu \nu}^c +
A_\mu^a A_\nu^b A_{\mu \nu}^c \right) =
\]
\[
= i g A_{\mu \nu}^a W_\mu^a W_\nu^a - i g A_{\mu \nu}^a (W_\mu^a W_\nu^a -
W_\mu^a W_\mu^a) \quad (8.1)
\]

In eq. (8.1) we have used the shorthand notation \(A_{\mu \nu}^a = \frac{2}{g} A_\mu^a - \frac{2}{g} A_\nu^a\), and, in the last line, we have used eq. (6.16), to express the interaction in terms of the fields associated to the charged particles \(W^\pm\). To get the photon coupling, we have to express \(A_{\mu \nu}^a\) in terms of the physical fields \(A_\lambda\) and \(Z_\lambda\), and keep the terms with \(A_\lambda\). To this aim, we use eqs. (6.9) and (6.17), to get finally:
\[
\mathcal{L}_{Y-W} = i e \left[ F_{\mu \nu}^a W_\mu^a W_\nu^a + A_\lambda^a (W_\mu^a W_\mu^a -
W_\mu^a W_\nu^a)ight] \quad (8.2)
\]

(\(F_{\mu \nu} = \mathcal{F}_{\mu \nu} - \mathcal{F}_{\nu \mu} = \) the e.m. field strength tensor). The remarkable feature of eq. (8.2) is the presence of the first term, where the \(W\) field is directly coupled to the electric and magnetic fields, contained in \(F_{\mu \nu}\).

To see why this is remarkable, let us suppose that, unlike the case we are considering, weak interactions were a completely independent phenomenon from electromagnetic interactions. We could still describe weak interactions as resulting from \(W\)-exchange, with a lagrangian of the form:
\[
\mathcal{L}_{\text{weak}} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} (W - \text{ph., W-qua.})
\]
\[
\mathcal{L}_{\text{free}} = -\frac{1}{2} \left( \partial_\mu W_\mu^a W^a_\mu + M_\mu W^a_\mu W^a_\mu \right)
\]

To describe the e.m. interactions of \(W\), we would then perform \(\mathcal{L}_{\text{free}}\) the "minimal substitution" appropriate to electromagnetism, namely we would
make the replacement
\[ B^\mu W^\nu \rightarrow (\partial^\mu - i e A^\mu) W^\nu \]

\( A^\mu \) being the photon field. If we do so, it is easy to compute what one would call the "minimal"
\( \gamma - W \) lagrangian, which turns out to be
\[ \mathcal{L}_{\gamma - W} = i e A^\mu (\gamma^\mu W^\nu - W^\nu \gamma^\mu) \]

Eq. (8.3) differs from eq. (8.2) precisely because it does not contain the "non minimal" \( \gamma^\mu W^\nu \) term. The way we derived eq. (8.2) should make it clear that the non minimal term is precisely associated to the fact that both weak and e.m. interactions are described by a unified Yang-Mills theory.

For this reason, when the \( W \) mesons will be experimentally detected, a precise study of the \( \gamma - W \) vertex will be a very important thing to do, and could give the first experimental indication that Yang-Mills theories are indeed at work in Nature.

The \( \gamma - W \) vertex could be determined by a study of the neutrino production of \( W \) (Fig.5) or of \( W \) pair production in colliding rings
\[ (e^- e^- \rightarrow \gamma \rightarrow W^+ W^-) \]

Fig. 5. Neutrino production of \( W^+ \) off a target nucleus.

It is possible to see that the non minimal term gives rise to a pointlike anomalous magnetic moment of the \( W \). The "minimal term" (8.3) gives rise to a normal magnetic moment:
\[ \mu_{\text{normal}} = \frac{e}{2 M_W} S \]

\( S \) being the \( W \) spin, while the non minimal term adds to \( \mu \) an anomalous term; \( \mu_{\text{anom}} = \mu_{\text{normal}} \) so that eq. (8.2) implies:
\[ \mu_{\text{tot}} = 2 \mu_{\text{normal}} \]

9. The lepton masses

Non vanishing masses for the leptons can be generated by suitably coupling leptons to the scalar isodoublet, \( \varphi \). The general idea goes as follows.

We introduce a Yukawa type interaction (the only one allowed, if we want to keep the theory renormalizable) which we write symbolically as:
\[ \mathcal{L}_{\text{yuk}} = g_s \psi \bar{\psi} \varphi \]

(9.1)

(we will see later on the the precise lagrangian is actually a sum of terms like (9.1), \( \varphi \) running over various lepton fields; each term has an independent coupling constant). Reexpressing (9.1) in terms of the shifted scalar field \( \sigma \), eq. (6.5), one obtains:
\[ \mathcal{L} = (g_s \sigma) \bar{\psi} \psi + g_s \frac{\sigma}{\sqrt{2}} \bar{\psi} \psi \]

(9.2)

The first term, which is bilinear in \( \psi \), can be added to the free lepton lagrangian, and it corresponds to a lepton mass term. The second term describes a residual, Yukawa type interaction of \( \sigma \) with the leptons, to be added to the weak interactions described in Section 7. The new interaction is characterized by the coupling constant \( g_s \); its strength can be easily compared to the strength of the weak interactions arising from \( W \)-exchange. Notice that, from eq. (9.2):
\[ g_s \sigma \approx m_e \]

(9.3)

\( m_e \) being a lepton mass (say \( m_e \) or \( m_\mu \))

while, from eq. (6.12) we have:
\[ g_s \sigma \approx M_W \]

(9.4)

\( g \) being the coupling constant of the gauge fields.

One therefore finds:
\[ g_s \approx g \frac{m_e}{M_W} \]

(9.5)

Furthermore, the lepton-lepton amplitude arising from \( \sigma \)-exchange at low energy is of order
\[ g_s^2 \frac{1}{H_W} \approx g^2 \frac{m_e^2}{M_W^2} \approx g \frac{m_e^2}{M_W^2} \]

(9.6)

As a consequence of (9.6), as far as the neutral Higgs boson \( \sigma \) is much heavier than leptons, the addition of the coupling (9.1) does not change appreciably the picture of weak interactions we have outlined previously, and the main effect of the new interaction (9.1) is that of generating lepton masses.

The same line will be followed in the case of quark (hence hadronic) masses, and the above considerations can be repeated for the interaction of quarks with the Higgs fields, replacing \( \varphi \) with the \( \sigma \)-quark coupling constant and \( m_q \) with \( m_q \)
(quark mass). It is usually assumed that $\sigma$-field is also much heavier than normal hadrons, including charmed particles. We will accept this hypothesis and, in what follows, we will completely neglect $\sigma$ exchange amplitudes. In this approximation, the only role of the interaction of $\varphi$ to normal matter is to provide non vanishing leptonic and hadronic mass scales.

We go now to more precise considerations, and determine the most general form of the coupling (9.1).

To this aim, we have to construct all the possible terms of the form (9.1), which are invariant under $SU(2) \otimes SU(1)$, and involve the isodoublet $\bar{E}_L$ and $M_L$ and the iso-singlets $E_R$, $\gamma_R$, $\gamma^\prime_R$.

Consider first $E_L$ and $\varphi$. The combination:

$$\bar{E}_L \varphi \equiv \bar{\nu}_L \kappa^+ + \bar{\nu}_L \kappa^0$$

is obviously an $SU(2)$ singlet. To make it Lorentz invariant we have to multiply it by some right-handed field. Since the weak hypercharge of $\varphi$ is $+1$ (compare eqs. (5.19) and (7.5)) and that of $\bar{E}_L$ is also $+1$, to obtain an $SU(1)$ invariant coupling we need a $Y = -2$ field, namely $E_R$ or $\gamma_R$. Repeating the same argument also for $M_L$, we find the invariant coupling:

$$\left( g_{d_L} \bar{E}_L E + g_{d_R} \bar{E}_L M - g_{\gamma_2} \bar{m}_L E_R + g_{\gamma_3} \bar{m}_L M_R \right) \varphi +$$

$$+ \text{hermitian conjugate}$$

the $g_i$'s being arbitrary coupling constants, which are a specification of the coupling constant $g_f$ in (9.1).

The term (9.7) is not, however, the only invariant coupling, due to the special property of the group $SU(2)$, that it admits only real representations. This means that we can construct a new isodoublet (to be called $\varphi'$) whose components are linear combinations of the components of $\varphi$, and which transforms precisely like $\varphi$. Indeed, if we define:

$$\varphi' = -i \left( \varphi^T \tau_3 \right)^T \tau_3 \tau_1 \tau_2 \equiv \left( \varphi^0 \right)$$

then:

$$\delta \varphi' = -i \left( \left( \delta \varphi^T \tau_3 \right)^T \equiv -i \left( \left( \varphi^T \tau_3 \right)^T \varphi \right) \right)$$

$$= \delta \varphi \left( \varphi^T \tau_3 \right)^T \equiv i \left( \varphi^T \tau_3 \right)^T \varphi$$

due to the identity:

$$\tau_L \tau_A \tau_2 = -\tau_A^T$$

The transformation rule (9.9) has precisely the same form as (1.6). Note further that $\varphi'$ has weak hypercharge $-1$. Repeating the argument which led to (9.7), one obtains therefore the further coupling:

$$\left( h_{\gamma_2} \bar{E}_L \varphi_R + h_{\gamma_3} \bar{E}_L \gamma_R + h_{\gamma_4} \bar{M}_L \varphi_R + h_{\gamma_5} \bar{M}_L \gamma_R \right) \varphi'$$

$$+ \text{hermitian conjugate}$$

with new, independent, coupling constants. The most general $\varphi$-lepton interaction is, finally, the sum of the lagrangians (9.7) and (9.10), which have to be added to (7.8), to obtain the full lepton weak lagrangian.

The resulting expression can be somehow simplified by the following observation. If we perform orthogonal transformations on corresponding electron-like and muon-like fields, i.e. if we set:

$$E_L = \cos \alpha \bar{E}'_L + \sin \alpha \bar{M}'_L$$

$$M_L = -\sin \alpha \bar{E}'_L + \cos \alpha \bar{M}'_L$$

and similarly for $E_R$, $M_R$ and for $\gamma_R$ and $\gamma^\prime_R$ (with independent angles $\beta$ and $\gamma$), the lagrangian (7.8) remains unaffected, while (9.7) and (9.10) go into similar expressions, with new constants $g'$ and $h'$, linearly related to the unpri-

med constants. By elementary considerations one can show that the parameters $\alpha$, $\beta$ and $\gamma$ can be chosen so that the $g'$ and $h'$ satisfy either the relations:

$$g'_{d_2} = g'_{d_4} = 0 \; ; \; h'_{\tau_1} = h'_{\tau_2}$$

or, equivalently:

$$g'_{d_2} = g'_{d_4} \; ; \; h'_{\tau_1} = h'_{\tau_2} = 0$$

Stated differently, the eight couplings appearing in (9.7) and (9.10) are redundant, and with no loss of generality we may require either (9.11) or (9.12) to hold. In what follows, we shall choose condition (9.11). It can also be shown that, with no loss of generality, the constants $g_i$'s and $h_i$'s can be chosen to be real (this corresponds to a $CP$ conserving interaction; the model is therefore intrinsically $CP$ conserving).

According to our previous discussion, we replace now $\varphi$ by the eq. (6.5) (and similarly for $\varphi'$) and keep only the terms linear in $\varphi$. The sum of (9.7) and (9.10), with the conditions (9.11), reduces then to:

- 44 -
\( L_{\text{mass}} = \frac{g_4}{2} \bar{e}(x) e(x) + \frac{g_2}{2} \bar{\nu}(x) \mu(x) + \frac{h_4}{2} \bar{\nu}(x) \nu(x) + \frac{h_2}{2} \bar{\nu}(x) \nu'(x) + \frac{h_{42}}{2} \left( \bar{\nu}(x) \nu(x) + \bar{\nu}'(x) \nu'(x) \right) \) \tag{9.13}

As anticipated, (9.13) corresponds to a mass term for all the lepton fields. In particular we have obtained two independent masses for electrons and muons:

\[ m_e = \frac{g_4}{2} \eta; \quad m_\mu = \frac{g_2}{2} \eta \] \tag{9.14}

In the model there is no way of predicting the \( p-e \) mass ratio: the only thing we can do is fit the observed masses with \( g_4 \) and \( g_2 \).

We consider now the neutrino mass terms. Experimentally, both \( \nu \) and \( \nu' \) mass are consistent with zero, the experimental upper bounds being:

\[ m_\nu < 60 \text{ eV}; \quad m_\nu < 1.2 \text{ MeV} \] \tag{9.15}

There is, however, no preference for massless neutrinos in (9.13). If we do so, we can get vanishing neutrino masses, simply by setting:

\[ h_4 = -h_2 = h_{42} = 0. \]

In this case, neutrino fields appear only in (7.8), with the consequence that the right handed neutrinos have no weak interaction at all. In this limit, our model is equivalent to the two component neutrino theory, and we could even drop \( \nu_R \) and \( \nu'_R \) from our lagrangian. In any case, even if righthanded neutrinos would exist, we would not see them, as they would interact with the other particles only through the gravitational interaction. In the same limit (\( m_\nu = m_\nu' = 0 \)) the lepton lagrangian admits the exact conservation of the electron and of the muon numbers.

It is interesting to study also the case of small, but non vanishing neutrino masses, allowing all \( h \)'s to be non zero. Since the neutrino mass term is not diagonal, \( \nu(x) \) and \( \nu'(x) \) do not correspond to freely propagating particles. Rather, we have to consider two orthogonal combinations:

\[ \nu_1 = \cos \varphi \nu + \sin \varphi \nu' \]
\[ \nu_2 = -\sin \varphi \nu + \cos \varphi \nu' \] \tag{9.16}

such that \( L_{\text{mass}} \) is diagonal when expressed in term of \( \nu_1 \) and \( \nu_2 \). This implies:

\[ \tan^2 (2 \varphi) = \frac{2h_{12}}{h_1 - h_2} \] \tag{9.17}

At the same time, we have to express also the weak lagrangian (7.8) or (7.10) in terms of \( \nu_1 \) and \( \nu_2 \). This changes the interaction terms involving \( \nu \) emission according to:

\[ L(\nu e, \nu' e) \rightarrow \]

\[ \frac{g}{\sqrt{2}} \left( \bar{\nu}_1 \gamma^\mu (1 - \gamma_5) (\cos \varphi e + i \sin \varphi \mu) + \bar{\nu}_2 \gamma^\mu (1 - \gamma_5) (-\sin \varphi e + i \cos \varphi \mu) \right) \] \tag{9.18}

When \( \varphi \neq 0 \), the interaction (9.18) violates electron and muon number conservation, in that \( e \) and \( \mu \) are coupled at the same time to \( \nu_1 \). A related phenomenon are the so called neutrino oscillations (much similar to \( K^0 - \bar{K} \) oscillations which we are now going to discuss...)

Suppose we have a source of neutrinos, located at the origin of coordinates, \( x = y = z = 0 \), and let us suppose that neutrinos are there produced in association with muons (e.g. from \( \pi^+ \rightarrow \mu^+ \nu \) decay). A decay at \( t = 0 \) gives rise to a linear superposition of \( \nu_1 \) and \( \nu_2 \), precisely corresponding to \( \nu' \). Hence the outgoing neutrino is in the state:

\[ \left| \nu', t = 0 \right> = \sin \varphi \left| \nu_1 \right> + \cos \varphi \left| \nu_2 \right> \]

If this state has a momentum \( p \) along the \( z \) direction, at a later time \( t \) (i.e. at a distance \( z = v t \approx t \), since we are assuming \( p >> m_\nu, m_\mu, m_e \)) we will observe the state:

\[ \left| \nu'(t) \right> = \sin \varphi \left| \nu_1 \right> e^{-i \left( E_1 t - p z \right)} + \cos \varphi \left| \nu_2 \right> e^{-i \left( E_2 t - p z \right)} \]

\[ \approx e^{-i p(t - z)} \left\{ \sin \varphi \left| \nu_1 \right> e^{-i m_\nu^2 t \over 2p} + \cos \varphi \left| \nu_2 \right> e^{-i m_\mu^2 t \over 2p} \right\} \]

where we have used the formula:

\[ E = p + \frac{m_\nu^2}{2p} + O \left( \frac{m_\mu^4}{p^2} \right) \]

The expression in brackets is now not exactly \( \nu' \) or \( \nu \), but rather a \( z \)-dependent combination of the two states. If we place at \( z \) a piece of iron, a charge exchange interaction of the neutrino with the iron may therefore give rise either to a muon or
to an electron, the probability for the two processes being proportional to:

\[ 1 - |a_{\nu'\nu}|^2 \quad (\text{for } \mu \text{ production}) \]  

(9.19)

\[ |a_{\nu'\nu}|^2 \quad (\text{for } e \text{ production}) \]  

(9.20)

where \( a_{\nu'\nu} \) is the neutrino-flip amplitude:

\[ |a_{\nu'\nu}|^2 = \left| \langle \nu \mid \nu' \rangle(t) \right|^2 = (\sin^2 \theta W)^2 \]  

(9.21)

and:

\[ \frac{1}{L_0} = \left| E_1 - E_2 \right| = \frac{|m_1^2 - m_2^2|}{2 \cdot P} \]

In conclusion, if we have a source of \( \mu^- \) -neutrinos at \( z = 0 \), one should see, in a matter target placed at \( z \), a non-vanishing number of \( e^- \)-producing (hence muon number violating) events, the ratio of abnormal to normal events being given by the ratio of (9.20) to (9.19). Similar arguments hold in the case of a source of electron neutrinos. In particular, if we consider a source of low-energy neutrinos (such as a nuclear reactor, or the sun) and in the case of maximal mixing \( (\theta = \frac{\pi}{3}) \), then when \( |a_{\nu'\nu}| \approx 1 \) the neutrinos cannot interact by producing a \( \mu^- \), because of energy conservation. At those distances the neutrinos are "sterile". This effect has been invoked to explain, at least partially, the lack of observation of solar neutrinos.

In the experiments performed thus far no evidence has been found for neutrino oscillations. A refinement of the experimental limits is of great importance, especially since, as our previous analysis indicates, in the framework of the Weinberg-Salam model such a phenomenon would be expected to arise quite naturally.

In conclusion, we have seen that the spontaneous breaking of the gauge symmetry can explain quite simply the observed mass spectrum of leptons.

A disappointing feature of the model is, however, that is is not able to shed any light on two fundamental issues: the relation between electron and muon mass, and the smallness or the vanishing of neutrino masses. Perhaps this could be an indication that this simple model is just the phenomenological manifestation of a more fundamental theory.

10. Hadronic interactions with four quarks.

In this and in the following section, we will describe theories of the weak and \( e.m. \) interactions of quarks. The reason for doing so is, of course, the quark model, whereby all hadrons are supposed to be bound states of these fundamental fermions.

In this framework, it is indeed quite natural to think that a simple theory can be achieved in terms of the fundamental constituents, rather than directly for the composite objects.

Having said that, however, one has to face the serious problem of obtaining meaningful predictions from the theory, applicable to real experiments where only composite hadrons (\( \pi, K, \) etc.) are produced and observed.

We will not discuss herein any detail how this can be done and shall limit ourselves to the following observations.

i) The very fact that one is dealing with a gauge theory requires (see Section 3) that the weak currents are the Noether currents of some strong interaction symmetry, which is exact, at least prior to spontaneous breaking of the gauge group. This is sufficient in certain instances to derive from the quark theory structure-independent predictions for hadronic processes. In particular this fact will allow us to derive the selection rules obeyed by the transition amplitudes.

ii) According to the parton picture, structure dependent effects become increasingly less important in deep inelastic processes, so that one is able to test there the quark weak couplings directly.

Our framework has to be further specified, and we will assume that:

i) Quarks come in different flavors;

ii) for each flavor, quarks come in three different colors, all observed hadrons being color singlets;

iii) weak and \( e.m. \) currents are color singlets and act only on flavor space.\(^(*)\)

To explain the hadronic spectroscopy, only three flavors were needed, prior to \( W^- \) discovery, corresponding to the fractionally charged \( SU(3) \) quarks: \( u, d, s \). It is by now well known, however, that it is impossible to construct a gauge theory of weak interactions based on three flavors only, without

\(^(*)\) This excludes the very important model developed by Pati and Salam. The option of neutral gluons is taken here for simplicity, and we refer the reader to the original papers, for a discussion of the Pati-Salam ideas.\(^4\)
predicting first order strangeness changing, neutral current processes, in striking conflict with the observed suppression of, e.g., $K_L \rightarrow \mu^+\mu^-$. To avoid this inconsistency, we shall follow the Glashow, Iliopoulos and Maiani (GIM) proposal, and introduce from the cutset a fourth flavor, associated to a new (charm carrying) quark $c$, with electric charge $+2/3$.

Now that we have stated our assumptions, we can proceed to assign appropriate $SU(3)\otimes U(1)$ transformation properties to the left and righthanded quark fields.

In doing so, we shall be guided by the lepton-quark symmetry (*). With the addition of the charmed quark, we have in fact, as many quarks as there are leptons, and the charge spectrum is the same, except for an unessential shift of $+2/3$ of quark charges with respect to lepton charges.

We can therefore put lepton and quark fields in a one-to-one correspondence, up to one, very essential, arbitrariness.

Suppose we associate $\nu \leftrightarrow u$ and $\bar{\nu} \leftrightarrow c$. Then we could associate $e \leftrightarrow d$ (and $\mu \leftrightarrow s$) or $e \leftrightarrow s$ (and $\mu \leftrightarrow d$). Actually, since $d$ and $s$ have the same charges, there is no reason to prefer one choice to the other, or to the more general choice:

$$
\begin{align*}
\nu &\leftrightarrow \cos \theta_c \ d + \sin \theta_c \ s \\
\mu &\leftrightarrow -\sin \theta_c \ d + \cos \theta_c \ s \\
\theta_c &\text{ being any angle.}
\end{align*}
$$

(10.1)

Up to this arbitrariness, the lepton-quark symmetry enables us to translate the lepton model of Section 7 into a quark model, and leads us to assume that:

$$
\begin{pmatrix}
u \\
\bar{d}_c \\
s_c
\end{pmatrix}_L
\begin{pmatrix}
u_R \\
\bar{d}_c_R \\
s_R
\end{pmatrix}
$$

are weak isodoublets

$$
\begin{pmatrix}
u \\
\bar{u}_c \\
\bar{c} \\
s_R
\end{pmatrix}_L
\begin{pmatrix}
u_R \\
\bar{u}_c_R \\
\bar{c}_R \\
\bar{s}_R
\end{pmatrix}
$$

are weak isosinglets

(Since weak currents are to be color singlets, (10.2) holds for all the three colors, with the same angle $\theta_c$). Furthermore, as in the lepton case, the transformation rule of the quark fields under $U(1)$ (i.e. the weak hypercharge) is uniquely fixed by the above ansatz and by the requirement that the photon field $A_\mu$ as defined in eq. (6.9) couples to the electric charge.

In conclusion, the lepton-quark symmetry leads us to a weak and e.m. coupling scheme for quarks uniquely determined, up to the yet unspecified angle $\theta_c$ As it is apparent from (10.2), $\theta_c$ is nothing but the Cabibbo angle.

In fact, the coupling of the charged $W$, to the uncharmed quarks, which we can derive from (10.2), is:

$$
-\frac{g}{\sqrt{2}} W^\mu_\mu \\
\nu \gamma^\mu (1-\gamma_5) \left( \cos \theta_c \ d + \sin \theta_c \ s \right) \\
+ h.c.
$$

which precisely coincides with the Cabibbo coupling for semileptonic, $\Delta S = 0, 1$, processes. As a consequence, the position (10.2) completely determines the additional weak interactions involving charmed particles (i.e. neutrino production and weak decays) and it enables us to predict intensity and selection rules for these new processes. Before doing this, however, we have still to examine the properties of weak, neutral current processes with $\Delta S \neq 0$, and the arising of quark masses.

$Z$-quark coupling. As remarked in Section 7, the current coupled to $Z_\mu$ is a linear combination of the e.m. current, and of the current associated to the 3rd component of the weak isospin, $L^3_\mu$. From (10.2) it follows that:

$$
L^3_\mu = \frac{1}{2} \left( \bar{u}_\mu \gamma^\mu u_\mu - \bar{d}_\mu \gamma^\mu d_\mu \right) + \\
+ \frac{1}{2} \left( \bar{c}_\mu \gamma^\mu c_\mu - \bar{s}_\mu \gamma^\mu s_\mu \right) = (10.3)
$$

Therefore $L^3_\mu$ obeys the selection rule $\Delta S = 0$.

Since also the e.m. current is strangeness conserving, we conclude that $K_L \rightarrow \mu^+\mu^-$ cannot occur, to lowest order, through $Z$ exchange.

Actually, the observed ratio:

$$
\frac{\Gamma(K_L \rightarrow \mu^+\mu^-)}{\Gamma(K^+ \rightarrow \mu^+\nu)} = 4.6^{+3}_{-1.5} \times 10^{-9}
$$

is so small as to put also a strict limit to possible higher order contributions.

A non vanishing $K_L \rightarrow \mu^+\mu^-$ amplitude does in fact arise from two $W$ exchange, as illustrated in Fig. 7. A priori, such diagrams should give rise to an amplitude of order:

---

(*) The introduction of a fourth quark to restore lepton-quark symmetry was considered by Y. Hara in 1963.
\[ A(k_{L} - \mu_{L}^2) \approx \frac{\sin \theta_{C}}{H_{W}^2} \approx G d \sin \theta_{C} \]  
(10.4)

This would still be too large, compared to the experimental result. It turns out, however, that in the model there is still an additional suppression. This is due to the fact that, as indicated in Fig. 6, there are two possible diagrams, one with u exchange and the other with c exchange, and the two diagrams exactly cancel in the limit where u and c have equal masses. The correct estimate must therefore contain a further factor which vanishes when \( m_u = m_c \), and indeed one gets:

\[ A \propto \sin \theta_{C} \frac{g^4}{H_{W}^2} \approx \frac{g^2}{H_{W}^2} \approx \sin \theta_{C} \frac{G^2}{(m_c^2 - m_u^2)} \]  
(10.5)

Fig. 6. Feynman diagrams for the \( K^0 \rightarrow \mu^+\mu^- \) amplitude from two W exchange. In the vertical quark line, both u and c exchange have to be considered.

We can now agree with the experiment, provided \( m_c \) is not too large. It was in fact estimated, in the GIM paper, that:

\[ m_c \approx 1 \div 3 \times 10^5 \text{ GeV} \]  
(10.6)

Eq. (10.6) can be interpreted as determining the mass scale of the lowest lying charmed particles. It is very satisfactory that all the "new physics" discovered in the last two years (dimuon events, the \( \Psi^- \), the raise in \( e^+e^- \) cross section, the narrow \( K\pi \) peak etc.) opens up precisely at this range of hadronic masses.

Quark masses. Since the transformation rules implied by (10.2) involve chiral transformations, in the limit where \( SU(2) \otimes U(1) \) is exact the quarks have to be massless. Quark masses must arise from the spontaneous breaking, similarly to lepton masses. To see what sort of mass spectrum arises by coupling quarks to the weak isodoublet \( Q \), we have just to repeat, word by word, the analysis given in Section 9, replacing everywhere \( \nu \rightarrow \mu, \nu \rightarrow c \) and \( e \) and \( \mu \) according to (10.1). Again the most general coupling can be described by eight coupling constants (for which we shall use the same notation even though they differ numerically from the previous ones).

The coupling constants can be chosen to obey either (9.11) or (9.12) and we shall choose (9.12). The mass lagrangian is therefore:

\[ L_{\text{mass}} = k_1 \gamma \cdot \bar{u}(x)u(x) + k_2 \gamma \cdot \bar{c}(x)c(x) + \]  
\[ + g_1 \gamma \cdot \bar{d}_c d_c + g_2 \gamma \cdot \bar{s}_c s_c + \]  
\[ + g_3 \gamma \cdot \bar{c}_d c_d + g_4 \gamma \cdot \bar{c}_s s_s \]  
(10.7)

The lagrangian is not diagonal in the charge - 1/3 fields. By definition, we have to require it to be diagonal in the fields \( d \) and \( s \), corresponding to the eigenstates of the quantum numbers conserved by strong interactions (i.e. the flavors).

This can be obtained, provided:

\[ t_{\alpha\beta}(2 \theta_C) = \frac{2 g_{\beta}}{g_{\alpha} - g_{\beta}} \]  
(10.8)

which defines \( \theta_C \), in terms of the unknown \( g \)'s.

In all, the five undetermined coupling constants in (10.7) give rise to four independent quark masses \( m_u, m_c, m_d, m_s \) and to one mixing angle \( \theta_C \). As was the case for leptons, no mass relation is found, and no explanation is given as to why \( m_u \approx m_d \) (which gives rise to isospin symmetry) or why \( m_u \approx m_s \ll m_c \) (corresponding to the observed broken \( SU(3) \) and to the much less respected \( SU(4) \) symmetry).

On the other hand, from the above considerations a quite elegant picture on the origin of the Cabibbo angle emerges. A non vanishing value of \( \theta_C \) arises because the fields which diagonalize the symmetry breaking lagrangian (chosen by the \( \Phi \)-quark interaction) do not agree with those that are matched by the lepton-quark symmetry to \( e \) and \( \mu \).

It is remarkable that the Cabibbo angle arises precisely through the same mechanism that could give rise to neutrino mixing and to electron and muon number non conservation.

This fact enhances further the interest for a refinement of the experimental limits on the latter phenomenon.
Selection rules for charm-changing weak processes.
We close this Section with a brief account of the selection rules one can derive from (10.2) for the weak production and decay of charmed particles. Only $W$ mediated processes are relevant, as we have seen that the $Z$ boson is coupled to a diagonal current. In what follows, we shall assign charm $C = +1$ to the $c$ quark, and strangeness $S = -1$ to the $s$ quark.

1) Semi-leptonic processes. They are induced by the elementary processes:

$$ C \rightarrow s + \ell^+ + \nu $$  \hspace{1cm} (10.9)

$$ C \rightarrow d + \ell^+ + \nu $$  \hspace{1cm} (10.10)

$\ell = e, \mu$. From the definition of $S_c$ we read that the amplitudes (10.9) and (10.10) are in the ratio $\cos \theta_c : \sin \theta_c$. In both cases, if $\Delta Q$ represents the net hadronic electric charge variation, we have the selection rule:

$$ \Delta C = \Delta Q $$  \hspace{1cm} (10.11)

and since $\cos \theta_c >> \sin \theta_c$, (10.9) is the dominant transition and it obeys

$$ \Delta C = \Delta S $$  \hspace{1cm} (10.12)

The selection rule (10.11) is relevant to the interpretation of the dimuon events seen at FNAL as due to the weak neutrino production of a charmed particle and its subsequent semi-leptonic decay. Since the initial and final states of the overall two-step process are both uncharmed, applying twice (10.11) we predict the total dimuon charge must be zero. This rule is indeed obeyed by the bulk of dimuon events, and also by the $e\mu$ events seen at Gargamelle.

There have been seen a few like-charge dimuons $\mu^-\mu^- \ or \ \mu^+\mu^+$. It is possible that they arise from the associated production of charmed particles, one of the two undergoing a semi-leptonic decay. In this case, no charge correlation is expected. Eq. (10.12) tells us that non-strange charmed particles (expected to be the easier to produce) should decay into a state with $S \neq 0$. This correlation has been observed in the $\mu\mu$ events at Gargamelle and at FNAL (even though in the FNAL events there may be too many $K^0\bar{K}^0$ as one would expect).

1) Non leptonic processes. They are induced by the elementary processes:

$$ C \rightarrow W^+ + d $$  \hspace{1cm} (10.13)

$$ C \rightarrow W^+ + s $$  \hspace{1cm} (10.14)

Since the virtual $W$ goes into hadrons through the Cabibbo coupling, we expect to get from $W^+$:

- $S = 0$ states with amplitude $\cos \theta_c$,
- $S = +1$ states with amplitude $\sin \theta_c$. We therefore get, for the overall transition:

$$ \Delta S = \Delta C \propto \cos \theta_c $$

$$ \Delta S = 0 \propto \sin \theta_c \cos \theta_c $$

$$ \Delta S = - \Delta C \propto \sin \theta_c \sin \theta_c $$

valid up to phase space corrections.

If we apply these considerations to the rates for non leptonic decays of the charmed non strange pseudoscalar mesons $D^0, D^+$ (respectively $c\bar{u}$ and $c\bar{d}$ states), we get the intensity rules:

$$ D_{0, D^+} \rightarrow \left\{ \begin{array}{l}
K^+ + \text{pions, } \bar{K}^0 + \text{pions } \sim \cos \theta_c \\
\text{pions } \sim (\sin \theta_c \cos \theta_c)^2 \\
K^+ + \text{pions, } \bar{K}^0 + \text{pions } \sim (\sin \theta_c)^2
\end{array} \right. $$

The two narrow states recently discovered at SPEAR at about 1.870 MeV nicely fit into this picture. The charge zero state has been seen to decay into $S \neq 0$ states. The charge $= +1$ state has been seen to decay into $K^+ + $ pions but not into $K^- + $ pions, or into multipions, in agreement with the above rules. An improvement in sensitivity so as to be able to check the $(\sin \theta_c)^2$ modes would be extremely important.

11. More flavors?

We know of no reason why quarks and leptons should appear in precisely four different flavors. In fact, there exist already arguments in favor of further hadronic and leptonic flavors. Among them:

1) The observation of the process:

$$ e^+ e^- \rightarrow e^\pm + \mu^\mp + \text{unseen neutrals} $$ (11.1)

has been reported, at SPEAR, at center of mass energies above 4 GeV. The experimental data seem to favor the interpretation of (11.1) as due to the
production of a pair of heavy charged leptons \( U^\pm (\nu_e, \sim 2 \text{ GeV}) \), each decaying into a charged lepton and two neutrinos. If this interpretation is confirmed, and if we accept the principle of lepton-quark symmetry, then one or more new hadronic flavors are implied.

ii) It may be desirable, for theoretical reasons, to formulate weak interactions in the framework of the so-called vectorlike theories' (see below). This is compatible with the present phenomenology only if there exist more quark flavors.

iii) A four-flavor model is basically CP conserving. It turns out that the observed CP violation can be naturally explained with more than four quark flavors.

iv) To account for the observed violations of scaling in neutrino deep inelastic processes, righthanded couplings of new quarks to the light quarks may be required. These couplings can be introduced only with more than four flavors.

Motivated by these or by other reasons, many models have been proposed and discussed in the recent literature.

The reason for the very large proliferation of models is of course to be found in the fact that our present experimental and theoretical knowledge about weak and e.m. interactions is not precise enough to uniquely determine one model, once one decides to put more flavors into the game. There are, however, a number of constraints that any particular model must satisfy, if it has to prove successful. These constraints can be stated as follows:

a) No Adler-Bell-Jackiw anomalies. The renormalizability of a theory where fermions are coupled through vector and axial vector currents can be spoiled by the so-called Adler-Bell-Jackiw anomaly. In the case of the gauge group we are considering, namely \( SU(2) \otimes U(1) \), and if fermions transform as weak isodoublets or isosinglets, the condition of no ABJ anomaly takes the very simple form:

\[
\sum_{\text{Lethanded doublets}} Q - \sum_{\text{Righthanded doublets}} Q = 0
\]

(11.2)

where we have indicated explicitly the contribution from each weak isodoublet, and the factors 3 for the quark doublets arise because any such doublet comes in three different colors.

b) No \( AS \neq 0 \), neutral current transitions, to order \( G \) and \( G \chi \). Though not universally accepted, it is my opinion that the fulfillment of this selection rule should be independent upon the specific values assigned to the parameters of the model (as it is the case e.g. for the four-flavor model).

c) Agreement with low energy phenomenology. To agree with well established facts, no appreciable righthanded couplings must be present between \( u \) quarks and both \( d \) and \( s \) quarks (as required by the Cabibbo theory). Similarly, no righthanded couplings of \( e \) and \( \mu \) to \( \nu \) or \( \nu' \) should be allowed.

d) Charm-strangeness correlation. According to the most recent data about charmed particle decay (see Section 10) the charmed quark must be coupled more strongly to \( s \) as to \( d \) quarks.

We discuss now a few models. Aiming at an illustration of the main underlying ideas, our discussion will be rather schematic. In particular, no details will be given about the structure of symmetry breaking and the resulting mass-spectrum. Only the couplings to the gauge-fields will be considered.

1) A five flavor model\(^5\)

The following isodoublets are considered:

\[
\begin{align*}
(U_L, \nu_L, c_L, \nu'_L, \mu_L) \ (U_R, \nu'_R, \mu_R, \nu''_R) \\
(\bar{d}_L, \bar{s}_L, \bar{L}_L, \bar{S}_L, \bar{\nu}_R) \ (\bar{b}_R, \bar{t}_R, \bar{W}_R, \bar{W}'_R)
\end{align*}
\]

(11.3)

all other fields being weak isosinglets. Here \( b \) stands for the quark field (charge -1/3) which carries the new conserved flavor (beauty?), and \( \nu'' \) is a combination of righthanded neutrinos

\[
\nu''_R = \cos \alpha \nu_R + \sin \alpha \nu'_R
\]

The model introduces only one additional flavor. It obeys the constraints (a) to (d), but the suppression of \( AS \neq 0 \) neutral current processes is "unnatural": a slight mixing of \( b \) with \( d \) and \( s \) quarks:

\[
b_R \to b'_R = \cos \beta b_R + \sin \beta (\cos \varphi d_R + \sin \varphi s_R)
\]
would give rise to $\Delta S \neq 0$ in the neutral current. This requires $\beta = 0$ to hold within an extraordinary accuracy.

This feature is actually shared by all models where there are more $Q = -\sqrt{3}$ than $Q = \sqrt{3}$ quark fields. To this class belong e.g., the superunified models considered by F. Gursey et al. The mass scale of the "beautiful" (b-containing) particles is not determined, if it is in the presently available energy range, anti neutrino b production off $u$ could help explaining the so-called "y anomaly" in $\nu'$ deep inelastic scattering. However, no trace of the $b\bar{b}$ analog of $\nu'$ is seen at SPEAR. If $M_b$ is much larger, one gets the same predictions as in the four-flavor model.

11) A conservative, six flavor model[7]

We introduce lefthanded weak isodoublets only:

$$
\begin{pmatrix}
    u \\
    d_L \\
    c_L \\
    s_L \\
    t \\
    b_L \\
\end{pmatrix}
\begin{pmatrix}
    v \\
    \nu_L \\
\end{pmatrix}
\begin{pmatrix}
    e \\
    \mu_L \\
    \nu_L \nu_L \\
\end{pmatrix}
\begin{pmatrix}
    \nu \\
    \nu_L \\
\end{pmatrix}
\begin{pmatrix}
    v \\
    \nu_L \\
\end{pmatrix}
(11.4)
$$

all righthanded fields being singlets. $b$ is the same as previously, $t$ is another new quark, carrying the sixth flavor (truth?). All our constraints are satisfied, and (b) holds "naturally", i.e. independent upon the value of the mixing angles. Actually to make both $t$ and $b$ quarks unstable, we have to allow in (11.4) for a small mixing of the new with the old quarks. This can be done, keeping the deviations from the Cabibbo theory within the experimental errors.

In this model, the spontaneous symmetry breaking can lead, in addition to Cabibbo like angles, to the arising of one CP violating phase in the weak currents. What results is a "milliweak" theory, which mimics very well a "superweak" model, including the prediction of a very small electric dipole moment for the neutron.\footnote{La}

The $t$ and $b$ mass are usually assumed to be above the SPEAR range. In this case no departure from the four-flavor theory is expected in present experiments. Whether this may cause trouble to explain the $y$ anomaly and other features of neutrino data, is at present unclear.

11) Vectorlike models[9]

Vectorlike models confront us with the new idea of a weak interaction theory which is basically parity conserving, the observed parity violation, in e.g. $\beta$-decays, arising from the spontaneous breaking of the gauge symmetry. Let us see how this works.

A parity transformation changes a one particle state of given momentum and helicity, into a one particle state with opposite momentum and opposite helicity (see Fig. 7). Applying further a 180° rotation around an axis orthogonal to the momentum, we get a state with the same momentum as the one we started with, but with opposite helicity.

$$
\begin{align*}
(q & \quad q' = -q) \\
(a & \quad b) \\
(c & \quad q'' = q)
\end{align*}
$$

Fig. 7. (a) initial state; (b) parity reflected state; (c) parity reflected and 180° rotated state.

So, if the coupling of quarks to the gauge fields is parity conserving, left and righthanded fields must appear in a symmetrical fashion. To each lefthanded weak isomultiplet there must correspond a righthanded multiplet with the same weak isospin and hypercharge. Suppose we have four flavors $(u,d,s,c)$ and we start with the lefthanded isodoublets:

$$
\begin{pmatrix}
    u \\
    d_L \\
\end{pmatrix}
\begin{pmatrix}
    c \\
    s_L \\
\end{pmatrix}
(11.5)
$$

what are the possible parity conserving assignments for the righthanded fields? If the fields are massive, there is a natural way in which we can find the righthanded state to be associated to a given lefthanded state: we go from one to the other by slowing down the particle to its rest-frame, reversing the spin, and then boosting it back to the original momentum. If we require this to be a symmetry of the weak interactions, the righthanded isodoublets to be associated with (11.5) are simply:

$$
\begin{pmatrix}
    u \\
    d_R \\
\end{pmatrix}
\begin{pmatrix}
    c \\
    s_R \\
\end{pmatrix}
(11.6)
$$

and the resulting theory is a vector theory, with no parity violation. If we disregard masses, however, and if there are particles with identical charges (like $d$ and $s$ quarks), ambiguities may arise, leading to different, equally acceptable parity definitions. Suppose we decide that, under a parity operation:
\[\begin{align*}
\underline{u}_L & \rightarrow \underline{u}_R \\
\underline{c}_L & \rightarrow \underline{c}_R
\end{align*}\]  \hspace{1cm} (11.7)

We could then complete (11.7) either by the choice implied by (11.6) or by:

\[\begin{align*}
\underline{d}_L & \rightarrow \underline{s}_R \\
\underline{s}_L & \rightarrow \underline{d}_R
\end{align*}\]  \hspace{1cm} (11.8)

or by the more general choice:

\[\begin{align*}
\underline{d}_L & \rightarrow \cos \theta_{DR} \underline{d}_R + \sin \theta_{SR} \underline{s}_R \\
\underline{s}_L & \rightarrow -\sin \theta_{DR} \underline{d}_R + \cos \theta_{SR} \underline{s}_R
\end{align*}\]  \hspace{1cm} (11.9)

All these choices are equally well suitable for the weak interactions and we have a continuous family of possible parity operations. If the one chosen by the weak interactions does not agree with the one defined by the masses (and by the strong interactions) we have a vectorlike theory. In a vectorlike theory, parity is violated by weak amplitudes. For example, choosing the definitions (11.7) and (11.8) one would construct the weak multiplets:

\[\begin{pmatrix}
\underline{u}_L \\
\underline{d}_L \\
\underline{s}_L \\
\underline{c}_L \\
\underline{d}_R \\
\underline{s}_R
\end{pmatrix}\]  \hspace{1cm} (11.10)

If we restrict e.g. to weak processes involving non strange, uncharmed particles, only the coupling \(\underline{u}_L \rightarrow \underline{d}_L\) is active, and we get out of (11.10) pure V-A (parity violating) amplitudes.

A notable property of any vectorlike theory is that it does never give rise to ABJ anomalies. Eq. (11.2) is always obeyed, since for any weak left isodoublet there is a right isodoublet with the same charge spectrum.

A vectorlike theory with four flavors only, necessarily violates our constraint (c), since the \(\underline{u}_R\) field up to be coupled to some linear combination of \(\underline{d}_R\) and \(\underline{s}_R\).

By introducing two more quark flavors (t and b) we can construct a six flavor model, which is actually uniquely determined by the requirements (c) and (d). The quark lefthanded isodoublets are:

\[\begin{pmatrix}
\underline{u}_L \\
\underline{d}_L \\
\underline{s}_L \\
\underline{c}_L \\
\underline{t}_L \\
\underline{b}_L
\end{pmatrix}\]

(for simplicity of notation, we are neglecting \(\Theta_c\) and other similar notation), while the righthanded doublets are:

\[\begin{pmatrix}
\underline{u}_R \\
\underline{c}_R \\
\underline{t}_R \\
\underline{s}_R \\
\underline{d}_R \\
\underline{b}_R
\end{pmatrix}\]

Leptons are more complicated and we will not discuss them. We cannot just translate the above formulae, with the substitution \(b \rightarrow \nu^{-}, t \rightarrow \nu_{\nu}\), since, in this case, \(\mu_R\) would turn out to be coupled to some combination of \(\nu_{\nu}\) and \(\nu_{\nu}'\) fields.

If we assume that \(m_{\nu}\) and \(m_{\nu}'\) are so high that we cannot presently produce them, the six flavor vectorlike model reduces to the conventional, four flavors, one, except for one very remarkable prediction.

Let us compute the current associated to the 3rd component of weak isospin:

\[L^3_{\mu} = \frac{1}{2} \left(\underline{u}_L \gamma_{\mu} \underline{u}_L - \underline{d}_L \gamma_{\mu} \underline{d}_L\right) + (u, d \rightarrow c, s) + (u, d \rightarrow t, b) + (u, b \rightarrow c, s) + (u, b \rightarrow t, d)\]

Assembling together terms involving the same fields, it is easy to see that \(L^3_{\mu}\) is a pure vector current, with no axial component. This holds true also for those components of \(L^3_{\mu}\) associated to the charged leptons. As a consequence:

1) the interaction of the neutral weak boson Z with hadrons, electrons and muons is exactly parity conserving;

2) inclusive and exclusive neutrino (neutral current) cross-sections are equal to the corresponding anti-neutrino cross-sections.

Statement (1) has to be contrasted with the simple Weinberg-Salam model, where P-violating effects are expected from Z exchange (Sect. 7). It is clear that the experiments sensitive to P-violating, Z related effects are very crucial, in particular those attempting to measure P-violation in atoms. No firm data are, unfortunately, still available.

There are, on the other hand, available data about the processes:

\[v'(|v'\rangle + \text{matter}) \rightarrow v'(|\bar{v}'\rangle + \cdots\]  \hspace{1cm} (11.11)

\[v'(|\bar{v}'\rangle + P \rightarrow v'(|\bar{v}'\rangle + P\]  \hspace{1cm} (11.12)

coming from Gargamelle and FNAL, for (11.11), and from Brookhaven, for (11.12). All present data dis-
agree with the vector-like model, but, perhaps, they are still not accurate enough to exclude it.

12. Conclusions

Gauge theories are emerging as the most likely candidate to describe weak and e.m. interactions.

On the theoretical side, the present status of gauge theories is comparable to that of the best understood examples of quantum field theory, namely quantum electrodynamics. On the experimental side, they are receiving quite impressive confirmations.

It would be premature, however, to say that a Yang-Mills picture of weak and e.m. phenomena has been established. To accomplish this, quite a number of experimental facts must be ascertained, first of all the existence of weak intermediate bosons (both charged and neutral).

The importance of the experimental observation of the weak bosons and of a study of their interactions with other particles and with themselves cannot be overemphasized. It stands out, at present, as the most important task for the next generation of accelerators.

Among other things, the mass spectrum of the weak bosons would tell us something useful about the gauge symmetry breaking, about which we have, at present, no experimental information whatsoever.

We have based our discussion on the gauge group $SU(2)\otimes U(1)$, which is a minimal choice, in the sense that it has the minimum number of gauge fields needed to describe the observed phenomena. Even though this model is receiving substantial support from the experiments (as indicated by the agreement of very different experimental results with the same Weinberg-Salam angle), few would concede that this is the final theory of weak interactions. For one thing, the unification of weak and e.m. interactions achieved in this model is not complete: we have still two independent coupling constants and the spectrum of the electric charge is not fixed by the theory, unlikely that of the weak isospin. The natural solution to this problem could be the embedding of $SU(2)\otimes U(1)$ into a larger gauge group, with no abelian factors and only one coupling. No significant progress has however been yet achieved along this line.

There are other unsatisfactory points:

1) the mass scale associated to quark and lepton flavors are completely undetermined, likewise the Cabibbo angle and all other similar mixing angles; as a consequence we have no explanation whatsoever of the approximate symmetries in strong interactions (e.g. isospin, $SU(3)$) and of the exact or nearly exact vanishing of the neutrino mass;

11) the number of lepton and quark flavors is undetermined; a theoretically sound model (e.g. with no Adler-Bell-Jackiw anomalies) could be constructed out of only one color triplet of $u$ and $d$ quarks and a lepton doublet $\nu, e$; so: why the muon, why the strange and charged particles (let alone new flavors) ?

111) the V-A structure of weak currents is put into the model by hand; no real explanation of parity violation is given;

iv) the observed CP violation can be introduced in the weak interaction in various ways; with only left-handed weak doublets, CP-violation would arise naturally with six quark flavors and this could give a rationale for the existence of such a large number of different quark types; we are still far, however, from a true comprehension of the problem.

Progress in the determination of a more satisfactory theory is conditioned by the acquisition of a number of very crucial experimental informations. In particular:

1) an extensive search should be done for new quark flavors (e.g. in $\omega^\pm \rightarrow \gamma \pi^0$, or $Y(f) \rightarrow P \rightarrow e^+ e^−$ anything) and for lepton flavors; in particular a more complete evidence for the proposed heavy lepton $U$ should be achieved;

11) neutrino and antineutrino processes could give valuable information about the quark weak couplings, as well as about new flavors;

111) more refined limits on CP-violation other than in the neutral $K$ decays, and in particular further improvements of the limits on the electric dipole of the neutron, are badly needed, to see whether the CP-violation is really a weak interaction phenomenon.

It goes without saying that the intensive study of the weak decays of charm particles is very important; besides testing weak interaction models, it could give us a general check of our capacity of extracting predictions for physical processes out of a quark-based theory.

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**Appendix I**

We consider first an electron with momentum \( \mathbf{p} \). Its wave function is a positive energy spinor, obeying the Dirac equation:

\[
(\gamma^0 \mathbf{p} - \mathbf{m}) \psi(r) = (\mathbf{E} \lambda_0 - \mathbf{p} \cdot \mathbf{A} - \mathbf{m}) \psi(r) = 0 \quad (I.1)
\]

There can be two possible polarization states, corresponding to states with spin component along \( \mathbf{z} \) (i.e. helicity) equals to \( \pm \frac{1}{2} \). The helicity operator is given by

\[
\mathbf{h} = \frac{1}{2} \mathbf{z} \cdot \mathbf{p} = \mathbf{M} \cdot \mathbf{p}
\]

In the representation we are using for the Dirac matrices (i.e. the representation used in the Bjorken and Drell book):

\[
\mathbf{z} = \begin{pmatrix} \sigma & \sigma \\ \sigma & -\sigma \end{pmatrix}
\]

(\( \sigma \) are the 2x2 Pauli matrices).

Using the explicit matrix form:

\[
\lambda_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \lambda_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\]

we see that

\[
\lambda_0 \cdot \lambda_0 = \mathbf{z}
\]

so that, by multiplying eq. (I.1) by \( \lambda_0 \psi \) from the left, we get:

\[
\lambda_0 \psi(r) = \left( \frac{\mathbf{z} \cdot \mathbf{p}}{\mathbf{E}} + \frac{\mathbf{m}}{\mathbf{E}} \lambda_0 \right) \psi(r) 
\]

This shows that, in the limit \( \mathbf{m} \to 0 \) (or equivalently \( \mathbf{E} \to \infty \)),

\[
\lambda_0 \psi(r) = \frac{\mathbf{z} \cdot \mathbf{p}}{\mathbf{E}} \psi(r) = 2 \mathbf{h} \psi(r) 
\]

Therefore the two matrices:

\[
\mathbf{a}_\pm = \frac{1 \pm \gamma_3}{2}
\]

project out righthanded, \( \mathbf{a}_+ \), or lefthanded, \( \mathbf{a}_- \), states. The two negative energy solutions of (I.1) are associated to positron states. If we set:

\[
\nu^-(E, \mathbf{p}) = \mathbf{u}(-E, -\mathbf{p}) \quad E = \pm \sqrt{\mathbf{p}^2 + m^2}
\]

then \( \nu \) is the appropriate spinor for a positron traveling with momentum \( \mathbf{p} \). From (I.1) it follows that \( \nu \) obeys the equation:

\[
(\lambda_0 \mathbf{E} - \mathbf{p} \cdot \mathbf{A} - \mathbf{m}) \nu(-E, -\mathbf{p}) = 0
\]

If we multiply again eq. (I.8) by \( \lambda_0 \psi \), we get, in the limit \( m = 0 \):

\[
\lambda_0 \psi(r) = \frac{\mathbf{z} \cdot \mathbf{p}}{\mathbf{E}} \psi(r) = 0
\]

so that, applied to a positron wave function, the operators \( \mathbf{a}_+ \) and \( \mathbf{a}_- \) given by (I.7) project out the lefthanded and the righthanded parts, respectively.

Finally, we consider the quantized field \( \mathbf{E}(O) \). At \( t = 0 \), we may expand \( \mathbf{E}(x, o) \) into creation and annihilation operators, according to:

\[
\mathbf{E}(x, o) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{\mathbf{E}} \sum_k \left( e^{-i\mathbf{p} \cdot \mathbf{r}} a_k(\mathbf{p}) u_\mathbf{k}(\mathbf{p}) +
\right.
\]

\[
+ e^{i\mathbf{p} \cdot \mathbf{r}} b_\mathbf{k}^+ \mathbf{u}_\mathbf{k}^+(\mathbf{r}) \right)
\]

(we normalize spinors according to: \( \mathbf{u}\mathbf{u}^+ = \mathbf{u}^+\mathbf{u} = 1 \), as it is appropriate for massless particles) where \( h \) is the helicity, and the operators \( \mathbf{a}_h(\mathbf{p}) \) (\( \mathbf{b}_h(\mathbf{p}) \)) annihilate an electron (create a positron) of momentum \( \mathbf{p} \) and helicity \( h \). We see therefore that \( \mathbf{e}_h = \mathbf{a}_h \mathbf{e} \) may annihilate only lefthanded electrons and create only righthanded positrons (and vice versa for \( \mathbf{e}_R = \mathbf{a}_- \mathbf{e} \)). These considerations, applied to the neutrino case, show that if in the weak interaction only the field \( \mathbf{\nu}_L = \mathbf{a}_n \mathbf{\nu} \) appears, righthanded neutrinos and lefthanded antineutrinos (associated to \( \mathbf{\nu}_R \)) would be never produced or absorbed in a weak process (and would therefore behave as free unobservable particles, except for gravity).

**Appendix II**

Following the considerations given in Sect. 2, we consider an isodoublet field \( \mathbf{\psi} \), such that:
$$\delta \psi = i \, \epsilon^a(x) \, \frac{\partial}{\partial x^a} \, \psi(x) \quad (\text{II.1})$$

We want to determine the transformation properties of the gauge fields $A^a_\mu$, under the requirement that the covariant derivative

$$\nabla_\mu \psi = \left( \frac{\partial}{\partial x^\mu} + i \, g \, A^a_\mu \, \frac{\partial}{\partial x^a} \right) \psi \quad (\text{II.2})$$

transforms according to:

$$\delta (\nabla_\mu \psi) = i \, \epsilon^a(x) \, \frac{\partial}{\partial x^a} \left( \nabla_\mu \psi \right) \quad (\text{II.3})$$

To simplify our formulae it is convenient to introduce the notation:

$$\epsilon = \epsilon^a \frac{\partial}{\partial x^a} \quad (\text{II.4})$$

$$A_\mu \equiv A^a_\mu \frac{\partial}{\partial x^a}$$

$\epsilon$ and $A_\mu$ are therefore $2 \times 2$ space-time dependent, matrices. We now compute the variation of $\nabla_\mu \psi$. Using (II.1) we find:

$$\delta (\nabla_\mu \psi) = \nabla_\mu (\delta \psi) + i \, g \, A_\mu (\delta \psi) + i g (\delta A_\mu) \psi =$$

$$= i \, \epsilon \, \nabla_\mu \psi + i (\epsilon \, \nabla_\mu) \psi - g \, \epsilon \, A_\mu \psi + i g (\delta A_\mu) \psi$$

and we want to determine $\delta A_\mu$, so that the r.h.s. coincides with:

$$i \, \epsilon \, \nabla_\mu \psi = i \, \epsilon \, \nabla_\mu \psi - g \, \epsilon \, A_\mu \psi$$

Equating the r.h.s. of the last two equations, we find:

$$i g \, \delta A_\mu \psi = - \left[ g (\epsilon \, A_\mu - A_\mu \, \epsilon) + i \, \epsilon \, A_\mu \right] \psi$$

namely:

$$\delta A_\mu \equiv \frac{\partial}{\partial x^a} \left[ i \, (\epsilon, A_\mu) - g \, \epsilon \right] \psi \quad (\text{II.5})$$

Equating the coefficients of $\epsilon \, \nabla_\mu / 2$, we get precisely eq. (2.5) of the text. Note that in the expression of $\delta A_\mu^a$, any reference to the transformation properties of $\psi$ has disappeared. We would have gotten the same result by considering the covariant derivative of any matter field $\Phi$ with non-trivial transformation properties (i.e., such that $\delta \Phi \neq 0$). This is obviously a very important point for the consistency of the whole scheme: the way $A^a_\mu$ transforms must be independent from the type of matter fields we are going to couple to it.

Using the representation of $A^a_\mu$ as $2 \times 2$ matrices, it is also easy to work out the transformation properties of $F_{\mu \nu}^a$. Define:

$$F_{\mu \nu}^a = F_{\mu \nu}^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial x^a} \left[ \partial_\nu A^a_\mu - \partial_\mu A^a_\nu - \partial_\rho A^b_\mu \partial^a_{\lambda} A^c_\nu \right] =$$

$$= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i \, g \, \left[ A^a_\mu, A^a_\nu \right]$$

Then:

$$\delta F_{\mu \nu} = \partial_\mu (\delta A_\nu) - \partial_\nu (\delta A_\mu) + i g \left[ \delta A_\mu, A_\nu \right] +$$

$$+ i g \left[ A_\mu, \delta A_\nu \right] =$$

$$= i \left[ \epsilon, \nabla_\mu - \nabla_\nu \right] -$$

$$- g \left( \left[ \epsilon, A_\mu \right], A_\nu \right) + \left[ \left[ A_\mu, \epsilon \right], A_\nu \right]$$

By using the Jacobi identity, valid for any three matrices $A, B, C$:

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$$

the double commutators can be simplified to give:

$$\delta F_{\mu \nu} = i \left[ \epsilon, \nabla_\mu - \nabla_\nu \right] - g \left[ \epsilon, \left[ A_\mu, A_\nu \right] \right] =$$

$$= i \left[ \epsilon, F_{\mu \nu} \right] = i \left( \epsilon^b \frac{\partial}{\partial x^b} \right) \frac{\partial}{\partial x^a} F_{\mu \nu} =$$

$$= \frac{\partial}{\partial x^a} \left( - \epsilon^b \epsilon^c \epsilon^{\mu \nu} \epsilon_{\mu \nu} \right) = \frac{\partial}{\partial x^a} \left( \epsilon^{\mu \nu} \epsilon_{\mu \nu} \right) =$$

Equating the coefficients of $\epsilon \, \nabla_\mu / 2$, we get eq. (2.8) of the text.

Finally we prove the invariance of the Yang-Mills lagrangian, given in eq. (2.9) of the text. To this aim we observe that we can write

$$L_{YM} = - i \frac{1}{4} \, F_{\mu \nu}^a \, F_{\mu \nu}^a \, F_{\mu \nu}^a =$$

$$= - \frac{1}{2} \, \text{Tr} \left( F_{\mu \nu}^a \, F_{\mu \nu}^a \right)$$

where, as before:

$$F_{\mu \nu} = \frac{\partial}{\partial x^a} F_{\mu \nu}$$

and:

$$\text{Tr} \left( \epsilon \, \nabla_\mu \right) = 2 \, \delta a_b$$

The variation of $L_{YM}$ under gauge transformation is therefore:
\[ \delta \mathcal{L}_W = - \frac{1}{2} \mathcal{T}( \epsilon, F_{\mu \nu} F^{\mu \nu}) = - \frac{i}{2} \mathcal{T}( \epsilon, F_{\mu \nu} F^{\mu \nu}) = 0 \]

Q.E.D.

The last step follows because the trace of the commutator of any two matrices is always zero.

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