IMPROVED ABSOLUTE BOUNDS ON THE $\pi^0\pi^0$ AMPLITUDE

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ABSTRACT

Axiomatic properties are used to derive new absolute upper and lower bounds for the $\pi^0\pi^0$ amplitude in the Mandelstam triangle. In particular, if the amplitude is so normalized that its value at threshold is the scattering length, the value at the symmetry point, which is considered as a measure of the $\pi\pi$ coupling, is shown to lie between -13.5 and 2.75.

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From the very general properties of unitarity, crossing symmetry and axiomatic analyticity, Łukaszyk and Martin ¹⁾ have derived some years ago absolute bounds for the $\pi\pi$ amplitude in the Mandelstam triangle. These bounds, which can be used to constrain various observables like the S wave scattering lengths ^{1),2)} or integrated cross-sections ^{1),3)} have an obvious interest in the absence of any complete theory. Thus many efforts have been devoted to their improvement ^{4),5)}, but they remain nevertheless generally far from the values expected from phenomenological estimates ⁶⁾ or models ⁷⁾.

However, a noticeable improvement on the lower bound of the S wave scattering length has been recently obtained by two of us ⁸⁾. In this letter we show that a convenient generalization of the method used in that case leads to absolute upper and lower bounds of the $\pi\pi$ amplitude at any point inside the Mandelstam triangle. These bounds are easy to obtain and really improve the old ones, as shown by the first numerical applications given here, where we focus on the values taken at the symmetry point which usually stands for the $\pi\pi$ coupling constant.

Let us first recall that the $\pi^0\pi^0$ amplitude $F(s,t,u)$ reads (s,t,u) Mandelstam invariants, $m_\pi=1$) :

$$F(s,t,u) = [F^{I=0}(s,t,u) + 2 F^{I=2}(s,t,u)] / 3 \quad (1)$$

and in the normalization where the partial waves (p.w.) are

$$f_\ell^I(s) = [\eta_\ell^I e^{2i\delta_\ell^I} - 1] / 2i\rho(s) ; \rho(s) = [(s-4)/s]^{1/2} \quad (2)$$

the unitarity constraints for the $\pi^0\pi^0$ p.w. $f_\ell(s)$ are :

$$\rho(s) |f_\ell(s)|^2 \leq \text{Im} f_\ell(s) \quad (3)$$

which implies

$$0 \leq \text{Im} f_\ell(s) \leq \rho(s)^{-1} \quad (4)$$

The method we use is to first express the amplitude at any point (s_0, t_0, u_0) inside the Mandelstam triangle $(s_0, t_0, u_0 < 4)$ as a dispersion relation smeared out in the s and t variables on some part of their physical range. Thus, starting from a twice subtracted fixed t dispersion

relation, we project it on the S wave and take then the average in s of this representation on $[4, s_1]$, where a priori $4 < s_1 < 60$, with an arbitrary "weight" function w normalized to unity :

$$\int_4^{s_1} w(s) ds = 1 \quad (5)$$

This gives the wanted identity :

$$F(s_o, t_o, u_o) = H_o(w; s_o, t_o) + H(w; s_o, t_o) \quad (6)$$

where

i) $H_o(w; s_o, t_o)$ is the S wave contribution :

$$H_o(w; s_o, t_o) = \int_4^{s_1} w(s) \operatorname{Re} f_o(s) ds - \int_4^{\infty} \overline{w}(s) \operatorname{Im} f_o(s) ds \quad (7)$$

with

$$\begin{aligned} \overline{w}(s) = & -\frac{1}{\pi} \int_4^{s_1} w(s') \left\{ \frac{1}{s-s'} + \frac{2}{s'-4} \operatorname{Log} \left(1 + \frac{s'-4}{s} \right) \right\} ds' \\ & - \frac{1}{\pi} \left\{ \frac{1}{s-s_o} + \frac{1}{s-t_o} + \frac{1}{s-u_o} \right\} \end{aligned} \quad (8)$$

ii) $H(w; s_o, t_o)$ is the contribution of the remaining p.w. with $l \geq 2$:

$$H(w; s_o, t_o) = - \sum_{\substack{l \geq 2 \\ l \text{ even}}} (2l+1) \int_4^{\infty} h_l(s) \operatorname{Im} f_l(s) ds \quad (9)$$

with

$$h_l(s) = \int_4^{s_1} ds' \frac{2 w(s')}{s'-4} \int_{(4-s')/2}^0 dt \left\{ \varphi_o(s, t) P_l \left(1 + \frac{2 t_o}{s-4} \right) + \varphi_1(s, s', t) P_l \left(1 + \frac{2 t}{s-4} \right) \right\} \quad (10)$$

and

$$\begin{aligned} \varphi_o(s, t) = & \left\{ (s-t)^{-1} + (s+t+t_o-4)^{-1} - (s-s_o)^{-1} - (s-u_o)^{-1} \right\} / \pi \\ \varphi_1(s, s', t) = & \left\{ (s-s')^{-1} + (s+s'+t-4)^{-1} - (s-t_o)^{-1} - (s+t+t_o-4)^{-1} \right\} / \pi \end{aligned} \quad (11)$$

Our next task is to bound these two contributions from the unitarity constraints (3) for the S wave, and only (4) for the others. For the first one, H_0 , the Lagrange multiplier technique gives a straightforward answer :

$$H_0^-(w; s_0, t_0) \leq H_0(w; s_0, t_0) \leq H_0^+(w; s_0, t_0) \quad (12)$$

with

$$H_0^\pm(w; s_0, t_0) = \int_4^{s_1} ds \rho(s)^{-1} \left\{ \pm [w(s)^2 + \bar{w}(s)^2]^{1/2} - \bar{w}(s) \right\} / 2 - \int_{s_1}^\infty ds \rho(s)^{-1} \bar{w}(s) \theta[\mp \bar{w}(s)] \quad (13)$$

where

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

[The weight has to be chosen to ensure the existence of $H_0^\pm$; this is always possible except for $H_0^+(w; s_0=4, t_0=0)$.]

The remaining component $H(w; s_0, t_0)$ can be bounded, as shown in Ref. 8), through a clever choice of the "weight" function, by :

$$H^-(w; s_0, t_0) \leq H(w; s_0, t_0) \leq H^+(w; s_0, t_0) \quad (14)$$

with

$$H^\pm(w; s_0, t_0) = - \sum_{\ell \geq 2} (2\ell+1) \int_4^\infty ds \rho(s)^{-1} h_\ell(s) \theta[\mp h_\ell(s)] \quad (15)$$

as far as the choice of w ensures the convergence of the integration and summation occurring in (15). For this, the necessary conditions are :

- i) only a finite number of p.w. must contribute for all finite fixed $s \geq 4$;
- ii) as $h_\ell(s) \sim (s-4)^{-\ell}$ when $s \rightarrow 4$, the $h_\ell(s)$ must not contribute in (15) on some neighbourhood of $s=4$, that is to say, they must have a definite sign in this neighbourhood.

When both s and ℓ are large

where the moments $\Psi_n(s)$ and Ψ_ℓ^n are known functions. These constraints are numerically studied in the space of the c_n . For example if $N=2$ we have two families of straight lines in the (c_1, c_2) plane depending upon the parameters ℓ and s , each line defining an admissible half-plane. In fact, it happens that this infinity of constraints can be fulfilled with a finite number of c_n because in practice the asymptotic regimes in ℓ and s are quickly reached.

One can then vary the weight parameters in the admissible domain thus found, as we have done for the points $(4/3, 4/3, 4/3)$ and $(2, 0, 2)$. This gives the following bounds, using $s_1=12$ and the parameters given in the Table :

$$-13.5 < F(4/3, 4/3, 4/3) < 2.75 \quad (28)$$

$$-4.85 < F(2, 0, 2) < 2.9$$

[We note that $H^\pm(w; s_0, t_0) = 0$, except $H^-(w; s_0=2, t_0=0) = -1.19$, so that only the positivity of $\text{Im} f_\ell$ is used for $\ell \geq 2$ to get bounds at the symmetry point.]

We recall the previous best results ⁵⁾ :

$$-36.5 < F(4/3, 4/3, 4/3) < 3.5 \quad ; \quad |F(2, 0, 2)| < 17. \quad (29)$$

which shows that the improvement is noticeable. So it would be satisfactory to find some systematics about optimal weights. For the time being, we only know that if the function w is optimal, it can be interpreted as the Lagrange multiplier associated with the original S wave dispersion relation, which must be exactly verified at the extrema for all s , $4 < s < s_1$. This could be checked, but we have not done it.

We also wish to point out that other bounds can be computed, and in particular for the difference of the amplitude between two points, which can give useful sum rules on cross-sections. Another source of improvement is the use of additional unitarity constraints. Let us recall that up to now we have not incorporated any constraint on the real parts of higher partial waves. So we know that by incorporating the unitarity of the higher waves, we shall get better results. Work along these lines is in progress.

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TABLE : Weight parameters associated to the bounds given in the text.

$F(4/3, 4/3, 4/3)$			$F(2, 0, 2)$	
	Lower bound	Upper bound	Lower bound	Upper bound
α	1.	1.5	1.	1.5
β	0.65	0.97	0.65	1.
s_p	3.65	1.4	3.65	1.33
c_1	-9.662	-0.783	-4.278	-0.715
c_2	11.138	0.553	-3.829	0.575
c_3	9.381	-0.246	2.430	0.087
c_4	-19.739	0.224	-1.638	0.056

R E F E R E N C E S

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