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OFF-SHELL STATES IN DUAL RESONANCE THEORY

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A B S T R A C T

We discuss off-shell states guided by an analogue model approach. This leads us to a more complete understanding of a model proposed recently by Schwarz with critical dimension 16. We are led, by algebraic considerations, to off-shell states in the Neveu-Schwarz-Ramond model, which obey the gauge conditions in the same critical dimension as the on-shell theory, the amplitudes factorizing on the usual positive definite states in 10 dimensions. Brief calculations reveal that some of the divergences present in the orbital model disappear in the fermion theory.

1. INTRODUCTION

The problem of defining off-shell states, and ultimately hadronic currents, in the dual resonance model has proved to be a difficult one mainly because of the highly constrained nature of the theory. This manifests itself in the requirement that at all poles of any amplitude the residues must factorize on a spectrum of positive definite states, even when some of the scattering states are off the mass shell.

Recently, Schwarz¹⁾ and Schwarz and Wu²⁾ have made progress towards constructing current amplitudes, in which the spectrum is not enlarged by explicitly constructing an off-shell state satisfying the gauge conditions of the theory. Using this state, one current and two (adjacent) current amplitudes were constructed and found to have some desirable analytic features such as scaling and reasonable asymptotic behaviour²⁾. However, the off-shell state would only respect the gauge condition in 16 dimensions of space-time rather than the critical dimension 26 of the on-shell model. The reason for this choice of dimension was obscure in the work of Ref. (1). The amplitudes constructed with these off-shell states were also divergent and had to be regularized by hand to obtain finite results.

In this paper we shall approach the problem from a somewhat different point of view and construct a model for off-shell states using the analogue approach³⁾. In the analogue description of dual resonance models amplitudes are associated with integrals over the energy per unit time dissipated by current flow in a two-dimensional resistive medium. This interpretation is related directly to the string formulation of Mandelstam⁴⁾, once we transform to imaginary time. In conventional dual theory, only Neumann boundary conditions, that is constant imaginary potential between the sources on the edge of the Koba-Nielsen upper half plane, have been used, in contrast to the fermion extension when mixed boundary value conditions for the spinor fields (serving to discriminate between meson and fermion strings)^{5,6)} are allowed. Here we shall incorporate mixed boundary conditions for the orbital model also, and interpret the strips along the boundary where the potential has a constant real part as sources corresponding to off-shell scalar states. The construction of the amplitudes will be explained in Section 2.

The analogue idea is used purely as a guide and we bear in mind that later on we shall have to check all the desirable properties of the amplitudes. However, we believe that the approach ought to inject a respectable degree of conformal invariance and hence retain the factorizing properties of the on-shell model.

In principle, we can construct amplitudes for any number of off-shell scalar states in this way, but in practice the technical details are formidable. Instead, having found an amplitude with one off-shell state we shall factorize it operatorially in Section 3, and discuss its properties with respect to the Virasoro gauges

in Section 4. In summary, we shall obtain an operator vertex for the emission of an off-shell state, having well-defined transformation properties under the projective group. In terms of this vertex we can obtain the scattering amplitudes for any number of off-shell states, not necessarily adjacent in a multiperipheral configuration. We shall see explicitly why the value $D = 16$ is picked out to ensure the positivity of pole residues.

Finally, because of the way that we see the space time requirement, we are able to overcome it by using the operators defined in Section 3 to construct an off-shell scalar or pseudoscalar state in the Neveu-Schwarz-Ramond model⁷⁾. We find that the reflection of G gauges is important for checking the absence of ghosts and is sufficient to allow the off-shell state to work in 10 dimensions instead of 16. As a bonus, some amplitudes with one off-shell state do not diverge [they all do in the conventional theory²⁾], in particular, all fermion form factors are finite as far as we have checked. An off-shell vector state can also be constructed having the same properties as the pseudoscalar state with respect to the gauges.

There are technical difficulties, however, which do not yet permit us to check everything that we should, for example, whether factorization in every channel is ghost free and whether amplitudes with many off-shell states enjoy the same desirable properties as they do in the orbital model. For this reason we shall not discuss the properties of amplitudes with two off-shell states in detail here.

Our off-shell states will have no internal quantum numbers, corresponding to $SU(2)$ or $SU(3)$, except those such as 'G parity' defined by the theory itself.

2. ANALOGUE MODEL CONSTRUCTION

The idea behind the analogue picture for ground state particles is to imagine the two-dimensional distribution of current in the upper half plane arising from sources of current (k_1^u) placed at the Koba-Nielsen points, z_i , on the real axis (Fig. 1)³⁾.

Fig. 1

The Koba-Nielsen integrand⁸⁾ is then the exponential of the energy dissipated per unit time in the upper half plane, regarded as having resistivity α' . Self energy is disregarded and the result is just

$$\exp \left\{ -\alpha' \sum_{i \neq j} k_i k_j \log |z_i - z_j| \right\} = \prod_{i \neq j} |z_i - z_j|^{-\alpha' k_i k_j}$$

Now, we imagine an off-shell scalar state to be associated with a strip on the real axis between the points z' , z'' (Fig. 2).

Fig. 2

The properties of the strip are defined to be such that the complex potential corresponding to the configuration, $f(z)$, has a constant real part when z is real and in the range z' to z'' . $\text{Im } f(z)$, of course, must jump by the amount of momentum Q which is distributed on this electrode as z goes from z' to z'' . As in the conventional case $\text{Im } f(z)$ jumps by an amount k_i as z moves past z_i on the real axis. Current is conserved -- thus $\sum_1^N k_i + Q = 0$ if there are N particles and one off-shell state.

It is easy to check that the complex potential with these properties is simply

$$\frac{1}{i\pi} \sum_i^N k_i \log \left[\frac{\sqrt{\frac{z_i - z'}{z_i - z''}} - \sqrt{\frac{z - z'}{z - z''}}}{\sqrt{\frac{z_i - z'}{z_i - z''}} + \sqrt{\frac{z - z'}{z - z''}}} \right] \quad (2.1)$$

and, following the procedure outlined above, the corresponding Koba-Nielsen integrand is

$$\prod_{i \neq j} \left| \frac{\sqrt{\frac{z_i - z'}{z_i - z''}} - \sqrt{\frac{z_j - z'}{z_j - z''}}}{\sqrt{\frac{z_i - z'}{z_i - z''}} + \sqrt{\frac{z_j - z'}{z_j - z''}}} \right|^{-\alpha' k_i k_j} \quad (2.2)$$

As usual we suppose that $\alpha' k_i^2 = -1$, and we shall take $\alpha' = 1/2$ for convenience.

To complete the prescription we choose a measure and integrate over the real axis, keeping three points fixed and maintaining the order of the variables indicated in Fig. 2, to obtain this particular planar contribution to the full amplitude. Notice that the factors (2.2) are projectively invariant, hence the measure must be by itself.

A convenient choice of measure, which also ensures the bootstrap property whenever we look at the residue of a ground state pole in the amplitude, is

$$\int \prod_{i=1}^N \left[\frac{dz_i (z' - z'')}{(z' - z_i)(z'' - z_i)} \right] \frac{dz' dz''}{(z' - z'')^2} \frac{1}{dV_3} \quad (2.3)$$

In this context the bootstrap property means that the residue of any ground state pole factors into an off-shell amplitude and the usual Koba-Nielsen amplitude.

Pictorially,

and this is encouraging. The amplitude (2.2), (2.3) has pole singularities in all channels including Q^2 . The poles in the variable Q^2 lie on the same trajectories as the particle poles. The states in this channel correspond to all the scalar daughters of the theory.

We should remark at this point that the amplitude we have defined is infinite (because of singularities in the measure if $z_i \rightarrow z'$ or z'') this infinity corresponding to that of Ref. 2. Indeed, if we calculate the scalar ground state form factor it comes out to be

$$\int_0^1 \frac{du}{1-u} \left(\frac{1 - \sqrt{1-u}}{1 + \sqrt{1-u}} \right)^{-\frac{1}{2}Q^2 - 2} \equiv \int_0^1 \frac{4dv}{1-v^2} v^{-\frac{1}{2}Q^2 - 2} \quad (2.4)$$

which is precisely the result of Ref. 2. However, we shall not worry too much about this singularity but rather remember that it will disappear when we eventually move to the fermion model and consider off-shell pion states and fermion form factors.

It would be possible, in principle, to construct amplitudes for any number of off-shell states but in practice this is very difficult. Consider the configuration of Fig. 3.

Fig. 3

where we have to deal with off-shell strips, momenta Q_i^μ , (y_i', y_i'') , and particle points z_l, k_l^μ . Let the complex potential arising from this configuration be $f(\xi)$; then we have

$$\frac{df}{d\xi} = \sum_l \frac{k_l}{\xi - z_l} \prod_i \left(\frac{(z_l - y_i')(z_l - y_i'')}{(\xi - y_i')(\xi - y_i'')} \right)^{1/2} P(\xi)$$

where $P(\xi)$ is a polynomial of degree $R - 1$ if R is the number of off-shell strips. The coefficients in this polynomial have to be designed (and this is the hard part) so that the contributions to $\text{Im } f(\xi)$ from the various sources come out correctly. For $R = 1$ we can do the integral and reproduce (2.1). For $R > 1$, it is not easy to do the integral. In this case the integral is an Abelian integral of the third kind. It can be performed in terms of elliptic integrals for the case $R = 2$ ⁹⁾ but we shall not attempt that here. In general, the integral defines a mapping from the Koba-Nielsen ξ plane to the string picture where $f^+ = \tau + i\sigma$, which is a generalization of Mandelstam's mapping ⁴⁾, in so far as the off-shell states terminate at finite times (Fig. 4).

Fig. 4

[Actually, there is a complex potential for each component of the momentum in the analogue model; the mapping above corresponds to the light-cone combination of components $X^+ = (X_0 + X_1)/\sqrt{2}$].

However, instead of attempting to construct more complicated amplitudes, we shall exploit the fact that we have a Koba-Nielsen amplitude with projective invariance to obtain a vertex for the emission of an off-shell state in the operator formalism. Using several of these vertices, we should be able to compute the complicated formulae at least with the reassurance that we always have factorizing amplitudes even though they are very complicated.

3. OPERATOR FACTORIZATION OF THE AMPLITUDE WITH ONE-PARTICLE OFF-SHELL

To facilitate the factorization of the amplitude with one off-mass shell state, namely,

$$\int \prod_{i=1}^N \left(\frac{dz_i (z' - z'')}{(z_i - z') (z_i - z'')} \right) \frac{dz' dz''}{(z' - z'')^2} \frac{1}{dV_3} \prod_{i+j} \left(\frac{\sqrt{\frac{z_i - z'}{z_i - z''}} - \sqrt{\frac{z_j - z'}{z_j - z''}}}{\sqrt{\frac{z_i - z'}{z_i - z''}} + \sqrt{\frac{z_j - z'}{z_j - z''}}} \right)^{-\frac{1}{2} k_i k_j} \quad (3.1)$$

we find it convenient to introduce some new formalism.

Consider the field $R^H(z)$:

$$R^H(z) = -i \sum_{\frac{1}{2}}^{\infty} \frac{1}{\sqrt{r}} \left(c_r^+ z^r - c_r z^{-r} \right) \quad (3.2)$$

where the c_r, c_r^+ operators are annihilation and creation operators labelled by half integers and defined by:

$$[c_r^+, c_s^+] = -g^{rs} \delta_{rs}, \quad [c_r^+, c_s] = 0, \quad c_r |0\rangle = 0 \quad r > 0 \quad (3.3)$$

These operators have been introduced before in a different context¹⁰.

We can define Virasoro generators in this Fock space

$$L_N^c = - \sum_{\frac{1}{2}}^{\infty} \sqrt{N+r} \sqrt{r} c_r^+ c_{N+r} - \frac{1}{2} \sum_{\frac{1}{2}}^{N-\frac{1}{2}} \sqrt{r} \sqrt{N-r} c_r c_{N-r} \quad (3.4)$$

$$L_{-N}^c = L_N^{c^T}$$

and

$$[L_N^c, L_M^c] = (N-M) L_{N+M}^c + \delta_{N+M,0} \frac{DN}{12} (N^2 + \frac{1}{2})$$

with respect to which the field $R(z)$ transforms as a scalar. That is to say

$$[L_N^c, R(z)] = z^{N+1} \frac{dR}{dz} \quad (3.5)$$

Notice that the generators $L_{\pm 1}, L_0$ generate the projective group without c numbers only if an extra $D/16$ is added to L_0^c . This is essentially the reason why the results of Ref. 1 work only in 16 dimensions, as will become more apparent later.

Now, define a vertex $V_R(k, z)$:

$$V_R(k, z) = e^{-ik \cdot R(z)} z^{\frac{1}{2} k^2} \quad (3.6)$$

Then, because of (3.5) we can deduce that $V_R(k, z)$ transforms in precisely the same way as the Fubini-Veneziano vertex $V_0(k, z)$, with respect to projective transformations¹¹⁾.

The vertex (3.6) is useful because it is easy to see that we have an identity

$$\prod_{i < j} \left(\frac{\sqrt{\frac{z_i - z'}{z_i - z''}} - \sqrt{\frac{z_j - z'}{z_j - z''}}}{\sqrt{\frac{z_i - z'}{z_i - z''}} + \sqrt{\frac{z_j - z'}{z_j - z''}}} \right)^{-k_i k_j} = \langle 0 | \prod_{i=1}^N : e^{-i k_i R \left(\frac{z_i - z'}{z_i - z''} \right)} : | 0 \rangle_c \quad (3.7)$$

Furthermore, we can use the projective properties of the right-hand side of (3.7) to remove the dependence on z' and z'' in the vertices. Thus we have, if $\frac{1}{2}k^2 = -1$,

$$\begin{aligned} \langle 0 | O(\mu) \prod_{i=1}^N V_R(k_i, z_i) O(\mu^{-1}) | 0 \rangle_c \\ = \prod_{i=1}^N \left(\frac{(z' - z'')}{(z' - z_i)(z'' - z_i)} \right) \langle 0 | \prod_{i=1}^N : e^{-i k_i R \left(\frac{z_i - z'}{z_i - z''} \right)} : | 0 \rangle_c \end{aligned} \quad (3.8)$$

where μ is the projective transformation $\xi \rightarrow (\xi - z')/(\xi - z'')$ and $O(\mu)$ is its representation in the c Fock space. The vacuum state is not kept invariant by projective transformations in this representation, but we can simplify the left-hand side of (3.8) a little since,

$$\begin{aligned} O(\mu^{-1}) | 0 \rangle &= (z' - z'')^{D/16} e^{z' L_{-1}} | 0 \rangle \\ \langle 0 | O(\mu) &= \left(\frac{z' - z''}{z''^2} \right)^{D/16} \langle 0 | e^{\frac{1}{z''} L_{+1}} \end{aligned}$$

The right-hand side of (3.8) is the integrand of our amplitude, when we make use of (3.7), so the next step is to write the left-hand side in a form which explicitly displays the usual vertices for the on-shell particle states. We write

$$\begin{aligned} \langle 0 | O(\mu) \prod_{i=1}^N V_R(k_i, z_i) O(\mu^{-1}) | 0 \rangle \\ \equiv \rho \langle 0 | \prod_{i=1}^L V_0(k_i, z_i) Z(z', z'', Q) \prod_{L+1}^N V_0(k_i, z_i) | 0 \rangle \end{aligned} \quad (3.9)$$

and use this equation to define the off-shell vertex $Z(z', z'', Q)$ ^{*)}. Notice, that

^{*)} ρ is an 'eigenvalue' to be obtained, see Eq. (3.12) below.

although the off-shell operator depends on two variables z' , z'' our construction implies that it transforms like,

$$O(\gamma) Z(z', z'', Q) O(\gamma^{-1}) = Z(\gamma(z'), \gamma(z''), Q)$$

since we have maintained the projective invariance throughout. Thus, provided we can calculate the operator Z we have found our amplitude in the form

$$\int_{\text{ordering}} \frac{\prod d\tau_i}{dV_3} \frac{d\tau' d\tau''}{(\tau' - \tau'')^2} \langle 0 | \prod_{i=1}^L V_0(k_i, \tau_i) Z(z', z'', Q) \prod_{i=1}^N V_0(k_i, \tau_i) | 0 \rangle_\alpha \quad (3.10)$$

as desired.

It is relatively straightforward to work out what $Z(z', z'', Q)$ has to be from (3.9). It has the form:

$$Z(z', z'', Q) = \left(\frac{z' - z''}{z''} \right)^{D/8} \langle 0 | \exp F^+(\frac{1}{2} z'') | 0 \rangle_\beta e^{\frac{1}{2} z'' L_{+1}} e^{-i q \cdot Q} \\ \times e^{z' L_{-1}} \langle 0 | \exp F(z') | 0 \rangle_\alpha \quad (3.11)$$

where $\langle 0 |$ and $| 0 \rangle_\beta$ are the vacuum for two identical Fock spaces corresponding to the ordinary Fubini-Veneziano creation operators lying to the right and left of Z , respectively, in Eq. (3.10) except that neither of these vacua are annihilated by p the momentum operator. This is necessary so that energy-momentum conservation works out in the usual way operatorially [in the literature the momentum δ function is usually not mentioned for convenience, but it is in fact automatically there in the operator formalism because of the zero modes¹¹⁾]. The operator $\langle 0 | \exp F(z') | 0 \rangle_\alpha$ has the property

$$\langle 0 | \exp F(z') | 0 \rangle_\alpha = \lambda e^{-ik \cdot q} V_R(k, z - z') \langle 0 | \exp F(z') | 0 \rangle_\alpha \quad (3.12)$$

$$\langle 0 | \exp F(z') | 0 \rangle_\alpha | 0 \rangle_\alpha | 0 \rangle_\beta = | 0 \rangle_\alpha$$

provided $F(z')$ is given by

$$F(z') = \frac{1}{2} \frac{1}{(2\pi i)^2} \oint_C \frac{dx dy}{xy} P(x) A(x, y) P(y) + \frac{1}{(2\pi i)^2} \oint_{C'} \frac{dx dy}{xy} P(x) B(x, \frac{1}{y}) S(y)$$

where $S^H(y) = -iydR^H(y)/dy$ and the functions A, B have to be:

$$A(x, y) = \sum_0^\infty x^n A_{nm} y^m = 2 \log (\sqrt{y-z'} + \sqrt{x-z'}) \quad (3.13)$$

$$B(x, \frac{1}{y}) = \log \left(\frac{\sqrt{y} - \sqrt{x-z'}}{\sqrt{y} + \sqrt{x-z'}} \right) = \sum_{n=0}^\infty \sum_{r=k}^\infty x^n \beta_{nr} y^{-r}$$

$$\lambda = 2k^2$$

The contours of integration must be such that on C $|x|$ and $|y| < |z'|$ but on C', $|x| < |z'|$ and $|y| > |x - z'|$. The eigenvalue λ entering into Eq. (3.12) comes out to be $2k^2$ and is thus $1/4$ for the conventional model where $1/2k^2 = -1$. Hence ρ , appearing in Eq. (3.9), must be 4^N .

We notice the similarity between the function A(x,y) and the matrices occurring in the formalism of Ref. 1. We also notice that the procedure we have been following is very similar to that used to deduce the correct form of a Fermion emission vertex^{12,13}). In fact when we come, in the next section, to consider the various gauges we shall be able to argue quite often by analogy.

A further piece of information that we will need, in order to be able to write the expression (3.12) for our amplitude in such a way as to explicitly display the poles corresponding to a particular multiperipheral configuration, is how to transform the quantity $e^{z'L^{-1}} \alpha \langle 0 | \exp F(z') | 0 \rangle_c$, projectively. It is straightforward to deduce that we must have

$$O^c(\gamma) e^{z' L_{-1}^c} \alpha \langle 0 | \exp F(z') | 0 \rangle_c O^a(\gamma^{-1})$$

$$= (cz' + d)^{-0/8} e^{\gamma(z') L_{-1}} \alpha \langle 0 | \exp F(\gamma(z')) | 0 \rangle_c \quad (3.14)$$

by considering the defining equation (3.12) and the simple case of a vacuum state on the right, remembering that

$$\alpha \langle 0 | \exp F(z') | 0 \rangle_c | 0 \rangle_{\alpha, p} \equiv | 0 \rangle_c$$

$O^c(\gamma)$, $O^a(\gamma^{-1})$ are the representations in the c, a Fock spaces, respectively, of the projective transformation $\gamma: z \rightarrow (az + b)/(cz + d)$, and $ad - bc = 1$. One must remember that in the c space $L_0^c | 0 \rangle = D/16 | 0 \rangle$ and not zero because of the redefinition necessary to preserve the correct commutation relations for the projective group.

Using the relation (3.14) we can deduce a similar transformation law for the other part of the vertex, thus,

$$\begin{aligned}
 O(\gamma) \langle 0 | \exp F^+ \left(\frac{1}{z''} \right) | 0 \rangle_c e^{\frac{1}{z''} L_{+1}^c} (z'')^{-\frac{D}{8}} O(\gamma^{-1}) \\
 = (c z'' + a)^{-\frac{D}{8}} \langle 0 | \exp F^+ \left(\frac{1}{\gamma(z'')} \right) | 0 \rangle_c e^{\frac{1}{\gamma(z'')} L_{+1}^c} (\gamma(z''))^{-\frac{D}{8}}
 \end{aligned}$$

and hence verify that the complete off-shell vertex $Z(z', z'', Q)$ transforms as we claimed it should.

Now, using (3.14) and the known way in which the conventional vertices transform we can rewrite (3.10) in a multiperipheral form:

$$\begin{aligned}
 \langle 0 | e^{-ik_1 \cdot q} V_0(k_2, 1) \frac{1}{L_0 - 1} \dots V_0(k_L, 1) \frac{1}{L_0 - 1} J(Q) \frac{1}{L_0 - 1} V_0(k_{L+1}, 1) \dots \\
 \times \frac{1}{L_0 - 1} V_0(k_{N+1}, 1) e^{-ik_N \cdot q} | 0 \rangle
 \end{aligned}$$

corresponding to Fig. 5.

Fig. 5

Here the off-shell state vertex has simplified to $J(Q)$, where

$$\begin{aligned}
 J(Q) = \int_0^1 \frac{du}{u(1-u)^{2(1-\frac{D}{16})}} \langle 0 | \exp F^+(1) | 0 \rangle_c e^{\frac{L_{+1}^c}{L_0 - 1}} e^{-iq \cdot Q} u^{\frac{L_0^c - D}{16}} \times \\
 \times e^{\frac{L_{-1}^c}{L_0 - 1}} \langle 0 | \exp F(1) | 0 \rangle_c
 \end{aligned} \tag{3.15}$$

and we can again see the singularity of the integrand making the integral undefined, but in quite a different way. Here it arises because if we write

$$\int_0^1 \frac{du}{u(1-u)^{2(1-\frac{D}{16})}} u^{\frac{L_0^c - D}{16}} = \sum_{r=0}^{\infty} \frac{\alpha_r}{L_0^c - \frac{D}{16} + r}$$

we see that there is a divergence (3.15) arising from the term

$$\langle 0 | \frac{1}{L_0^c - \frac{D}{16}} | 0 \rangle_c$$

Also note that for $D = 16$ the above integral is trivial and becomes simply

$$\frac{1}{L_0 - 1}$$

and looks like all the other propagators. Of course this quantity is not a propagator because it has no dependence on momentum, but it is convenient to write everything this way because we shall see in the next section that it facilitates the calculation of gauge properties of the vertex $J(Q)$. $J(Q)$ contains poles in the variable Q^2 but we have not been able to obtain an operatorial way of displaying this fact.

It is clear that, in principle, we can now write down the amplitudes for any number of off-shell states and there is no reason why they would have to be adjacent in the multiperipheral configuration. However, they do diverge and so some regularization procedure such as that of Ref. 2 would have to be adopted.

One other remark that should be made is that if we had chosen to factorize in a configuration in which our off-shell state is at one end of the tree instead of in the middle (Fig. 5) the amplitude would have simplified to

$$\langle J(Q) | \frac{1}{L_0 - 1} V_0(k_1, 1) \frac{1}{L_0 - 1} \dots \frac{1}{L_0 - 1} V_0(k_{N-1}, 1) e^{-ik_N \cdot q} | 0 \rangle \quad (3.16)$$

where the off-shell state is defined by

$$\langle J(Q) | = \langle 0 | e^{-iq \cdot Q} \propto_P \left\{ \frac{1}{2} \frac{1}{(\pi i)^2} \int_C \frac{dx dy}{xy} P(x) A(x, y) P(y) \right\}$$

with

$$A(x, y) = 2 \log (\sqrt{x-1} + \sqrt{y-1})$$

and is essentially the state discovered by Schwarz¹⁾.

4. GAUGE RELATIONS

In this section we shall consider the behaviour of our off-shell vertex with respect to the Virasoro gauges. We can easily see what should happen if we look at the infinitesimal version of Eq. (3.14):

$$\begin{aligned} L_N^c e^{z' L_{-1}^c} \langle 0 | \propto_P F(z') | 0 \rangle_c &= e^{z' L_{-1}^c} \langle 0 | \propto_P F(z') | 0 \rangle_c L_N^a \\ &= z'^N \left(z' \frac{d}{dz'} + (N+1) \frac{D}{16} \right) e^{z' L_{-1}^c} \langle 0 | \propto_P F(z') | 0 \rangle_c \end{aligned} \quad (4.1)$$

for $N = \pm 1, 0$. Clearly, this is the relation we expect to be true for every N since this is so much like the fermion case¹³⁾. In fact, the reasoning employed there^{13,14)} to deduce such a relation as (4.1) works here too so we do not need to repeat the argument, but merely say that the field $P(z)$ is transformed into the field $S(z)$ in exactly the same way as $H(z)$ was converted to $\Gamma(z)$ in the fermion theory¹³⁾. We shall take the parallel further later on.

Using Eq. (4.1) we can deduce that

$$(\tilde{L}_N - \tilde{L}_0) e^{\tilde{L}_{-1}} \langle 0 | \exp F(1) | 0 \rangle_c = e^{\tilde{L}_{-1}} \langle 0 | \exp F(1) | 0 \rangle_c (\tilde{L}_N - \tilde{L}_0 + \frac{ND}{16}) \quad (4.2)$$

whilst complex conjugation provides the corresponding relation for the other half of the off-shell vertex $J(Q)$. Then, exploiting the fact that when $D = 16$ we have what looks like a propagator $1/(\tilde{L}_0^C - 1)$ in the C space, we can deduce from Eqs. (4.2) and (3.15) that

$$(\tilde{L}_N - \tilde{L}_0 - N + 1) J(Q) = J(Q) (\tilde{L}_N - \tilde{L}_0 + 1) \quad (4.3)$$

In other words the off-shell vertex behaves in the same way with respect to the gauges as a conventional vertex $V_0(k, 1)$ but without requiring $\frac{1}{2} Q^2 = -1$! The price that we have to pay for this is that it only works when $D = 16$ and there seems to be a mismatch between the on- and off-shell critical dimension¹⁾. Nevertheless, we know that for $D \leq 26$ equations like (4.3) are sufficient to ensure the absence of ghosts, since an arbitrary tree state satisfies the usual Virasoro gauge conditions¹⁵⁾.

However, now that we have an understanding of the $D = 16$ requirement we can push the analogy with the fermion theory a little further. Let us look at the consequences of (4.2) for the off-mass shell state when it occurs at the end of a tree. Clearly we shall have from Eq. (4.2)

$$\langle J(Q) | (\tilde{L}_{-N}^C - \tilde{L}_0^C + (1-N) \frac{D}{16}) = 0 \quad N = 1, 2, \dots \quad (4.4)$$

remembering that ${}_c \langle 0 | \tilde{L}_0^C = {}_c \langle 0 | D/16$ and that ${}_c \langle 0 | \tilde{L}_{-N}^C = 0$. Now, a gauge condition of this type has occurred before in the fermion theory. The ground state massless fermion-antifermion pair $\langle \bar{F} F |$ satisfies just such a condition, though the gauges are constructed in the combined b and a Fock spaces¹⁶⁾. However, it is quite trivial to change Eq. (4.4) by the addition of a vacuum ${}_b \langle 0 |$ on the left. This is the

vacuum of the anticommuting annihilation and creation operations of the Neveu-Schwarz model ⁷⁾. Then we genuinely have a correspondence

$$\langle F\bar{F} | \longleftrightarrow \langle 0 | \langle J(Q) | \equiv \langle \pi(Q) |$$

in the sense that they both satisfy the same Virasoro gauge condition, Eq. (4.4).

Remarkably, the correspondence remains true when we consider the G gauges acting on the state $\langle \pi(Q) |$. We find that

$$\langle \pi(Q) | \left(G_{-r} - \sum_{\frac{1}{2}}^{\infty} \alpha_{rs} G_s \right) = 0 \quad (4.5)$$

just as in the fermion theory ¹⁶⁾, where

$$\alpha_{rs} = (-)^{r+s+1} \frac{r}{r+s} \binom{-\frac{1}{2}}{r-\frac{1}{2}} \binom{-\frac{1}{2}}{s-\frac{1}{2}} \quad r, s = \frac{1}{2}, \frac{3}{2}, \dots \quad (4.6)$$

and

$$\sum_{\frac{1}{2}}^{\infty} \alpha_{rs} y^s = y^{-r} + i \sqrt{1-\frac{1}{4}} \sum_{k=0}^{r-\frac{1}{2}} y^{k+\frac{1}{2}-r} \binom{-\frac{1}{2}}{k} (-)^k$$

Expression (4.5) is deduced from the formula (compare Refs. 14, 16), which is easily checked using the defining equations for $F(z')$, Eqs. (3.13),

$$\begin{aligned} e^{z' L_{-1}^C} \langle 0 | \exp F(z') | 0 \rangle_C &= \oint_{\frac{0}{2\pi i z}} \frac{dz}{z} f(z) \sqrt{\frac{z-z'}{z}} P(z) \cdot H(z) \\ &= \oint_C \frac{dz}{2\pi i z} f(z) \sqrt{\frac{z-z'}{z}} S(z) \cdot H(z) e^{z' L_{-1}^C} \langle 0 | \exp F(z') | 0 \rangle_C \end{aligned} \quad (4.7)$$

where $H(z)$ is the usual Neveu-Schwarz field, by a suitable choice of $f(z)$ and putting $z' = 1$. The expression (4.7) is true provided the function $f(z)$ is regular at $z = z'$ and has poles at the origin only. A suitable choice is

$$f(z) = \sum_0^{r-\frac{1}{2}} \binom{-\frac{1}{2}}{k} (-)^k z^{k-r+\frac{1}{2}}$$

The contour 0 encircles the origin and avoids the branch point at z' , the contour C encloses the cut from 0 to z' in the integrand of the right-hand side of (4.7). Notice that $H(z)$ is unaltered, but $P(z)$ changes from an integrally moded field to a half integrally moded one. Using (4.5) twice reproduces the Virasoro gauge condition (4.4) for $N = 2r$.

The gauge conditions (4.4) and (4.5) suggest that we try to use this state as a negative 'G-parity' off-shell state in the Neveu-Schwarz model -- that is as a 'pion' off the mass shell. So we now consider amplitudes such as:

$$\langle \pi(Q) | \frac{1}{L_0 - \frac{1}{2}} V_H(k_1, i) \frac{1}{L_0 - \frac{1}{2}} \dots \frac{1}{L_0 - \frac{1}{2}} V_H(k_{N-1}, i) e^{-ik_N q} | 0 \rangle \quad (4.8)$$

and investigate their properties. It is clear, however, that by doing this we are losing the generality of having a vertex for the emission of the off-shell state and that more work is required to recover that from (4.8). On the other hand, we can see that at least in this simple case the residues of any pole will factorize onto the positive definite transverse states in 10 dimensions.

If we look at the residue of a pole in the propagator adjacent to the off-shell state we can make use of the physical state projection operator¹⁸⁾ to check that there are only physical states coupling. This calculation is algebraically identical to the calculation in the fermion model^{16,17)} and will not be repeated. What can be shown is

$$\langle \pi(Q) | (1 - \tau) \mathbb{M} V_H(k_1, i) \frac{1}{L_0 - \frac{1}{2}} \dots \frac{1}{L_0 - \frac{1}{2}} V_H(k_{N-1}, i) e^{-ik_N q} | 0 \rangle = 0$$

where $\tau(k)$ is the physical state projection operator depending on a lightlike k , and $\mathbb{M}$ is the projection onto a mass shell¹⁸⁾,

$$\mathbb{M} = \frac{1}{4\pi i} \oint \frac{dz}{z} z^{L_0 - \frac{1}{2}}$$

Hence τ is effectively one when inserted at this point of the tree.

A slightly more subtle argument is needed if we try to use the projection operator technique to check the absence of ghosts at other pole residues. Usually it is necessary to rely on the fact that a cancelled propagator gives zero⁷⁾, but now this is no longer true for all propagators, only those non-adjacent to the off-shell states. This is because factors like (2.2) appear when we calculate the amplitude, and by analytic continuation we can argue that if two Koba Nielsen variables, say z_i and z_{i+1} , are forced to be the same because a propagator is missing in the operator expression, then that contribution is zero. However, when a propagator is next to the off-shell state, cancelling it is the same as forcing z_i , say, to be equal to either z' or z'' and that does not produce a zero. However, because of the reflection property of the G gauges on the off-shell state, there will always be two cancelled propagator contributions; that is in detail

$$\begin{aligned}
 G_r V_H \frac{1}{L_0 - \frac{1}{2}} |\pi(Q)\rangle &= (L_0 + r - \frac{1}{2}) V_0 \frac{1}{L_0 - \frac{1}{2}} |\pi(Q)\rangle \\
 + \sum_s \alpha_{rs} \left[G_{-s} V_H \frac{1}{L_0 + r + s - \frac{1}{2}} - (L_0 + r - \frac{1}{2}) V_0 \frac{1}{L_0 + r + s - \frac{1}{2}} \right] |\pi(Q)\rangle &+ \left(\sum_s \alpha_{rs} - 1 \right) V_0 |\pi(Q)\rangle \quad (4.9)
 \end{aligned}$$

Each term but the last in Eq. (4.9) is accounted for in the usual fermion argument, and the last term is usually said to be zero because the propagator is cancelled. Actually it is zero anyway, because of the definition of α_{rs} , [Eq. (4.6)] since

$$\sum_{s=\frac{1}{2}}^{\infty} \alpha_{rs} \equiv 1$$

To conclude, substituting $\mathcal{M}(\tau - 1)$ for any propagator in the tree represented by Eq. (4.8) gives zero, as desired.

We ought to emphasize at this point that although we are encouraged by what we have found we should obtain the off-shell vertex and check the absence of ghost states in all multiperipheral channels. We have not been able to do this yet because of technical details involving moving from the $\mathcal{Y}_2$ to $\mathcal{Y}_1$ picture in order to deduce the projectively invariant vertex.

We remark, at this stage, that we can also find an off-shell vector state with positive 'G parity' satisfying the gauge conditions (4.4) and (4.5). It is simply the state,

$$\langle 0 | \sum_{r=\frac{1}{2}}^{\infty} b_r^{\sim} | \mathcal{J}(Q) \rangle \equiv \langle e^r(Q) | \quad (4.10)$$

To see this, look at the integrated version of Eq. (4.7) more closely. Writing

$$K_N = \oint \frac{dz}{2\pi i z} z^N S(z) \cdot H(z)$$

we have, choosing the function $f(z)$ as indicated previously,

$$\begin{aligned}
 e^{\mathcal{L}_1} \langle 0 | \exp F(1) | 0 \rangle_c &= (G_{-r} - \sum_{\frac{1}{2}}^{\infty} \alpha_{rs} G_s) \\
 &= \sum_0^{\infty} \beta_{rN} K_{-N} e^{\mathcal{L}_1} \langle 0 | \exp F(1) | 0 \rangle_c \quad (4.11)
 \end{aligned}$$

with

$$\beta_{rN} = (-)^{N+\frac{1}{2}-r} \sum_{k=0}^{r-\frac{1}{2}} \binom{\frac{1}{2}}{N-k} \binom{-\frac{1}{2}}{r-\frac{1}{2}-k} \quad \begin{matrix} r = \frac{1}{2}, \frac{3}{2}, \dots \\ N = 0, 1, \dots \end{matrix} \quad (4.12a)$$

and

$$\sum_{N=0}^{\infty} z^{r-\frac{1}{2}} \beta_{rN} y^N = \sqrt{\frac{1-y}{1-zy}} \frac{1}{1-zy} \quad (4.12b)$$

Then we shall have

$$\langle P^r(Q) | (G_{-r} - \sum_{s \neq r} G_{rs} G_s) = \langle 0 | \langle 0 | \sum_{\frac{1}{2}}^{\infty} C_k \sqrt{k} \sum_k^{\infty} \beta_{r, l-k} e^{L_{-1}^C} \langle 0 | \langle P^r(1) | 0 \rangle_C \quad (4.13)$$

noting that $\langle 0 | K_{-N} = 0$, $N = 0, 1, \dots$ and that $\langle 0 | L_{-1}^C = 0$ also. The right-hand side of (4.13) is identically zero because

$$\sum_{N=0}^{\infty} \beta_{rN} = 0, \quad r \geq \frac{1}{2}$$

following directly from Eq. (4.12b).

5. FORM FACTORS AND OFF-SHELL AMPLITUDES

In this section we shall calculate the fermion pseudoscalar form factor and the four-'pion' amplitude, with one state off the mass shell, as examples. Consider first the form factor corresponding to Fig. 6.

Fig. 6

We can calculate this by taking advantage of the fact that the off-shell state satisfies the same gauge conditions as the fermion-antifermion ground-state pair in 10 dimensions and hence the correction factor to the propagator will be the same as that calculated by Olive and Scherk for the four-fermion amplitude¹⁷⁾. Since the algebraic manipulations, using the on-shell physical state projection operator, are identical in the two cases we shall omit them, but note that the corrected amplitude is simply

$$\langle \pi(Q) | \int_0^1 du \frac{u^{L_0 - \frac{3}{2}}}{\Delta(u)} | F \bar{F} \rangle \quad \text{and}^{19)} \quad \Delta(u) = (1-u)^{1/4} \quad (5.1)$$

The expectation value in (5.1) is readily evaluated using the expression for the fermion emission vertex¹²⁻¹³⁾, the defining equation (3.12) and remembering that the fermions are massless, $k_1^2 = k_2^2 = 0$, so that $\lambda = 1$ in this case. A further piece of information that is needed is

$$\exp F(1) e^{-ik \cdot q} |0\rangle = \exp \left\{ -k \cdot \sum_{\frac{1}{2}}^{\infty} \frac{c_r^+}{\sqrt{r}} (-1)^r \right\} e^{-ik \cdot q} |0\rangle \quad (5.2)$$

which follows from the definition of $B_{\text{or}}(1)$, Eq. (3.13). The result of this calculation is simply

$$\begin{aligned} F(Q) &= \int_0^1 \frac{du}{u^{3/2}} \frac{1}{(1-u)^{1/4}} \left(\frac{1-\sqrt{1-u}}{1+\sqrt{1-u}} \right)^{-k_1 \cdot k_2} \bar{u}_2 \gamma^5 u_1 \\ &= \frac{1}{2} \int_0^1 dv \sqrt{1-v} (1+v)^{3/2} v^{-1/2} Q^2 - \frac{3}{2} \bar{u}_2 \gamma^5 u_1 \end{aligned} \quad (5.3)$$

where we have made the change of variables $u = 4v/(1+v)^2$ to provide the second expression.

Expression (5.3) is perfectly finite, in contrast with the ground-state form factor in the orbital model which diverges²⁾. It has poles at positions on the pion trajectory and is a pseudoscalar coupling. We could equally well have found the scalar coupling (recall that the fermion theory requires the existence of degenerate parity doublets for odd G states^{6,19)} which would be identical to the above apart from the γ^5 . For large space-like Q^2 we have the behaviour

$$f(Q) \sim \frac{1}{(\cdot Q^2)^{3/2}} \sqrt{1\pi} \bar{u}_2 \gamma^5 u_1$$

to leading order. It must be said, however, that such power behaviour may be misleading since nowhere in our derivation of the off-shell state are we prevented from multiplying by an exponentially falling function of Q^2 . We suspect, however, that such arbitrary functions may not be necessary although, for example, Cremmer and Scherk²⁰⁾ did require them for their current scheme.

The vector form factor is equally easy to calculate and is

$$F^{\mu}(Q) = \int_0^1 dx \sqrt{\frac{1+x}{1-x}} x^{-1/2} Q^2 - 1 \bar{u}_2 \gamma^{\mu} u_1 \quad (5.4)$$

This expression is also finite and has poles on the ' ρ ' trajectory. Note also that $Q_{\mu} F^{\mu}(Q) \equiv 0$ since the fermions are massless. However, neither (5.4) or (5.3) are correctly signaturized and we have not been able to understand the reflected gauge conditions when there is a twisting operator in the way.

Finally, we shall calculate a contribution to the off-shell amplitude of Fig. 7 for four 'pions'.

Fig. 7

Thus we have

$$\begin{aligned}
 A_4(Q) &= \langle \pi(Q) | \frac{1}{L_0 - 1/2} V_H(k_3, 1) \frac{1}{L_0 - 1/2} V_H(k_2, 1) e^{-i k_1 \cdot q} | 0 \rangle \\
 &= -k_3 \cdot k_2 \int_0^1 \frac{du dv}{\sqrt{1-u} \sqrt{1-nv}} \frac{1}{(1-v)} \left(\frac{1 - \sqrt{1-u}}{1 + \sqrt{1-u}} \right)^{-k_1 \cdot k_3} \left(\frac{\sqrt{1-u} - \sqrt{1-nv}}{\sqrt{1-u} + \sqrt{1-nv}} \right)^{-k_2 \cdot k_3} \\
 &\quad \times \left(\frac{1 - \sqrt{1-nv}}{1 + \sqrt{1-nv}} \right)^{-k_1 \cdot k_2}
 \end{aligned}$$

where we have used the defining relation Eq. (3.12) and the identity (5.2) to perform the expectation values. Changing variables by

$$u = \frac{4x}{(1+x)^2} \quad uv = \frac{4xy}{(1+xy)^2}$$

we obtain

$$A_4(Q) = -(\frac{1}{2}t+1) \int_0^1 dx dy x^{-\frac{1}{2}Q^2 - \frac{3}{2}} y^{-\frac{1}{2}s-1} (1-y)^{-\frac{1}{2}t-2} (1-x^2y)^{\frac{1}{2}t} \quad (5.5)$$

This is crossing symmetric in s and t , has the correct poles in all channels, is signaturized in the Q^2 channel, the residue at $\frac{1}{2}Q^2 = -\frac{1}{2}$ is just the Neveu-Schwarz four-point function⁷⁾; but it is logarithmically divergent. The divergence is easily seen if one writes, performing the y integral,

$$A_4(Q) = \int_0^1 dx x^{-\frac{1}{2}Q^2 - \frac{3}{2}} F(x^2, s, t)$$

and notices that $F(x^2, s, t) \sim 1/(1 - x^2)$ as $x \sim 1$. However, the divergence is independent of s and t and a consistent prescription may exist for subtracting the divergences in this and other amplitudes though we have not found it.

6. DISCUSSION

We have been led by a rather tortuous path via the analogue picture of dual models, to a better understanding of a model for off-shell states proposed by Schwarz and, from there, to a model for off-shell states in the fermion model in ten dimensions. Whereas the off-shell states in the orbital model require $D = 16$ in order for the gauges to work, the gauges work perfectly well in 10 dimensions for the fermion extension. We have nothing to say, however, on the meaning of the 16 dimensional requirement in the string picture nor why whilst all amplitudes with one off-shell state diverge in the orbital model, this is not always the case when we have off-shell mesons coupling to fermions. Considerably more work needs to be done to show that amplitudes with many off-shell states remain ghost free and that it is possible to regularize the divergences in a consistent fashion so that amplitudes are completely finite.

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