PARASTATISTICS AND THE LIE ALGEBRAICAL METHODS USED THERE *)

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A B S T R A C T

We review some of the properties of the parafield operators and discuss in some details where the difference between the ordinary and paraquantization comes from. Particular attention is paid to the many-vacuum representations of the para-Fermi operators.

*) Invited Talk at the Symposium on New Mathematical Methods in Physics and Problems in General Relativity, Bonn (July 1973).

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In the present talk I would like to discuss some properties of the parastatistics and in particular to draw your attention to the very close connection between the representations of the paraFermi operators and the algebra of the orthogonal group.

To begin with let me remind you that parastatistics was introduced in 1953 by Green ¹⁾ who observed that the commonly accepted rules of second quantization are, although enough, not necessary, however, for satisfying all physical requirements, and pointed out how the quantization axioms can be generalized. It is worth discussing here in more detail the question of where the generalization comes from. For simplicity, let us consider the quantization on an example of a real scalar free field $\varphi(x)$.

A) - Classical case

One starts with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi(x) \cdot \partial^\mu \varphi(x) - \frac{1}{2} m^2 \varphi^2(x) \quad (1)$$

The Lagrange-Euler equation in this case is the Klein-Gordon equation with positive and negative frequency solutions $\varphi^\pm(x)$ of the form :

$$\varphi^\pm(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{2k^0}} e^{\pm i\mathbf{k}\cdot\mathbf{x}} \varphi^\pm(\mathbf{k}), \quad k^0 = \sqrt{\mathbf{k}^2 + m^2} \quad (2)$$

Noether's theorem together with the invariance of $\mathcal{L}$ under the Poincaré group $\mathcal{P}$ gives the invariant quantities : the energy-momentum vector P^μ and the angular-momentum tensor $M^{\mu\nu}$. In particular, P^μ may be represented as

$$P^\mu = \frac{1}{2} \int d\mathbf{p} p^\mu [\varphi^\dagger(\mathbf{p}), \varphi(\mathbf{p})]_+ \quad (3)$$

B) - Quantization of the field

The quantization can be performed in different equivalent ways ; for instance, by postulating the equal-time commutation relations. For what we want to show, however, it is more convenient to follow the quantization

procedure given in the book of Bogoliubov and Shirkov ²⁾. It is based on the following postulates :

- 1) the field $\Psi(x)$ becomes an operator ;
- 2) the energy-momentum vector P and the angular-momentum tensor M are expressed in terms of the operator-field functions by the same expressions of the type (3), i.e., as in the classical case with proper ordering of the operator factors.

It follows now from 1) and 2) together with the requirements that the field transforms according to a unitary representation of $\mathfrak{P}$ and the compatibility of the transformation properties of the field and the state vectors, that $\Psi(x)$ satisfies the commutation relation

$$[P^\mu, \Psi^\pm(\underline{x})] = \pm \kappa^\mu \Psi^\pm(\underline{x}) \quad (4)$$

Inserting (3) in (4), we have

$$[[\Psi^\pm(\underline{p}), \Psi^\mp(\underline{p})]_\pm, \Psi^\pm(\underline{\kappa})] = \pm 2 \delta(\underline{p} - \underline{\kappa}) \Psi^\pm(\underline{\kappa}) \quad (5)$$

Generalizing this equation, Green has postulated the three-linear structure relation for the parafields

$$\begin{aligned} [\Psi^\pm(\underline{q}), \Psi^\pm(\underline{p})]_\pm, \Psi^\pm(\underline{\kappa}) &= 0 \\ [[\Psi^\pm(\underline{q}), \Psi^\mp(\underline{p})]_\pm, \Psi^\pm(\underline{\kappa})] &= \pm 2 \delta(\underline{p} - \underline{\kappa}) \Psi^\pm(\underline{\kappa}) \end{aligned} \quad (6)$$

where the anticommutator (commutator) corresponds to the paraBose (paraFermi) field.

In the ordinary quantization, one has to add two more postulates :

- 3) the commutator or the anticommutator of two-field operators is a C number ;
- 4) all dynamical variables should be written in a normal product form.

From 3), it follows that

$$[\varphi^+(\underline{p}), \varphi^+(\underline{k})] = [\varphi^-(\underline{p}), \varphi^-(\underline{k})] = 0$$

and therefore the expression now corresponding to (5) is

$$\begin{aligned} [:[\varphi^+(\underline{p}), \varphi^-(\underline{p})]:, \varphi^+(\underline{k})] &= 2 [\varphi^+(\underline{p}) \varphi^-(\underline{p}), \varphi^+(\underline{k})] = \\ &= 2 \delta(\underline{p} - \underline{k}) \varphi^+(\underline{k}) \end{aligned} \quad (5')$$

Hence

$$[\varphi^-(\underline{p}), \varphi^+(\underline{k})] = \delta(\underline{p} - \underline{k}) \quad (6')$$

and we obtain the Bose commutation relations. We see that the difference between the parastatistics and ordinary statistics is mainly due to the postulate 3) which does not follow from some general physical requirements. However, the postulate greatly simplifies the practical calculations. So (modulo small details) we can conclude that the question of statistics depends mainly on whether the (anti) commutator of the fields is a C number (= ordinary statistics) or operator (= parastatistics).

Since the moment the parastructure relations were introduced, they were studied by several authors from different points of view. The general conclusion was, that in the real world at least, the known particles appear to satisfy only the ordinary statistics. Only at present the situation seems to be changing. There are indications that parastatistics may be relevant in physics and its abstract mathematical structure may have applications in some branches of mathematics. As an example, let me mention here the prequantization methods of Konstant-Souriau³⁾. In the present symposium, we have heard how these authors, reformulating in a proper way the physical quantization procedure, go a long way towards constructing all irreducible unitary representations of connected Lie groups and, in particular, define all such representations for a large class of solvable groups. In this respect, it is interesting to ask whether the more general paraquantization provides an extension of the prequantization.

From a physical point of view the interest in parastatistics is now stimulated a great deal by the circumstance that the elementary particles seem to be formed out of constituents as, for example, in the coloured quark model ⁴⁾. In this model, whose predictions are in reasonable agreement with experiment, the quarks seem to behave as paraFermions of order 3 ⁵⁾.

The order of the parastatistics is a concept which characterizes a certain class of representations of the paraoperators (PO), namely the Fock-type representations. One way to obtain a p -order representation is to express the paraoperators in terms of another set of p operators, namely ¹⁾

$$\varphi^{\pm}(\underline{p}) = \sum_{i=1}^p \varphi^{i\pm}(\underline{p}) \quad (7)$$

where the so-called Green ansatz $\varphi^{i\pm}(\underline{p})$ satisfy anomalous commutation relations :

$$\begin{aligned} [\varphi^{i\pm}(\underline{p}), \varphi^{j\pm}(\underline{q})]_{\epsilon} &= 0 \quad i \neq j \\ [\varphi^{i-}(\underline{p}), \varphi^{i+}(\underline{q})]_{-\epsilon} &= \delta(\underline{p} - \underline{q}) \\ [\varphi^{i-}(\underline{p}), \varphi^{i-}(\underline{q})]_{-\epsilon} &= [\varphi^{i+}(\underline{p}), \varphi^{i+}(\underline{q})] = 0 \end{aligned} \quad (8)$$

The sign $\epsilon = +(-)$ corresponds to the paraBose (paraFermi) case. Let $\mathcal{O}_p$ be the Fock space of the Green ansatz with vacuum $|0\rangle$. This space is in general reducible with respect to PO. The representation of order p is realized in the irreducible subspace $W_p \subset \mathcal{O}_p$ of PO, which contains $|0\rangle$. The vector $|0\rangle$ is the only state in W_p which is annihilated by $\varphi^{-}(\underline{p})$ and therefore these representations contain unique vacuum state. All representations of this type were found and classified by Greenberg and Messiah ⁶⁾ and in the context we shall refer to them as canonical representations. The space $\mathcal{O}_p$, however, contains also PO irreducible subspaces with more than one linearly independent state vanishing under the action of the annihilation operators $\varphi^{-}(\underline{p})$, i.e., in these subspaces the vacuum is degenerate. Previously such representations were rejected on the ground of the uniqueness of the vacuum. The concept of the vacuum, however, depends on the physical meaning ascribed to the operators involved. If one accepts the attitude that the elementary particles have no internal structure, then the vacuum

should be the only state annihilated by $\varphi^-(\underline{p})$, and hence one is forced to admit only canonical representations. On the other hand, if the particles are assumed to have internal structure, as for instance to be composed of quarks, it is quite natural to demand the vacuum to be unique with respect to the constituents. As an example let us assume that the field is of para-statistics 3. Then, one can write

$$\varphi^{\pm}(\underline{x}) = \frac{1}{(2\pi)^{3/2}} \sum_{i=1}^3 \int \frac{d\underline{p}}{\sqrt{2p^0}} e^{\pm i\underline{p}\underline{x}} \varphi^{i\pm}(\underline{p}) \quad (9)$$

and demand that the representation space contains only one state annihilated by the constituent annihilation operators $\varphi^{i-}(\underline{p})$. This means that one has to consider as representation space the Fock space $\mathcal{O}_3$ of the Green ansatz. The space $\mathcal{O}_3$ is irreducible with respect to $\varphi^{i\pm}(\underline{p})$ and, as we have already observed, contains several states annihilated by the field $\varphi^-(\underline{x})$. The vacuum is, however, unique. This point of view was first suggested by Govorkov ⁷⁾ and was properly realized in a recent paper of Bracken and Green ⁸⁾. The former authors propose to consider generalized parastatistics of order p , which makes use of all representations of PO in $\mathcal{O}_p$. The vectors from $\mathcal{O}_p$ which are annihilated by $\varphi^-(\underline{p})$ are called in this case reservoir or vacuum-like states.

So we conclude that not only the canonical representations may be relevant for physical applications. Therefore, we proceed to study all representations of PO and, in particular, to analyse the properties of these representations with respect to the reservoir states. We shall, however, restrict our considerations only to the case of paraFermi operators (PFO). The reason for this is a purely technical one. It is due to the circumstance that to determine all PFO representations we make essential use of the underlying Lie algebraical structure of the PFO, namely of the fact that every irreducible representation of n pairs of PFO can be extended to an irreducible representation of the classical Lie algebra B_n and vice versa. This allows us in the case of PFO operators to formulate the whole problem in a purely Lie algebraical language, whereas this seems to be impossible for the paraBose case.

For later use, we first consider a finite number a_i, b_i , $i \in N = \{j \mid j = 1, \dots, n\}$ of n pairs of paraFermi operators. The full set of relations they satisfy is the following :

$$\begin{aligned}
 [[a_p, b_q], a_r] &= 2\delta_{qr}a_p \\
 [[a_p, b_q], b_r] &= -2\delta_{pr}b_q \\
 [[b_p, b_q], a_r] &= 2(\delta_{qr}b_p - \delta_{pr}b_q) \\
 [[a_p, a_q], b_r] &= 2(\delta_{qr}a_p - \delta_{pr}a_q) \\
 [[a_p, a_q], a_r] &= [[b_p, b_q], b_r] = 0
 \end{aligned} \tag{10}$$

Let T be the associative free algebra of $a_i, b_i, i \in N$, and J be the two-sided ideal in T generated by the relations (10). The factor algebra $U = T/J$ is called paraFermi algebra. It is an infinite dimensional Lie algebra with respect to the commutator $[x, y] = xy - yx, x, y \in U$. To find a representation of the PFO means to find a representation of the associative algebra U in some Hilbert (or, more generally, in some linear) space. Denote by B_n the subspace of all antisymmetrized second-order polynomials of PFO in U . From (10) one can easily see that B_n is a Lie subalgebra of the paraFermi algebra and also that U is the universal enveloping algebra of B_n . Thus the paraFermi algebra coincides with the universal enveloping algebra of B_n . Since, by definition, a representation of B_n is a homomorphism of U in an algebra of operators in some Hilbert space, we conclude the following.

Proposition 1

Every representation of the PFO is a representation of B_n . The representation of the paraFermi operators is irreducible if and only if the representation of B_n is irreducible.

Thus the problem of constructing all representations of PFO is reduced to a Lie algebraical one. Let us remark that one is interested in considering not all representations but only those for which the Hermitian conjugate of the annihilation operator a_i equals b_i . Therefore, one needs to determine all representations of U considered as a *- algebra

with involution defined by the condition $(a_i)^* = b_i$, $i \in N$. The set of all irreducible representations of the operators a_i, b_i , $i \in N$ is now completely determined from the observation that as a complex algebra B_n is isomorphic to the classical algebra denoted in the same way ⁹⁾, whereas, as a real algebra, it is $so(n, n+1)$ ¹⁰⁾. Moreover, the condition $(a_i)^* = b_i$ may be fulfilled only for the finite dimensional representations of B_n which are known ¹¹⁾. It is convenient to choose the basis in B_n in the following way :

$$\omega_i = \frac{[a_i, b_i]}{8n-4} \quad i, j \in N$$

$$e_{\omega_i} = a_i, \quad e_{-\omega_i} = b_i$$

$$e_{\omega_i + \omega_j} = [a_i, a_j], \quad e_{-\omega_i - \omega_j} = [b_i, b_j]$$

$$e_{\omega_i - \omega_j} = [a_i, b_j] \quad (11)$$

The generators $[a_i, b_j]$, $i, j \in N$, span a basis for a subalgebra of B_n isomorphic to $U(n)$. In the following, when talking about $U(n)$, we shall understand this particular realization. The vectors $\Omega = (\omega_1, \dots, \omega_n)$ are orthogonal with respect to the Cartan-Killing form, namely

$$(\omega_i, \omega_j) = \frac{\delta_{ij}}{4n-2} \quad (12)$$

and can be taken as a basis of the Cartan subalgebra of B_n . In this case the root system Σ is given by the vectors

$$\Sigma = (\pm \omega_i \pm \omega_j, \pm \omega_i \mid i, j \in N) \quad (13)$$

Hence the weights corresponding to a_i (b_i) have positive first non-vanishing co-ordinate in Ω . Therefore, we obtain

Proposition 2

The Cartan subalgebra of B_n can be chosen in such a way that all paraFermi annihilation (creation) operators a_i (b_i), $i \in N$, are positive (negative) root vectors.

Let us turn now to the representations of PFO. Consider an irreducible representation realized in a space W and let x_Λ be the highest weight of the representation with weight

$$\Lambda = \sum_{i=1}^n L_i \omega_i \equiv (L_1, \dots, L_n)$$

As is known, the co-ordinates of Λ which are all non-negative (half) integers such that $L_1 \geq \dots \geq L_n$ define the representation of B_n and hence of PFO up to an isomorphism. A vector $(\ell_1, \dots, \ell_n)$ is a weight if and only if it has (half) integer co-ordinates satisfying the inequality

$$\sum_{k=1}^m |\ell_{i_k}| \leq \sum_{k=1}^m L_{i_k} \quad \forall \quad i_1 \neq i_2 \neq \dots \neq i_m \in N, \quad m = 1, \dots, n \quad (14)$$

Relation (12) together with the equality $\omega_i \cdot x_\Lambda = (\Lambda, \omega_i) x_{i_1}$ gives that $\frac{1}{2}[a_i, b_i] x_\Lambda = L_i x_\Lambda$ so that L_i are the eigenvalues of $\frac{1}{2}[a_i, b_i]$ on the highest weight vector.

Proposition 2 greatly simplifies the problem of finding all reservoir states contained in W . We recall that $x \in W$ is said to be a reservoir state if $a_i \cdot x = 0 \quad \forall \quad i \in N$. Let $P \subset U$ be the set of all polynomials of the $U(n)$ generators. Denote by $V \subset W$ the $U(n)$ representation space, which contains x_Λ , namely $V = (px_\Lambda \mid p \in P)$.

Proposition 3

The vector $x \in W$ is a reservoir state if and only if $x \in V$. The subspace V is the linear envelope of all weight vectors with weights $(\ell_1, \dots, \ell_n)$ such that

$$\sum_{i=1}^n \ell_i = \sum_{i=1}^n L_i \quad (15)$$

Omitting the details of the proof [see Ref. 12)] we observe that the weight vector x_ℓ with weight $\ell = (\ell_1, \dots, \ell_n)$ satisfying (15) is necessarily annihilated by all a_i since otherwise the weight corresponding to $a_i x_\ell$ would violate the condition (14). It is also clear from the construction

that V carries an irreducible $U(n)$ representation. Consider now the important special case of representation with highest weight $\Lambda = (L, L, \dots, L)$ of x_Λ . Clearly, for $i \neq j$ $[a_i, b_j] x_\Lambda$ is not a weight vector and therefore it is zero. Combining this with the formula for $i = j$ and taking into account that $a_i \cdot x_\Lambda = 0 \quad \forall i \in N$ we obtain :

$$a_i b_j \cdot x_\Lambda = \delta_{ij} p x_\Lambda \quad (16)$$

where we have put $p = 2L$.

It follows from (16) that $V = P x_\Lambda = x_\Lambda$ and therefore we obtain in this case a representation with a single reservoir state, i.e., a canonical representation of PFO. In fact Eq. (16) is the defining relation for representation of parastatistics p . So we see that the canonical representations of PFO are kinds of most degenerate representations of B_n . The representation of n pairs of usual Fermi operators corresponds, for instance, to a highest weight $(\frac{1}{2}, \dots, \frac{1}{2})$. It is also easy to show that only the representations with $L_1 = \dots = L_n$ contain a single reservoir state.

Let us now examine in more detail the reservoir states in an irreducible representation of a countable number $a_i, b_i, i \in I = (j | j = 1, 2, \dots)$ of PFO. We define, as before, the paraFermi algebra and the algebra B of all second-order antisymmetrized polynomials. In this case B is an infinite dimensional Lie algebra, and the paraFermi algebra $U(B)$ is its universal enveloping algebra in complete analogy with the finite case. The elements $p \in U(B)$ are polynomials of finite number of PFO.

Consider an irreducible representation of the PFO in some Hilbert space W and let $x_0 \in W$ be a reservoir state, i.e., $a_i \cdot x_0 = 0 \quad \forall i \in I$. From the irreducibility it follows that the subspace $U_0 = (p x_0 | p \in U(B))$ is dense in W . Let $x_1 = p_0 x_0, p_0 \in U(B)$ be another reservoir state. Since p_0 is a polynomial of finite, say, n paraFermi operators, we can put without loss of generality that p_0 is a function of $a_i, b_i, i \in N$. The representation of $a_i, b_i, i \in N$ in W can in general be reducible. From the full reducibility of the representations of B_n we have that x_0 can be represented in a unique way as a sum $x_0 = \sum_{\alpha} x_0^{\alpha}$, where x_0^{α} belongs to the B_n irreducible subspace W^{α} of W , and the subspaces W^{α} are linearly independent. Since $a_i \cdot x_0^{\alpha} \in W^{\alpha}, i \in N, a_i \cdot x_0 = 0$ implies that $a_i \cdot x_0^{\alpha} = 0 \quad \forall \alpha$ and $i \in N$. Hence x_0^{α} is a B_n reservoir state.

On the other hand $a_i p_0 x_0^\alpha \in W^\alpha$, $i \in N$ and therefore $a_i p_0 x_0 = 0$ if and only if $a_i p_0 x_0^\alpha = 0$ for all α and $i \in N$. Applying now Proposition 3 we have that $a_i p_0 x_0^\alpha = 0$, $i \in N$, only in the case where p_0 is a polynomial of $[a_i, b_j]$, $i, j \in N$. On the other hand, let p_0 be a polynomial of $[a_i, b_j]$, $i, j \in N$. For $K \notin N$ $[a_K, [a_i, b_j]] = 0$ and hence $[a_K, p_0] = 0$. Therefore, $a_K p_0 x_0 = p_0 a_K x_0 = 0$ and $p_0 x_0$ is a reservoir state. We have proved that for a given reservoir state x_0 the vector $p_0 x_0$ is also a reservoir state if and only if p_0 is a polynomial of $[a_i, b_j]$, $i, j \in I$. The set of all such elements spans a basis of an infinite dimensional Lie algebra A , which is a subalgebra of B . Denote by $U(A)$ its universal enveloping algebra, $U(A) \subset U(B)$. We have now that all reservoir states are contained in the subspace $V \subset W$, which is the closure of all vectors $p x_0$, $p \in U(A)$. The algebras B and A satisfy the commutation relations of the orthogonal and unitary algebras correspondingly, and therefore they can be considered as infinite dimensional analogue of the classical algebras B_n and A_n . The representation of A in V is an irreducible one. Indeed by construction, x_0 is a cyclic vector of A in V . Let $x_1 \in U(A)x_0$. Interchanging the places of x_0 and x_1 in the above considerations, we find that there exists an operator $p \in U(B)$ such that $x_0 = p x_1$. Moreover, since x_0 and x_1 are reservoir states, $p \in U(A)$ and therefore $U(A)$ contains a projection operator on x_0 for any vector from a dense set in V . Repeating the same argument for x_0 and any other vector x_2 from the dense set in V , we obtain that for any two vectors x_1, x_2 from the dense set of V , there exists an operator $q \in U(A)$ such that $x_2 = q x_1$. Hence the representation of A in V is irreducible. In fact, we have proved the Haag-Schroer lemma¹³⁾. So we have :

Proposition 4

Any irreducible representation of the algebra B defines an irreducible representation of an infinite number a_i, b_i , $i \in I$ of PFO and vice versa. All reservoir states from the representation space W of PFO are contained in a subspace $V \subset W$ which carries an irreducible representation of the algebra A .

In conclusion, let me stress that the question whether parastatistics exists in nature depends mainly on whether the field (anti) commutators are C numbers. The requirement that the field (anti) commutators be C numbers does not follow from any general physical requirements. If parastatistics exists, then the kind of representation of PO operators to be used depends on the physical meaning ascribed to these operators. It is worth mentioning that the extension to parastatistics preserves the essential properties of several of the field-theoretical models. For instance, one can easily verify that in any current algebra model the generalization to parastatistics preserves the commutation relations between the currents. This is due to the fact that the currents can always be expressed as functions of $[a_i, b_j]_{\epsilon}$ with $\epsilon = +(-)$ for the Bose (Fermi) case ¹⁴). It could be interesting to observe that the Bose and paraFermi statistics can be viewed as quite similar mathematical structures. The Bose operators are generators of a solvable Lie algebra, the Heisenberg algebra, whereas the PFO are generators of the simple Lie algebra of the rotation group. In both cases, the dynamical variables are functions of Lie algebra generators.

ACKNOWLEDGEMENTS

I would like to thank Prof. K. Bleuter for the warm hospitality during the Symposium in Bonn and Prof. H. Doebner for the valuable discussion of the subject at his seminar at the University of Clausthal. I would like to thank also Dr. J. Weyers for the careful reading of the present preprint and the CERN Theoretical Study Division for the hospitality.

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