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SUPERGAUGE INVARIANCE AND THE
GELL-MANN - LOW EIGENVALUE

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A B S T R A C T

The connections among supergauge, scale and conformal invariance are elucidated on the example of a renormalizable field theory model. For the massless model, at the Gell-Mann - Low eigenvalue, a non-perturbative argument leads to the contradictory result that it can only describe a free field theory. A study of the Callan-Symanzik equations for the massive model clarifies the situation from the point of view of perturbation theory. It is argued that the eigenvalue equation has no non-trivial solution and that the effective coupling constant increases without bound with energy.

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1. INTRODUCTION

In two recent papers ^{1),2)} a field theory model invariant under supergauge transformations ^{3),4)} was shown to be renormalizable. The model contains a scalar field A , a pseudoscalar field B , a Majorana spinor ψ and two auxiliary fields F and G . The model was renormalized in such a way as to satisfy to all orders in perturbation theory the Ward identities which follow from supergauge invariance. As a consequence all fields have the same wave function renormalization constant and there is also only one common mass renormalization and only one coupling constant renormalization. Furthermore, a more detailed study of the renormalization procedure, making use not only of the Ward identities, but more directly of the particular form of the Lagrangian, showed the occurrence of remarkable cancellations of divergences among perturbation theory diagrams. These cancellations make it actually unnecessary to introduce a mass and a coupling constant counterterm, so that one is left with only one renormalization constant, the wave function renormalization common to all fields. This renormalization constant, on the other hand, appears to be divergent to all orders in perturbation theory, so that the theory does not seem to be super-renormalizable.

A number of questions come naturally to mind at this point. Since the model is much less divergent than the general theory involving the same fields (with arbitrary masses and coupling constants), one wonders whether it has special properties in the limit of zero mass, in particular whether in that limit it provides perhaps an example of a truly scale and conformal invariant theory. A related question is that of the high (Euclidean) momentum behaviour of the massive theory. It is the aim of the present paper to answer these questions. We first consider, in Section 2, the massless theory assuming that the coupling constant satisfies the Gell-Mann-Low eigenvalue equation. With this assumption it is justified to consider the theory as satisfying all the constraints coming from scale and conformal invariance ⁵⁾. We make therefore use of the particular form of the two- and three-point functions required by these constraints, with the added requirement, coming from supergauge invariance, that the anomalous part of the dimension of all fields is the same. Furthermore, we assume that certain terms are absent because of the discrete γ_5 invariance of the massless theory. A simple non-perturbative calculation leads then to the result that the dimension of all fields must actually be canonical. Since, as is well known ⁶⁾, this implies that the theory is a free-field theory, one must come to the conclusion that the original assumption, that the Gell-Mann-Low eigenvalue equation was satisfied, must have been invalid. In order to clarify further this rather puzzling situation we study, in Section 3, the

Callan-Symanzik equations ⁷⁾ for the model with mass. They are found to have some interesting special features. In particular, from the fact that only a wave function renormalization is necessary, it follows that the functions $\beta(g)$ and $\gamma(g)$ of the Callan-Symanzik equations must be proportional to each other. Consequently, if the coupling constant satisfies the eigenvalue equation $\beta(g)=0$, $\gamma(g)$ must also vanish at that value and the dimension of all fields must be canonical. This confirms and extends the non-perturbative result of Section 2. The only logical conclusion seems to be that the function $\beta(g)$ has no zero other than $g=0$, since the theory is certainly not a free field theory when the coupling constant g is different from zero. More precisely, the usual positivity restriction on γ implies in our model a positivity restriction on β . The effective coupling constant $g(q^2)$ must then increase indefinitely without bound with q^2 .

2. PROPERTIES OF THE MASSLESS THEORY

In the present section we consider the implications of supergauge invariance for the massless theory at the Gell-Mann-Low eigenvalue g_ω , which satisfies $\beta(g_\omega)=0$. The Callan-Symanzik equations, or the renormalization group equations, imply that the theory is scale and conformal invariant when the coupling constant is equal to a Gell-Mann-Low eigenvalue. The different fields have anomalous dimensions which are in general unrelated. However, in a supergauge invariant theory, the dimensions of all fields belonging to a supergauge multiplet are related in a well defined way ²⁾. For instance, for the scalar multiplet, if we call d the dimension of the ψ field, the fields A and B have dimensions $d-\frac{1}{2}$ and the fields F and G have $d+\frac{1}{2}$. This fact is confirmed by the perturbation theory study of the renormalization of the model of Ref. 1), which shows that there is only one $\gamma(g)$ function of all fields of the scalar multiplet.

We consider the three-point function $V_{ijk}(x,y,z)$ of the scalar multiplet, where the indices i,j,k refer to the various fields of the multiplet. By conformal invariance the functional forms of the three-point functions are completely fixed at the eigenvalue, apart from a normalization coefficient. So we can write

$$V_{ijk}(x,y,z) = c_{ijk} f_{ijk}(x,y,z) \quad (1)$$

where f_{ijk} are known functions ⁵⁾ which depend on the anomalous dimension of the fields. In the following we need only the three-point functions

$$\begin{aligned} \langle 0 | \psi(x) \bar{\psi}(y) A(z) | 0 \rangle \\ = i c_{\psi \bar{\psi} A} (\cancel{x} - \cancel{z}) (\cancel{y} - \cancel{z}) [(x-y)^2 (x-z)^2 (y-z)^2]^{-\frac{1}{2}d - \frac{1}{4}} \end{aligned} \quad (2)$$

and

$$\begin{aligned} \langle 0 | F(x) A(y) A(z) | 0 \rangle \\ = c_{FFA} [(x-y)^2 (x-z)^2]^{-\frac{1}{2}d - \frac{1}{4}} [(y-z)^2]^{-\frac{1}{2}d + \frac{3}{4}}. \end{aligned} \quad (3)$$

An additional term which would be allowed in (2) by conformal invariance, vanishes because the massless theory is invariant under the discrete γ_5 transformation

$$\psi \rightarrow \gamma_5 \psi, \quad A \rightarrow -A, \quad B \rightarrow -B, \quad F \rightarrow F, \quad G \rightarrow G. \quad (4)$$

Here, without indicating it explicitly, we assume to be working with time ordered products, although the argument could be carried out just as easily with Wightman functions. The remaining three-point functions are easily seen to be related to (2) and (3) by supergauge invariance. Here $d = \frac{3}{2} + \gamma(g_\infty)$ is the anomalous dimension of the spinor field.

The infinitesimal change of the fields of the scalar multiplet under a supergauge transformation with constant parameter α , is given by

$$\begin{aligned} \delta A &= [\bar{\alpha} Q, A] = i \bar{\alpha} \psi \\ \delta B &= [\bar{\alpha} Q, B] = i \bar{\alpha} \gamma_5 \psi \\ \delta \psi &= [\bar{\alpha} Q, \psi] = \partial_r (A - \gamma_5 B) \gamma^r \alpha + (F + \gamma_5 G) \alpha \\ \delta F &= [\bar{\alpha} Q, F] = i \bar{\alpha} \gamma^r \partial_r \psi \\ \delta G &= [\bar{\alpha} Q, G] = i \bar{\alpha} \gamma_5 \gamma^r \partial_r \psi. \end{aligned} \quad (5)$$

Constant parameter supergauge invariance implies Ward identities for the Green functions ¹⁾. We shall use the Ward identities

$$i \langle 0 | \psi(x) A(y) \bar{\psi}(z) | 0 \rangle + i \langle 0 | \psi(x) \bar{\psi}(y) A(z) | 0 \rangle + \langle 0 | F(x) A(y) A(z) | 0 \rangle = 0 \quad (6)$$

and

$$-i \partial_\mu^z \langle 0 | \psi(x) A(y) \bar{\psi}(z) | 0 \rangle \gamma^\mu + \not{\partial}^x \langle 0 | A(x) A(y) F(z) | 0 \rangle = 0. \quad (7)$$

Here again we have set equal to zero some terms which would be present only for non-vanishing mass, for instance

$$\partial_\mu^x \langle 0 | A(x) A(y) A(z) | 0 \rangle ,$$

which vanishes because of the invariance of the Lagrangian under the transformation (4).

Using the conformal invariant expressions (2) and (3) in the Ward identity (6), one obtains

$$i^2 C_{\psi \bar{\psi} A} \left[(\cancel{x} - \cancel{z})(\cancel{y} - \cancel{z}) + (\cancel{x} - \cancel{y})(\cancel{z} - \cancel{y}) \right] \left[(x-z)^2 (y-z)^2 (x-y)^2 \right]^{-\frac{1}{2}d - \frac{1}{4}} + C_{FAA} \left[(x-y)^2 (x-z)^2 \right]^{-\frac{1}{2}d - \frac{1}{4}} \left[(y-z)^2 \right]^{-\frac{1}{2}d + \frac{3}{4}} = 0 , \quad (8)$$

which gives $C_{\psi \bar{\psi} A} = C_{FAA}$, as a consequence of the identity $(\cancel{x} - \cancel{z})(\cancel{y} - \cancel{z}) + (\cancel{x} - \cancel{y})(\cancel{z} - \cancel{y}) = (y-z)^2$. We now use the other Ward identity (7), which gives

$$\partial_\mu^z \left\{ (\cancel{x} - \cancel{y})(\cancel{z} - \cancel{y}) \left[(z-x)^2 (x-y)^2 (z-y)^2 \right]^{-\frac{1}{2}d - \frac{1}{4}} \right\} \gamma^\mu + \not{\partial}^x \left\{ \left[(x-y)^2 \right]^{-\frac{1}{2}d + \frac{3}{4}} \left[(z-y)^2 (x-z)^2 \right]^{-\frac{1}{2}d - \frac{1}{4}} \right\} = 0 . \quad (9)$$

Performing the differentiation and putting $y=0$, one has

$$\begin{aligned}
 & 4 \cancel{x} [(z-x)^2 x^2 z^2]^{-\frac{1}{2}d-\frac{1}{4}} + 2\left(\frac{d}{2}+\frac{1}{4}\right) (x^2 z^2)^{-\frac{1}{2}d-\frac{1}{4}} [(z-x)^2]^{-\frac{1}{2}d-\frac{5}{4}} \cancel{x} \cancel{z} (\cancel{x}-\cancel{z}) \\
 & - 2\left(\frac{d}{2}+\frac{1}{4}\right) [(z-x)^2 x^2 z^2]^{-\frac{1}{2}d-\frac{1}{4}} \cancel{x} - 2\left(\frac{d}{2}-\frac{3}{4}\right) [x^2 z^2 (x-z)^2]^{-\frac{1}{2}d-\frac{1}{4}} \cancel{x} \\
 & - 2\left(\frac{d}{2}+\frac{1}{4}\right) (x^2)^{-\frac{1}{2}d+\frac{3}{4}} (z^2)^{-\frac{1}{2}d-\frac{1}{4}} [(x-z)^2]^{-\frac{1}{2}d-\frac{5}{4}} (\cancel{x}-\cancel{z}) = 0.
 \end{aligned} \tag{10}$$

Finally, using the identity $\cancel{x}(\cancel{x}-\cancel{z}) = x^2(\cancel{x}-\cancel{z}) - \cancel{x}(x-z)^2$, one can eliminate the terms in $\cancel{x}-\cancel{z}$ and what remains is

$$3\left(\frac{3}{2}-d\right) \cancel{x} [x^2 z^2 (x-z)^2]^{-\frac{1}{2}d-\frac{1}{4}} = 0. \tag{11}$$

This equation is inconsistent for $d \neq \frac{3}{2}$, its canonical value. As explained in the Introduction, this shows that the assumption that the Gell-Mann-Low eigenvalue equation can be satisfied for a non-vanishing value of the coupling constant is invalid.

3. THE CALLAN-SYMANZIK EQUATIONS

In the previous section we have arrived at the surprising result that the Gell-Mann-Low eigenvalue equation for our model cannot have a solution other than $g=0$. This is a rather strong statement and we shall try now to elucidate it by studying the Callan-Symanzik equations for the model with mass.

It has been shown in Ref. 2) that all divergences can be absorbed in a wave function renormalization common to all fields. No mass or coupling constant counterterms need to be introduced. This property explains already the absence of a non-trivial Gell-Mann-Low eigenvalue. In fact, the renormalized mass and coupling constants (m and g) are given in terms of the bare quantities (m_0 and g_0) by the relations

$$m = Z m_0, \quad g = Z^{\frac{3}{2}} g_0, \tag{12}$$

where Z is the wave function renormalization constant ^{*)}. If $\beta(g)$ is the Callan-Symanzik function and $\gamma(g)$ the anomalous dimension of the fields, it follows that the ratio β/γ is given by

$$\frac{\beta(g)}{\gamma(g)} = 2 \frac{\frac{\partial g}{\partial m_0}}{\frac{\partial \log Z}{\partial m_0}} = 3g. \quad (13)$$

Therefore β and γ are not independent functions and in particular they have the same zeros. Hence the result of the previous section is understood from the point of view of perturbation theory. It is a consequence of the cancellation of divergences which in turn follows from the supergauge invariance and the particular structure of the Lagrangian, as explained in Ref. 2). It is interesting to observe that our model contains only Yukawa couplings and tri- and quadrilinear couplings among scalar and pseudoscalar fields. For the general theory of this class, one usually assumes that a non-trivial solution of the eigenvalue equation exists. In our case, supergauge invariance imposes definite relations among the masses and the coupling constants, which can be viewed as defining a submanifold in the parameter space. One must therefore conclude that, if a non-trivial eigenvalue exists for the general theory (without supergauge invariance), it must be outside the submanifold.

It is interesting to write out the Callan-Symanzik equations which, in the present model, take a particularly simple and elegant form. Let $\Gamma_{\phi_1 \dots \phi_n}^{\text{unren}}(p_i; m_0, g_0, \xi)$ be the unrenormalized but regularized one-particle irreducible Green function with n external lines of the type $\phi_1, \dots, \phi_n$, where ϕ_i stands for one of the field A, B, F, G or ψ , and ξ is the regularizer parameter. The renormalizability of the model implies that

$$\lim_{\xi \rightarrow 0} Z^{\eta/2} \Gamma_{\phi_1 \dots \phi_n}^{\text{unren}}(p_i; m_0, g_0, \xi) = \Gamma_{\phi_1 \dots \phi_n}(p_i; m, g), \quad (14)$$

where in the right-hand side appears the renormalized one-particle irreducible n point function. On the other hand, it was shown in Ref. 2) that the derivative of Γ^{unren} with respect to m_0 , keeping g_0 and ξ fixed, amounts to the insertion of a zero momentum A field

*) The notation we use here is different from that of Ref. 2) in that we use subscripts for the unrenormalized, rather than for the renormalized, mass and coupling constant.

$$\frac{\partial}{\partial m_0} \Gamma_{\phi_1 \dots \phi_n}^{\text{unren}}(p_i; m_0, g_0, \xi) = \frac{1}{2g_0} \Gamma_{A, \phi_1 \dots \phi_n}^{\text{unren}}(0, p_i; m_0, g_0, \xi) \quad (15)$$

(notice the peculiar feature of the model in which the familiar mass insertion is reduced to the insertion of one of the fundamental fields of the theory).

Using (14) and the chain rule of differentiation, we can write (15) as

$$\begin{aligned} & \left[m \frac{\partial}{\partial m} + \beta(g) \frac{\partial}{\partial g} - n \gamma(g) \right] \Gamma_{\phi_1 \dots \phi_n}(p_i; m, g) \\ &= \frac{m}{2g} \delta(g) \Gamma_{A, \phi_1 \dots \phi_n}(0, p_i; m, g) \quad , \end{aligned} \quad (16)$$

where

$$\begin{aligned} \beta(g) &= \frac{3}{2} g \frac{f}{1+f} \\ \gamma(g) &= \frac{1}{2} \frac{f}{1+f} \end{aligned} \quad (17)$$

$$\delta(g) = \frac{1}{1+f}$$

and

$$f(g) = \lim_{\xi \rightarrow 0} m_0 \frac{\partial \log Z}{\partial m_0} = \frac{g^2}{4\pi^2} + \dots \quad (18)$$

It is easy to see that the limit (18) is finite. As explained in Ref. 2), one has

$$\Gamma_{AA}(p) = p^2 + \dots \quad (19)$$

where the dots denote terms of higher order in p . Since the coefficient of the p^2 term is one, it follows from (16) that

$$f = - \frac{m}{2g} \left[\frac{\Gamma_{AAA}(0, p)}{p^2} \right]_{p=0} \quad (20)$$

Actually, one of the Ward identities ensures that $\Gamma_{AAA}(0, p)$ vanishes like p^2 for small p^2 .

Equation (16) presents some interesting features. All three functions usually appearing, namely β , γ and δ , are expressed in terms of the single function $f(g)$. The right-hand side is again a Green function of the theory with one more external line instead of the usual mass insertion. Notice that since the added line carries zero momentum, the Green function in the left-hand side still dominates in the deep Euclidean region. Finally, we emphasize the fact that the insertion is an A field. Therefore the set of vertex functions not containing the auxiliary fields F and G satisfy the Callan-Symanzik equations among themselves in the sense that if we start in the left-hand side with vertex functions involving only the normal fields A, B and ψ , we will not generate in the right-hand side any ones involving F and G. This property of "closure" is not true for the connected Green functions because, as it is shown in Ref. 2), they satisfy an equation analogous to (15), but with an F field insertion in the right-hand side. Finally, we remark that (17) and (18) show that, as is expected from the type of couplings involved, the theory under consideration is not asymptotically free.

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