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TUNE SHIFTS FOR PARTICLE BEAMS CROSSING AT

SMALL ANGLES IN THE LOW- β SECTION OF A

STORAGE RING

by

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Abstract

The linear tune shift of a particle crossing an azimuthally uniform cylindrically symmetric Gaussian beam is evaluated to first order in the line density, taking into account the variation of the amplitude function β through the crossing region.

1. INTRODUCTION

In storage rings having low- β crossing regions, small crossing angle, and either unbunched beams, long bunches or many closely spaced bunches, there can be significant electromagnetic beam-beam interaction even when the beams do not spatially overlap.

The contribution per unit length to the tune shift does not decrease with the distance from the crossing point but is essentially constant over the whole length where the two beams interact electromagnetically without overlapping spatially. This is so because β increases quadratically with the distance from the crossing point while the beam-generated field gradient asymptotically decreases like the inverse square of that distance.

We present the general formalism required to compute the beam-induced tune shift of a particle crossing an unbunched beam and then evaluate the effect for the case of cylindrically symmetric beams. Formulas for the tune shift are given for a number of limiting cases; in the general situation, the tune shifts can be expressed in terms of two integrals which were evaluated numerically.

2. PROBLEM DEFINITION

We shall study the weak-strong interaction of two beams. In particular, we compute $\Delta\nu_x$ and $\Delta\nu_y$ for a single particle crossing a prescribed strong beam. In making this calculation we shall :

- a. Work to first order in the line density of the strong beam.
- b. Assume that the equilibrium orbit of the weak beam is a straight line, i.e. we neglect the change in the equilibrium orbit coming from beam-beam interaction. This is permitted because of our first assumption.
- c. Assume that wall effects are negligible; i.e., we calculate beam fields as if the beam is in free space.
- d. Assume the strong beam is unbunched; i.e., that the bunch length is much greater than the interaction length ℓ .

- e. Assume the beams cross in the (y, s) plane at a small crossing angle $\alpha \ll 1$.
- f. Assume that the electromagnetic field of the strong beam acts upon the weak beam only between $-\ell/2$ and $+\ell/2$. Outside this range the beams can be shielded electromagnetically, if they are well separated; otherwise they have to be separated quickly by bends at $\pm \ell/2$.
- g. Assume that derivatives of the beam profile with distance along the beam can be neglected when computing beam fields.

3. FORMALISM

We use the coordinate system shown in Fig. 1. To simplify the notation we use z when we need not distinguish between x and y . To first order in the perturbed focusing parameter $k_z(s)$ the tune shift Δv_z is given by

$$\Delta v_z = \frac{1}{4\pi} \int_{-\ell/2}^{\ell/2} \beta_z(s) k_z(s) ds, \quad (1)$$

where

$$k_z(s) = \frac{e(1 + \beta^2)}{m_0 c^2 \gamma \beta^2} \frac{\partial E_z}{\partial z}, \quad (2)$$

and s is the distance, measured from the crossing point, along the equilibrium orbit of the particle intersecting the strong beam. In (2), the electric field E can be derived from the potential V

$$E_z(x, y) = - \frac{\partial V(x, y)}{\partial z}, \quad (3)$$

which in turn follows from the beam density function ρ :

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) V(x, y) = - 4\pi\rho \left(\frac{x'}{a} \frac{y'}{b} \right). \quad (4)$$

In (4), ρ is assumed to depend on s' only through the dependence of beam dimensions a and b upon s' which is

$$a = a_0 \left[\beta_x(s') / \beta_{ox} \right]^{\frac{1}{2}}, \quad b = b_0 \left[\beta_y(s') / \beta_{oy} \right]^{\frac{1}{2}}. \quad (5)$$

Here a_0 and b_0 are the beam dimensions at the crossing point.

The dependence of $\beta(s')$ upon s' is given by :

$$\beta_z(s') = \beta_{0z} + s'^2/\beta_{0z} \quad (6)$$

where β_{0z} is the β -value at the crossing point.

We shall not attempt the solution of Eqs. (1) through (6) in the general case, but, rather, specialize the problem to a situation in which more progress can be made by analytical methods.

4. CYLINDRICALLY SYMMETRIC GAUSSIAN BEAM : FORMULAS

For a cylindrically symmetric beam Eq. (4) may be replaced by Gauss's theorem and the field perpendicular to the beam is given by

$$E_r(r) = \frac{4\pi}{r} \int_0^r \rho(r') r' dr' \quad (7)$$

with components

$$\begin{aligned} E_{x'} &= E_r \cos \phi \\ E_{y'} &= E_r \sin \phi \end{aligned} \quad (8)$$

in the beam coordinate system. To obtain the field components E_x and E_y in the particle coordinate system we use the transformations :

$$\begin{aligned} x' &= x \\ y' &= y \cos \alpha + s \sin \alpha \\ s' &= s \cos \alpha - y \sin \alpha, \end{aligned} \quad (9)$$

and

$$\begin{aligned} x &= x' \\ y &= y' \cos \alpha - s' \sin \alpha \\ s &= s' \cos \alpha + y' \sin \alpha. \end{aligned} \quad (10)$$

We find

$$E_x = E_{x'} = E_r(r) \cos \phi \quad (11)$$

$$E_y = E_{y'} \cos \alpha = E_r(r) \sin \phi \cos \alpha .$$

We now specialize to a Gaussian beam; i.e.

$$\rho(r) = \frac{\lambda e}{2\pi\sigma^2} \exp(-r^2/2\sigma^2) , \quad (12)$$

where σ is the rms half-width of the beam and hence a function of s' .

The density $\rho(r)$ is normalized so that

$$\int_0^\infty \rho(r) 2\pi r dr = \lambda e . \quad (13)$$

Therefore, λ is the number of particles per unit length of the strong beam.

From (7) and (12) we obtain :

$$E_r(r) = \frac{2\lambda e}{r} [1 - \exp(-r^2/2\sigma^2)] , \quad (14)$$

and, hence, using (11)

$$E_x = \frac{2\lambda e \cos \phi}{r} [1 - \exp(-r^2/2\sigma^2)] , \quad (15)$$

$$E_y = \frac{2\lambda e \sin \phi \cos \alpha}{r} [1 - \exp(-r^2/2\sigma^2)] .$$

In evaluating (15) one recalls that

$$\sin \phi = y'/r$$

$$\cos \phi = x'/r \quad (16)$$

$$r^2 = x'^2 + y'^2$$

and that the primed quantities are related to the unprimed variables by (9).

We need $k_z(s)$, which using (2), (9), (15) and (16), may be written in the form :

$$k_x = \left[\frac{2(1 + \beta^2)\lambda r_0}{\gamma\beta^2} \right] \frac{\partial}{\partial x} \left\{ \frac{x}{r^2} [1 - \exp(-r^2/2\sigma^2)] \right\} , \quad (17)$$

$$k_y = \left[\frac{2(1 + \beta^2)\lambda r_0 \cos \alpha}{\gamma\beta^2} \right] \frac{\partial}{\partial y} \left\{ \frac{y'}{r^2} [1 - \exp(-r^2/2\sigma^2)] \right\} , \quad (18)$$

with $y' = y \cos \alpha + s \sin \alpha$, (19)

$$r^2 = x^2 + (y \cos \alpha + s \sin \alpha)^2 \quad (20)$$

and $r_0 = e^2/m_0 c^2$, the classical particle radius.

Actually, we want k_x and k_y evaluated at $x = y = 0$; i.e. on the trajectory of the test particle. We obtain

$$k_x = \left[\frac{2(1 + \beta^2)\lambda r_0}{\gamma \beta^2} \right] \frac{1}{s^2 \sin^2 \alpha} \left[1 - \exp \left(- \frac{s^2 \sin^2 \alpha}{2\sigma^2} \right) \right] \quad (21)$$

and

$$k_y = \left[\frac{2(1 + \beta^2)\lambda r_0 \cos^2 \alpha}{\gamma \beta^2} \right] \left[\left(\frac{1}{s^2 \sin^2 \alpha} + \frac{1}{\sigma^2} \right) \exp \left(- \frac{s^2 \sin^2 \alpha}{2\sigma^2} \right) - \frac{1}{s^2 \sin^2 \alpha} \right]. \quad (22)$$

In Eqs. (21) and (22) it must be remembered that σ is a function of s' , given by

$$\sigma = \sigma_0 [\beta(s')/\beta_0]^{\frac{1}{2}} \quad (23)$$

with

$$\beta(s') = \beta_0 + s'^2/\beta_0, \quad (24)$$

and

$$s' = s \cos \alpha \quad (25)$$

The horizontal and vertical amplitude functions β_0 have to be equal in order to have cylindrical symmetry for all values of s' .

Combining (1), (6), (21) through (25), we obtain

$$\Delta v_x = \frac{\lambda r_0 \beta_0 (1 + \beta^2)}{2\pi \gamma \beta^2 \sin^2 \alpha} \int_{-\ell/2}^{\ell/2} \left(\frac{1}{s^2} + \frac{1}{\beta_0^2} \right) \left\{ 1 - \exp \left[- \frac{s^2 \sin^2 \alpha}{2\sigma_0^2 \left(1 + \frac{s^2 \cos^2 \alpha}{\beta_0^2} \right)} \right] \right\} ds \quad (26)$$

and

$$\Delta v_y = \frac{\lambda r_0 (1 + \beta^2) \beta_0 \cos^2 \alpha}{2\pi \gamma \beta^2} \int_{-\ell/2}^{\ell/2} \left(1 + \frac{s^2}{\beta_0^2} \right) \left\{ \left[\frac{1}{s^2 \sin^2 \alpha} + \frac{1}{\sigma_0^2 \left(1 + \frac{s^2 \cos^2 \alpha}{\beta_0^2} \right)} \right] \times \exp \left[- \frac{s^2 \sin^2 \alpha}{2\sigma_0^2 \left(1 + \frac{s^2 \cos^2 \alpha}{\beta_0^2} \right)} \right] - \frac{1}{s^2 \sin^2 \alpha} \right\} ds. \quad (27)$$

Formulas (26) and (27) cannot be evaluated analytically. If we make use of the smallness of the crossing angle ($\alpha \ll 1$) and introduce a change of variables, they take the form :

$$\begin{aligned}\Delta v_x &= \frac{\lambda r_0 (1 + \beta^2) \ell \beta_0}{4\pi \gamma \beta^2 \sigma_0^2} I_x \left(\frac{\ell}{2\beta_0}, \frac{\beta_0 \alpha}{\sigma_0} \right), \\ \Delta v_y &= \frac{\lambda r_0 (1 + \beta^2) \ell \beta_0}{4\pi \gamma \beta^2 \sigma_0^2} I_y \left(\frac{\ell}{2\beta_0}, \frac{\beta_0 \alpha}{\sigma_0} \right),\end{aligned}\quad (28)$$

where

$$\begin{aligned}I_x(\xi, \eta) &= \frac{1}{\xi \eta^2} \int_{-\xi}^{\xi} \frac{x^2 + 1}{x^2} \left\{ 1 - \exp \left[-\frac{1}{2} \eta^2 x^2 / (1 + x^2) \right] \right\} dx \\ I_y(\xi, \eta) &= \frac{1}{\xi \eta^2} \int_{-\xi}^{\xi} \frac{x^2 + 1}{x^2} \left\{ \left[1 + \frac{\eta^2 x^2}{1 + x^2} \right] \exp \left[-\frac{1}{2} \eta^2 x^2 / (1 + x^2) \right] - 1 \right\} dx\end{aligned}\quad (29)$$

This last form is convenient for numerical evaluation, as the dependence upon all machine parameters has been reduced to a dependence upon only two parameters ξ and η . For $\xi \gg 1$, ξ and η have simple physical meanings. ξ becomes the ratio of the beam widths at the ends and at the centre of the intersection region; and η becomes the distance between the beam centres at the ends of the intersection region, expressed in rms radii there.

Fig. 2 and 3 show graphs of I_x and I_y .

5. CYLINDRICALLY SYMMETRIC GAUSSIAN BEAMS : SPECIAL CASES

In the cases considered below and the examples in the next section we usually calculate only Δv_x because in all cases $|\Delta v_x| \leq |\Delta v_y|$.

5.1 HEAD-ON COLLISIONS

When the beam separation is very small, $\eta \ll 1$, the integrals of Eq. (29) may be readily evaluated since the exponent is bounded by $\frac{1}{2}\eta^2$ and consequently can be approximated by the series expansion. One finds :

$$I_x(\xi, \eta) \rightarrow 1 ; \quad I_y(\xi, \eta) \rightarrow 1 \quad (30)$$

and hence, from (28) with $\beta \approx 1$

$$\Delta v_x \big|_{\eta \ll 1} = \Delta v_y \big|_{\eta \ll 1} = \frac{\lambda r_0 \beta_0 \ell}{2\pi \gamma \sigma_0^2} \quad (31)$$

5.2 LARGE CROSSING ANGLE

For a machine with well separated beams, $\eta \gg 1$, we may obtain a closed expression for Δv_x . In this case the biggest contributions to the integral (29) comes from the interval with $|x| \ll 1$. Hence, we may neglect x^2 compared to unity in the exponent and obtain approximately :

$$I_x(\xi, \eta \gg 1) \approx \frac{2}{\eta^2} - \frac{2}{\xi^2 \eta^2} \left[1 - \exp(-\frac{1}{2} \xi^2 \eta^2) \right] + \frac{\sqrt{2\pi}}{\xi \eta} \operatorname{erf}(\xi \eta / \sqrt{2}) \quad (32)$$

If, furthermore, the crossing region is so long that also $\xi \eta \gg 1$, I_x tends towards

$$I_x \rightarrow 2/\eta^2 + \sqrt{2\pi}/\xi \eta \quad (33)$$

From (28) and (33) we obtain for $\beta \approx 1$ and $\eta \gg \xi$:

$$\Delta v_x \big|_{\eta \gg 1, \xi} = \sqrt{\frac{2}{\pi}} \frac{\lambda r_0 \beta_0}{\gamma \alpha \sigma_0} \quad (34)$$

In this situation, $\eta \gg 1$, and $\eta \gg \xi$, the tune shift Δv_x is independent of the length of the interaction region - because in the approximation $\eta \gg \xi$ the terms giving rise to the long-range tune shift were removed. A comparison between (31) and (34) shows that the short range part of a large angle crossing is equivalent to head-on collisions over a length $l = 2\sigma_0 \sqrt{2\pi}/\alpha$ which is plausible.

5.3 LONG-RANGE CONTRIBUTION TO Δv

If on the other hand, we have $1 \ll \eta \ll \xi$ in (33), we find from (28) and (33) for $\beta \approx 1$:

$$\Delta v_x \big|_{1 \ll \eta \ll \xi} = \frac{\lambda r_0 l}{\pi \gamma \beta_0 \alpha^2} \quad (35)$$

This estimate may be improved for $\eta \gg 1$ by observing that the integrals (29) become trivial for $x = s/\beta_0 \gg 1$. Introducing a cut-off at $x_0 = s_0/\beta_0 \geq 2$ we find for the contribution to Δv_z for $|x| > x_0$, assuming $\beta \approx 1$ and $\alpha \ll 1$:

$$(\Delta v_x)_\ell = \frac{\lambda r_o (\ell - s_o)}{\pi \gamma \alpha^2 \beta_o} \left\{ 1 - \exp \left[-\frac{1}{2} (\alpha \beta_o / \sigma_o)^2 \right] \right\} \quad (36)$$

$$(\Delta v_y)_\ell = \frac{\lambda r_o (\ell - s_o)}{\pi \gamma \alpha^2 \beta_o} \left\{ \left[1 + \left(\frac{\alpha \beta_o}{\sigma_o} \right)^2 \right] \exp \left[-\frac{1}{2} (\alpha \beta_o / \sigma_o)^2 \right] - 1 \right\} \quad (37)$$

These formulas, Eqs. (36) and (37) are the contributions to Δv_z from the region between $(s_o/2)$ and $(\ell/2)$ and between $(-s_o/2)$ and $(-\ell/2)$.

The accurate value of the contribution from $(-s_o/2)$ to $(+s_o/2)$ must be evaluated numerically. A rough estimate may be obtained from (31) with $\ell = 2\sigma_o\sqrt{2\pi}/\alpha$. The ratio between the long-range contribution (36) and the short range contribution then becomes :

$$\frac{(\Delta v_x)_{\text{long-range}}}{(\Delta v_x)_{\text{short-range}}} = \frac{\ell}{\beta_o} \frac{1 - \exp(-\frac{1}{2}\eta^2)}{\eta\sqrt{2\pi}} \quad (38)$$

For low- β machines $\beta_o \ll \ell$. Therefore, a major part of Δv_x is contained in (36) and (37) if the beam separation factor is within the rather wide limits $2\sqrt{2\pi} \beta_o/\ell \ll \eta \ll \ell/\sqrt{2\pi} \beta_o$.

6. EXAMPLES AND DISCUSSION

6.1 THE CERN INTERSECTING STORAGE RINGS (ISR) 1)

The ISR have a large crossing angle and Δv_x is adequately approximated by Eq. (34). Taking $\lambda = N/2\pi R$, with $N = 2 \times 10^{14}$, $R = 150$ m, $\gamma = 30$, $\alpha = 0.25$, $\sigma_o = 1.4$ mm (from luminosity measurements) and $\beta_o = 13$ m, one obtains $\Delta v_x \Big|_{\text{ISR}} = 3.5 \times 10^{-4}$.

6.2 BNL INTERSECTING STORAGE ACCELERATOR (ISA) 2)

In the high-luminosity mode of operating this machine the following parameters apply :

$$\begin{aligned} \ell &= 20 \text{ m}, \quad \beta_o = 0.7 \text{ m}, \quad \alpha = 2 \times 10^{-4}, \quad \gamma = 200, \\ \sigma_o^2 &= 0.07 \text{ mm}^2, \quad N = 6 \times 10^{14}, \quad R = 300 \text{ m}. \end{aligned}$$

From (28) and (29) we obtain

$$\xi = \ell/2\beta_0 = 14, \quad \eta = \beta_0\alpha/\sigma_0 = 0.3$$

Since $\eta^2 \ll 1$ we can employ (31) to evaluate Δv_x and find

$$\Delta v_x|_{ISA} = 0.12$$

The high value for Δv_x results from the assumptions of unbunched beams interacting electromagnetically over the full length, 20 m, of the intersection region; neither assumption needs to be maintained in future designs. An optimisation taking into account long-range tune shifts ³⁾ shows that one can reduce the tune shift substantially without losing much luminosity.

6.3 DISCUSSION

Although the analysis has been done for cylindrically symmetric Gaussian beams, it is reasonable to expect, because of Gauss's law, that the long-range contribution to Δv would be essentially the same for other beam cross sections. Of course, the local term is quite sensitive to beam profile.

The beam-beam limit in storage rings is considered to be due to the overlapping of non-linear resonances⁴⁾, the degree of overlapping being determined by the resonance widths. The long-range contribution to the resonance width was estimated by G.A. Voss⁵⁾. The linear tune shift appears as a proportionality factor in the widths of all non-linear resonances when the beams collide head-on ($\eta \ll 1$), when they are well separated ($\eta \gg 1$) and when the phase advance in the crossing region is small ($\xi < 1$). When the phase advance is large ($\xi \gg 1$) and the beams are just separated ($\eta \approx 1$) the average width of non-linear resonances is less than that expected from proportionality with the linear tune shift. In addition, phase factors appear which may make individual resonances even weaker for some combinations of ξ and η ⁶⁾. Therefore, the beam-beam limit obtained from the linear tune shift including the long-range contribution, Eq. (28), is pessimistic. An optimistic beam-

beam limit may be obtained by only including short-range contribution to the linear tune shift, Eq. (34). The true situation is bounded between these limits, which often are not too far apart. In cases where the two bounds are far apart, more detailed calculations, such as presented in Ref. 6, are required.

7. ACKNOWLEDGEMENT

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8. References

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Fig. 1 - Coordinate systems. The angle between the beam and the trajectory of the test particle is α , r - ϕ is a cylindrical coordinate system about the beam direction.

Fig. 2 - $I_X(\xi, \eta)$ as defined in (29). The full curves show $I_X(\xi, \eta)$.

$I_X(\xi, \eta) \rightarrow 1$ for $\xi, \eta \rightarrow 0$. The dashed curves show

$$I_X^* = I_X \xi \eta \left(1 + \frac{\xi}{\eta} \sqrt{\frac{2}{\pi}}\right)^{-1} (2\pi)^{-\frac{1}{2}}. \quad I_X^*(\xi, \eta) \rightarrow 1 \text{ for } \xi, \eta \rightarrow \infty.$$

Fig. 3 - $I_y(\xi, \eta)$ as defined in (29). The full curves show $I_y(\xi, \eta)$. $I_x(\xi, \eta) \rightarrow 1$ for $\xi, \eta \rightarrow 0$.
The dashed curves show $I_y^* = -\frac{1}{2}\eta^2 I_y$. $I_y^*(\xi, \eta) \rightarrow 1$ for $\xi, \eta \rightarrow \infty$.