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PHENOMENOLOGICAL UPPER AND LOWER BOUNDS

ON THE $\pi\pi$ S WAVE SCATTERING LENGTHS

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A B S T R A C T

Starting from a given phenomenological input (upper and lower bounds of the real and imaginary parts of the $\pi^0\pi^0$ and $\pi^+\pi^0$ amplitudes on the region $0.45 \text{ GeV} \leq E_{\text{cm}} \leq 1.9 \text{ GeV}$; $0 \leq t \leq 4 \mu^2$) we derive rigorous upper and lower bounds on the $\pi\pi$ S wave scattering lengths.

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1. INTRODUCTION

Some recent developments of the low-energy $\pi\pi$ phenomenology¹⁾ have shown the weakness of the model-dependent results in the framework of general principles. Even in the present resolution of Roy's equations²⁾ the implications of phenomenological assumptions such as the existence of the ρ -meson or Regge asymptotic behaviour are difficult to distinguish from the consequences of the approximations done.

Fortunately, phenomenology can sometimes be done rigorously and leads to some "improvements" of the axiomatic results. An example of it can be found in the story of the lower bounds of the $\pi\pi$ S-wave scattering lengths, which were first purely axiomatic³⁾ and afterwards reduced, thanks to some estimate of the D-wave scattering lengths⁴⁾.

In this work we want also to derive rigorous phenomenological bounds for the S-wave scattering lengths, but we adopt the position of using the maximal amount of available experimental data as directly as possible, and not only through the D-waves. As a result, a new class of bounds appear (upper bounds) and the lower ones are improved. In this approach, however, one has to take care of the consistency of the chosen phenomenology with general principles.

The plan of this study is then the following. In Section 2 we derive some basic sum rule inequalities which are applied to the determination of the scattering lengths in Section 3. The phenomenological input is detailed in Section 4, where our numerical results are given.

2. SUM-RULE INEQUALITIES

First we briefly introduce the notation and normalizations.

The s-channel partial wave expansion of the s-channel isospin (I) $\pi\pi$ amplitude $F^I(s, t, u)$ reads ($m_\pi = 1$):

$$F^I(s, t, u) = \sum_{\ell=0}^{\infty} \left[\frac{1 + (-)^{I+\ell}}{2} \right] (2\ell+1) f_{\ell}^I(s) P_{\ell} \left(1 + \frac{2t}{s-4} \right) \quad (2.1)$$

with

$$f_{\ell}^I(s) = (\eta_{\ell}^I e^{2i\delta_{\ell}^I} - 1) / 2i\vartheta \quad \vartheta = \left[(s-4)/s \right]^{\frac{1}{2}} \quad (2.2)$$

where δ_{ℓ}^I and η_{ℓ}^I are the real phase shifts and inelasticities. In our normalization the optical theorem is

$$\text{Im } F^{\text{I}}(s, 0) = \sqrt{s(s-4)} \sigma_{\text{TOT}}^{\text{I}} / 16 \pi \quad (2.3)$$

and the S-wave scattering lengths a_0 and a_2 are given by

$$a_{\text{I}} = f_0^{\text{I}}(4) = F^{\text{I}}(4, 0, 0) \quad (2.4)$$

In the following we consider for $t = t_0$ fixed, $0 \leq t_0 \leq 4$, the $\pi^0\pi^0$ and $\pi^+\pi^0$ elastic amplitudes F_N and F_S , respectively:

$$\begin{aligned} F_N &= F^0 + 2 F^2 \\ F_S &= F^1 + F^2 \end{aligned} \quad (2.5)$$

These ($s \leftrightarrow u$) crossing even amplitudes are real analytic functions in the complex z -plane cut for $z \geq 1$, where

$$z = \left(\frac{s-u}{4+t_0} \right)^2 = \left(\frac{2s+t_0-4}{4+t_0} \right)^2 \quad (2.6)$$

We then assume, as explained in Section 4, that on some part $[s_1, s_2]$ of the physical region ($4 < s_1 < s_2$), which through Eq. (2.6) is mapped onto $[z_1, z_2]$ ($1 < z_1 < z_2$), the available data are sufficiently reliable to yield upper and lower bounds of the real and imaginary parts of F_N and F_S . As has been shown by Common⁵⁾ in a slightly different context, this kind of information can be rigorously used in a powerful way in the framework of general principles.

For this we introduce first the mapping

$$v = v(\bar{z}) = \frac{1 - \sqrt{\omega}}{1 + \sqrt{\omega}}, \quad \omega = \frac{\bar{z} - z_1}{\bar{z} - z_2} \quad (2.7)$$

which maps the cut z -plane onto a cut unit disk D_v of the v -plane in the following way:

1. The low-energy region $1 \leq z \leq z_1$ is mapped onto the cut $v_1 \leq v \leq 1$, with

$$v_1 = v(\bar{z}=1) \quad v(\bar{z}=z_1) = 1 \quad 0 < v_1 < 1 .$$

2. The intermediate energy region $z_1 \leq z \leq z_2$, where both the real and imaginary parts of the amplitudes have phenomenological bounds, is mapped onto the circumference of the unit circle:

$$v = e^{i\theta} \quad v(z = z_2) = -1 .$$

3. The high-energy region $z \geq z_2$ is mapped onto the cut $-1 \leq v \leq 0$, in such a way that when $z \rightarrow +\infty$, $v \rightarrow 0^-$ like

$$v(z) \sim - \frac{(z_2 - z_1)}{4z} \sim - \frac{(z_2 - z_1)(4 + t_0)^2}{16 s^2} \quad (2.8)$$

We denote by I the union of the regions $[-1, 0]$ and $[v_1, 1]$.

Now, with the help of two arbitrary real analytic functions $P(v)$ and $Q(v)$, we construct

$$L(v) = v \left[P(v) F_N(v) + Q(v) F_S(v) \right] \quad (2.9)$$

which is real analytic in D_v , and bounded on the boundary. In fact the only trouble can come from $v = 0$ which corresponds to infinity in the s -plane, but using formula (2.8) and the Froissart-Martin bound for the amplitudes, one has $L(0) = 0$ since

$$|L(v)| \leq C \text{st.} (\sqrt{|v|} \log^2 |v|) \quad \text{as } |v| \rightarrow 0 . \quad (2.10)$$

Then, for any point w_0 inside D_v ($0 \leq w_0 \leq v_1$) one has

$$L(w_0) = \frac{1}{2i\pi} \oint \frac{L(v') dv'}{v' - w_0} \quad (2.11)$$

and the integral can be split into two contributions $L_0(w_0)$ and $L_I(w_0)$, which correspond to the contributions of the circle and of the region I , respectively:

$$L(w_0) = L_0(w_0) + L_I(w_0) \quad (2.12)$$

with

$$L_0(w_0) = \frac{1}{\pi} \int_0^\pi \operatorname{Re} \left\{ \frac{e^{i\theta} L(e^{i\theta})}{e^{i\theta} - w_0} \right\} d\theta \quad (2.13)$$

[we have used the reality of $L(v)$] and

$$L_I(w_0) = \frac{1}{\pi} \left\{ \int_{-1}^0 + \int_{v_1}^1 \right\} \left[\frac{v' dv'}{v' - w_0} \left(P(v') \Im_m F_N(s(v')) + Q(v') \Im_m F_S(s(v')) \right) \right] \quad (2.14)$$

Owing to the positivity of the absorptive parts of F_N and F_S on I , Eq. (2.14) implies that

$$\begin{aligned} & \text{— if } P \text{ and } Q \geq 0 \text{ on } I, \text{ then } L_I(w_0) \geq 0 \\ & \text{— if } P \text{ and } Q \leq 0 \text{ on } I, \text{ then } L_I(w_0) \leq 0. \end{aligned} \quad (2.15)$$

On the other hand, from our phenomenological assumption about the intermediate energy region, we are able to compute for all P and Q upper and lower bounds for $L_0(w_0)$:

$$L_0^m(w_0) \leq L_0(w_0) \leq L_0^M(w_0) \quad (2.16)$$

Combining formulae (2.12), (2.15), and (2.16) we obtain the inequalities sum rules that we shall use in the next section:

For any t_0 and w_0 , $0 \leq t_0 \leq 4$, $0 < w_0 \leq v_1$

$$\begin{aligned} L(w_0) & \geq L_0^m(w_0) \text{ for all } P, Q \geq 0 \text{ on } I \\ L(w_0) & \leq L_0^M(w_0) \text{ for all } P, Q \leq 0 \text{ on } I. \end{aligned} \quad (2.17)$$

3. BOUNDS ON THE SCATTERING LENGTHS

Inequalities (2.17) can be applied to any unphysical values, but we focus here on the scattering lengths.

3.1 "old" bounds

With the choice $t_0 = 0$, $w_0 = v_1$

$$F_N(v_1) = F^0(4,0) + 2 F^2(4,0) = a_0 + 2a_2$$

$$F_S(v_1) = F^1(4,0) + F^2(4,0) = a_2$$

and then

$$L(w_0) = a_0 v_1 P(v_1) + a_2 v_1 (Q(v_1) + 2 P(v_1)) \quad (3.1)$$

Inequalities (2.17) applied on (3.1) then give information on some linear combinations of a_0 and a_2 . But since (2.17) needs $PQ \geq 0$ on I and since v_1 belongs to I (as end point) it is, for example, impossible to bound a_0 alone, which would require $2P(v_1) + Q(v_1) = 0$. This is possible on a_2 [$P(v_1) = 0$, $Q(v_1) = 1$] but then $Q(v_1) = 1$ implies $P, Q \geq 0$ on I which allows one to compute only lower bounds. This is true also for $a_0 + 2a_2$, and we recover here the lower bounds already given^{4,6,8}. However, the lower bounds can now be positive and optimized owing to the freedom allowed in the weight functions P and Q .

3.2 "New" bounds

In the previous analysis only $s \rightleftharpoons u$ symmetry was used. In fact a noticeable improvement can be obtained if we add some $s \rightleftharpoons t$ crossing. For this we now choose $t_0 = 4$, $w_0 = v(z=0) = v_0$ ($0 < v_0 < v_1$). Then

$$F_N(v_0) = F^0(0,4) + 2 F^2(0,4) = F^0(4,0) + 2 F^2(4,0) = a_0 + 2a_2$$

$$F_S(v_0) = F^1(0,4) + F^2(0,4) = \frac{2}{3} [F^0(4,0) - F^2(4,0)] = \frac{2}{3} (a_0 - a_2)$$

and

$$L(w_0) = a_0 v_0 \left(P(v_0) + \frac{2}{3} Q(v_0) \right) + 2a_2 v_0 \left(P(v_0) - \frac{Q(v_0)}{3} \right) \quad (3.2)$$

Since v_0 do not belong to I , the values of P and Q at v_0 can be now prescribed independently of their required behaviour on I . Through Eqs. (3.2), this allows us to compute upper and lower bounds for any linear combination of the scattering lengths, and in particular for the scattering lengths themselves.

As examples, we have chosen to bound a_0 , a_2 , $2a_0 - 5a_2$, and $a_0 + 2a_2$. [The corresponding values of $P(v_0)$ and $Q(v_0)$ are given in Table I.]

We come now to the choice of the weight functions $P(v)$ and $Q(v)$. We have taken polynomials, but instead of testing a (finite) subset of the infinite set of polynomials which give necessary and sufficient conditions for the solutions of the associated "moment problem", as done in Ref. 5, we have constructed:

$$P(v) = P(v_0) R_P(v) \prod_{j \in J} \frac{(v - v_P^j)(v - \overline{v_P^j})}{(v_0 - v_P^j)(v_0 - \overline{v_P^j})} \prod_{k \in K} \frac{(v - v_P^k)}{(v_0 - v_P^k)} \quad (3.3)$$

[$Q(v)$ has an analogous representation] where

$$\begin{aligned} \text{i)} \quad v_P^j &= r_P^j e^{i\theta_P^j}, \quad \overline{v_P^j} = r_P^j e^{-i\theta_P^j}, \quad r_P^j \geq 0, \quad 0 < \theta_P^j < \pi \text{ for all } j \\ \text{ii)} \quad v_P^k &\leq -1 \quad \text{or} \quad v_P^k \geq 1 \quad \text{for all } k \end{aligned} \quad (3.4)$$

iii) $R_P(v)$ is equal to one or to $R(v) = v(v - v_1)/v_0(v_0 - v_1)$ according to the sign of $P(v_0)$ and $Q(v_0)$ and to the nature of the wanted bound. (These different possibilities are listed in Table 1.)

The parametrization (3.3) gives the most general expression for a polynomial of fixed degree ($\approx 2J + K$) submitted to our constraints. The number and location of real and complex zeros entering these representations are parameters which are optimized to give the best bounds for a given set of data. [The extrema of $L_0^m(w_0)$ and of $L_0^M(w_0)$ are found by MINUITL, with at most 48 parameters.]

4. PHENOMENOLOGICAL INPUT

As pointed out in the introduction, the bounds $L_0^m(w_0)$ and $L_0^M(w_0)$ cannot be constructed from any set of data since, for example, they have to fulfil the obvious constraints

$$L_0^m(w_0) < L_0^M(w_0)$$

More generally, the chosen data must satisfy the sum rule inequalities of Common⁵⁾ which express $s \rightleftharpoons u$ crossing and positivity. Following the methods of Sections 2 and 3, one can also add to these requirements some $s \rightleftharpoons t$ crossing sum rules, for all t_0 between 0 and 4.

As this study has not been done, we do not specialize in any phase-shift analysis. However, on the energy range $0.45 \text{ GeV} \leq E_{\text{cm}} \leq 1.9 \text{ GeV}$, the main features of $\pi\pi$ scattering are common to most of the recent analyses⁷⁾ and one can define a "central" family of S, P, D, and F phase shifts with associated "errors". To cover the spread of the data, we have multiplied the errors (between 0.45 GeV and 0.95 GeV) by a scaling factor ϵ ($0 \leq \epsilon \leq 1$) such that $\epsilon = 0$ gives the central family and $\epsilon = 1$ the bands of maximal expanse, as given in Table 2.

For any fixed value of ϵ one can easily compute the lower and upper bounds of the $\pi\pi$ amplitudes and then derive the bounds on the scattering lengths. Owing to the present status of phenomenology, only the results with $\epsilon \approx 1$ can be taken with confidence. However, it is interesting to follow the variations of the bounds with ϵ , as it is sketched in Table 3. (We recall that ϵ is not a measure of the errors in a given analysis, but an estimate of the discrepancies between various analyses.)

5. CONCLUSION

All the values given in Table 3 depend strongly on ϵ but, contrary to the upper bounds, the lower ones remain always small.

The lower values for $\epsilon = 1$ therefore stay on a rather firm basis. However, as has already been pointed out⁸⁾, they can increase [if some peculiar phase-shift analysis are used⁹⁾] sufficiently to rule out the Weinberg values of the scattering lengths. Conversely, this smallness of the lower bounds in all cases weakens the phenomenological interest of their present axiomatic values.

On the other hand, if the upper bounds for $\epsilon = 1$ are weak, they are surprisingly constraining for $\epsilon = 0$. This perhaps implies that in order to obtain a better estimate of the scattering lengths one can stay rather far from threshold (here 450 MeV), but then one has to reduce the errors. In this respect, the present results of K_{e4} seem weak¹⁰⁾ and have not been included in this analysis.

Finally, we want to point out that on a definite set of coherent data, this method yields a model-independent measurement (with true errors) of the scattering lengths.

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Table 1

Values of $P(v_0)$, $Q(v_0)$, $R_P(v)$, and $R_Q(v)$ for computing the upper (U.B.) and lower bounds (L.B.) of a_0 , a_2 , $2a_0 - 5a_2$, and $a_0 + 2a_2$.

		$R_P(v)$	$R_Q(v)$	$P(v_0)$	$Q(v_0)$
a_0	U.B.	R	R	$\frac{1}{3}$	1
	L.B.	1	1		
a_2	U.B.	R	1	$\frac{1}{3}$	$-\frac{1}{2}$
	L.B.	1	R		
$2a_0 - 5a_2$	U.B.	1	R	-1	$\frac{9}{2}$
	L.B.	R	1		
$a_0 + 2a_2$	U.B.	R	R	1	0
	L.B.	1	1		

Table 2

Central values and errors used to compute the bounds for the real and imaginary parts of the amplitude on the intermediate energy region $0.45 \text{ GeV} \leq E_{\pi\pi}^{\text{cm}} \leq 1.9 \text{ GeV}$. The phase shifts are given in degrees and the inelasticities fulfil

$$\text{Sup}\{0, \eta - \Delta\eta\} \leq \eta \leq \text{Inf}\{-1, \eta + \Delta\eta\}$$

$E_{\text{cm}} \text{ (GeV)}$	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95
δ_0^0	40 ± 9	44 ± 9	50 ± 9	58 ± 9	63 ± 11	70 ± 11	75 ± 11	82 ± 11	89 ± 13	97 ± 14	130 ± 36
$-\delta_0^2$	7 ± 3	10 ± 4	12 ± 4	14 ± 5	15 ± 5	16 ± 5	17 ± 5	18 ± 7	19 ± 7	20 ± 7	20 ± 7
δ_1^1	3.5 ± 1	6.5 ± 1.5	10 ± 1.5	15.5 ± 1.5	25 ± 1.5	42 ± 2	75 ± 2.5	115 ± 3.5	140 ± 2.5	150 ± 2.5	154 ± 2.5

$E_{\text{cm}} \text{ (GeV)}$	1	1.05	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
δ_0^0	180 ± 60	240 ± 20	250 ± 15	270 ± 15	290 ± 15	312 ± 15	332 ± 15	358 ± 15	380 ± 15	400 ± 15	405 ± 15
η_0^0	0.25 ± 0.25	0.5 ± 0.25	0.7 ± 0.25	0.9 ± 0.3	0.95 ± 0.3	0.95 ± 0.3	0.9 ± 0.3	0.85 ± 0.3	0.75 ± 0.3	0.7 ± 0.3	0.6 ± 0.3
$-\delta_0^2$	20 ± 7	20 ± 7	21 ± 7	21 ± 9	22 ± 9	22 ± 11	23 ± 11	23 ± 11	24 ± 15	25 ± 15	25 ± 15
η_0^2	1	1	1	1	1	1	1	1	1	1	1
δ_1^1	154 ± 10	157 ± 10	159 ± 10	163 ± 11	165 ± 11	166 ± 11	167 ± 11	168 ± 11	169 ± 11	169 ± 11	169 ± 11
η_1^1	1 ± 0.15	1 ± 0.15	0.95 ± 0.15	0.95 ± 0.15	0.9 ± 0.15	0.85 ± 0.15	0.7 ± 0.15	0.5 ± 0.15	0.85 ± 0.15	1 ± 0.15	1 ± 0.15
δ_2^2	8 ± 5	12 ± 6	17 ± 6	42 ± 10	100 ± 13	142 ± 11	157 ± 11	160 ± 11	160 ± 11	160 ± 11	160 ± 11
η_2^2	1 ± 0.2	1 ± 0.2	1 ± 0.2	0.9 ± 0.2	0.65 ± 0.2	0.75 ± 0.2	0.82 ± 0.2	0.86 ± 0.2	0.9 ± 0.2	0.94 ± 0.2	0.98 ± 0.2
δ_3^1	1 ± 3	1 ± 3	2 ± 3	2 ± 3	3 ± 3	4 ± 3	6 ± 3	10 ± 3	5 ± 3	-10 ± 3	-8 ± 3
η_3^1	1 ± 0.5	1 ± 0.5	1 ± 0.5	1 ± 0.5	1 ± 0.5	1 ± 0.5	0.92 ± 0.5	0.8 ± 0.05	0.5 ± 0.05	0.7 ± 0.05	0.9 ± 0.05

Table 3

Numerical values (for different values of ε) of the bounds on a_0 , a_2 , $2a_0 - 5a_2$, and $a_0 + 2a_2$ obtained using data of Table 2. The bounds are approximately linear with ε ($0 \leq \varepsilon \leq 1$).

		$\varepsilon = 0$	$\varepsilon = \frac{1}{2}$	$\varepsilon = 1$
a_0	U.B.	0.86	6.8	12.2
	L.B.	-0.07	-0.31	-0.47
a_2	U.B.	0.60	2.65	4.5
	L.B.	-0.02	-0.17	-0.27
$2a_0 - 5a_2$	U.B.	1.71	17.2	32
	L.B.	-2.8	-9	-16
$a_0 + 2a_2$	U.B.	1.28	7.2	12.5
	L.B.	0.126	-0.12	-0.22