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FINITE ENERGY BOUNDS FOR πN SCATTERING

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Upper bounds on energy averaged πN cross-sections are given. Using low energy data and data from πN backward scattering and $N\bar{N} \rightarrow \pi\pi$ annihilation, we get typically $\sigma_{\text{tot}} \lesssim 80$ mb and $\sigma_{\text{inel}} \lesssim 30$ mb, for energies just above the phase shift region. Our bounds are based on assumptions similar to those underlying Froissart's bound and are equal to it asymptotically. However, at finite but large energies, they increase much slower than what might have been anticipated on purely numerical grounds. Related problems in pp and Kp scattering are also discussed.

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1. INTRODUCTION

In the last few years, the methods underlying Martin's proof ¹⁾ of the Froissart limit ²⁾ have been generalized to yield upper bounds on total cross-sections at finite energies. More precisely, one can get bounds on energy averaged cross-sections at high energies, provided some low energy parameter is given. This was first shown by Martin himself ³⁾, using axiomatic estimates for the $\pi^0\pi^0$ amplitude inside the Mandelstam triangle. Later on, these bounds on $\pi^0\pi^0$ total cross-sections were considerably improved by using phenomenological low energy input : Common and Yndurain ⁴⁾ and others ⁵⁾ used the D wave scattering length [and effective range parameters ⁶⁾], as it is obtained either from extrapolation of the f meson tail or from its Froissart-Gribov representation. Two of us ⁷⁾ used an integral over the D wave in the physical region that was dominated by the f meson. In this way, we got typically $\sigma_{\text{tot}} \lesssim 100$ mb at centre-of-mass energies of $\gtrsim 2$ GeV. When supplied with some further phenomenological information ⁸⁾, this method gave in the same region upper bounds of ~ 20 -30 mb for σ_{tot} in all charge states, which is already of the estimated order of magnitude.

Motivated by this success, we shall in the present paper apply essentially the same method to πN scattering. This case has already been treated by Common and Yndurain ⁹⁾ and Steven ⁵⁾, using the $\pi\pi \rightarrow N\bar{N}$ D wave "scattering length" at $t=4m_\pi^2$ as input. This is of course a highly theoretical quantity, and the most direct way to measure it would consist in extrapolating the πN amplitudes (with the Born term subtracted) from the physical region $s > (M+m_\pi)^2$, $t \leq 0$, down to $s=M^2-m_\pi^2$, $t=4m_\pi^2$. In view of the uncertainty inherent in such a procedure, these authors preferred to calculate it from its Froissart-Gribov representation. Of course, in the latter, also the very same high energy πN amplitude contributes that one is later going to derive a bound on. Thus there is, strictly speaking, a vicious circle in the arguments of Refs. 5) and 9) (the seriousness of which is, of course, debatable).

In our method, instead, we avoid any use of high energy forward data in getting the necessary parameter, which is now a certain integral over low energy, backward, and t channel amplitudes. The most important (and least well known) contribution to this integral stems from the $\pi\pi \rightarrow N\bar{N}$ D wave. For values of t in the interval $4m_\pi^2 < t < 4M^2$, it is of course not directly measurable, and we used the results from an analysis of FESR in the backward direction ¹⁰⁾. As there are rather strong assumptions involved in the latter (as, e.g., a narrow resonance approximation), our bounds are not rigorous either. On the other hand, the $\pi\pi \rightarrow N\bar{N}$ D waves seem to be fairly well understood (in

contrast to the S wave). Moreover, all the uncertainty enters only via a single constant. Thus, changes due to better backward and $N\bar{N} \rightarrow \pi\pi$ data are trivially implemented. Anyhow, we feel justified by the numerical tightness of our bounds, which are shown in Figs. 1, 2 and 4.

Technically, we start with comparing fixed $-t$ with hyperbolic dispersion relations ^{10),11)}. In this way, we get a superconvergence-type sum rule on absorptive parts, which is derived in Section 2. The bounds on σ_{tot} which follow from using this sum rule as a constraint in an optimization program are derived in Section 3. Finally, in Section 4, bounds on the total inelastic cross-section, and bounds involving σ_{el} , are given.

2. THE SUM RULE

Let us consider an amplitude $F(s,t)$ that is even under s - u crossing. For the moment, we shall furthermore assume unsubtracted dispersion relations at fixed t and in the near backward direction.

By dispersing along fixed t , we have ^{*})

$$F(s,t) = F^{\text{Born}} + \frac{1}{\pi} \int_{(M+1)^2}^{\infty} ds' \left(\frac{1}{s'-s} + \frac{1}{s'-u} \right) \text{Im} F(s',t). \quad (2.1)$$

On the other hand, dispersing along a hyperbola

$$s \cdot u = R, \quad (2.2)$$

that is

$$t = \tau_R(s) \equiv 2(M^2+1) - s - \frac{R}{s}, \quad (2.2a)$$

$$s = \sigma_R(t) \equiv M^2+1 - \frac{t}{2} + \sqrt{(M^2+1 - \frac{t}{2})^2 - R} \quad (2.2b)$$

and

$$u = \nu_R(t) \equiv M^2+1 - \frac{t}{2} - \sqrt{(M^2+1 - \frac{t}{2})^2 - R}, \quad (2.2c)$$

^{*}) We use the units with $\hbar = c = m_\pi = 1$; M is the nucleon mass.

we get ¹¹⁾

$$F(s, t) = F(\sigma_R(t), t) = F^{\text{Born}} + \frac{1}{\pi} \int_4^\infty \frac{dt'}{t' - t} \text{Im} F(\sigma_R(t'), t') \\ + \frac{1}{\pi} \int_{(M+1)^2}^\infty ds' \left(\frac{1}{s' - \sigma_R(t)} + \frac{1}{s' - \nu_R(t)} - \frac{1}{s'} \right) \text{Im} F(s', \tau_R(t')). \quad (2.3)$$

Notice that the first integral on the right-hand side involves the $\pi\pi \rightarrow N\bar{N}$ amplitude both in the physical ($t > 4M^2$) and unphysical ($4 < t < 4M^2$) regions.

Comparing Eqs. (2.1) and (2.3), we obviously get a class of sum rules

$$\Delta(t, R) \equiv \int_{(M+1)^2}^\infty ds' \left\{ \left(\frac{1}{s' - \sigma_R(t)} + \frac{1}{s' - \nu_R(t)} \right) \cdot \text{Im} [F(s', t) - F(s', \tau_R(s'))] - \frac{1}{s'} \text{Im} F(s', \tau_R(s')) \right\} \\ - \int_4^\infty \frac{dt'}{t' - t} \text{Im} F(t', \sigma_R(t')) = 0. \quad (2.4)$$

In practice, we shall have to deal with amplitudes which require subtractions. In order to get rid of them, we shall in the following not work with Eq. (2.4) itself, but with a sum rule obtained by differentiating Eq. (2.4) with respect to t and R . Furthermore, we shall take the maximal value of t for which fixed- t DR hold, i.e., $t=4$, and hyperbolas in the backward direction, the latter corresponding to $R=(M^2-1)^2$:

$$\delta \equiv \frac{\partial^2}{\partial t \partial R} \Delta(t, R) \Big|_{\substack{t=4 \\ R=(M^2-1)^2}} = 0. \quad (2.5)$$

After some algebra (and apart from an irrelevant constant factor) we get :

$$\delta = \int_{(M+1)^2}^\infty ds' \frac{1}{(s' - M^2 + 1)^5} \left\{ s' \text{Im} [F^{\text{back}}(s') - F(s', 4)] + 2 \left(\frac{E'_\pi}{q'} \right)^2 \frac{\partial}{\partial z_s} F(s', 4) \right\} +$$

$$+ \frac{s' + M^2 - 1}{2} \cdot \text{Im} \left[F^{\text{backw}}(s') - F(s', 4) + 2 \left(\frac{E'_\pi}{q'} \right)^2 \frac{\partial}{\partial z_s} F^{\text{backw}}(s') \right] \Big\} \\ - \int_4^\infty \frac{dt'}{(t' - 4)^3 (t' - 4M^2)} \frac{\partial}{\partial z_t} \text{Im} F^{\text{forw}}(t') = 0. \quad (2.6)$$

Here, $F^{\text{forw}}(t')$ is the forward $\pi\pi \rightarrow N\bar{N}$ amplitude, while $F^{\text{backw}}(s')$ is the backward $\pi N \rightarrow \pi N$ amplitude. z_s and z_t are the c.m. scattering angles in the s and t channels, and q' and E'_π are the c.m. momentum and energy of the pion.

We should mention three essential points :

- i) as is easily seen, Eq. (2.6) is absolutely convergent if all amplitudes have normal threshold behaviour, if

$$\text{Im} F(s, 4) \text{ and } \frac{\partial}{\partial t} \text{Im} F(s, t=4) <_{s \rightarrow \infty} s^2 / \ln s \quad (2.7)$$

and if

$$\frac{\partial}{\partial u} \text{Im} F(t, z_t=1) <_{t \rightarrow \pm \infty} t^2 / \ln t. \quad (2.8)$$

Indeed, if F is an amplitude with positivity properties, the estimate (2.7) follows from the other conditions, yielding a slight improvement over the Jin-Martin estimate

$$\text{Im} F(s, t=4-\epsilon) <_{s \rightarrow \infty} \text{const.} \times s^2. \quad (2.9)$$

- ii) those parts of $\text{Im} F(s, t)$ that are constant or linear in $\cos \theta$ (either in the s or t channel) do not contribute to the sum rule (2.6). This property is quite important. Because of it, e.g., the large uncertainty in estimations of the $\pi\pi \rightarrow N\bar{N}$ S wave does not cause us any trouble. Also, it means that low energy contributions are relatively small.

iii) Born terms do not contribute.

[These properties make sum rules similar to Eq. (2.6) a very useful tool in determining absorptive parts in the t channel, using s channel absorptive parts as input. Indeed, this problem is being studied presently¹²⁾].

Our next task is to specify the function $F(s,t)$. Our choice

$$F(s,t) = A^{(+)}(s,t) + \frac{s-u}{4M} B^{(+)}(s,t) \quad (2.10)$$

is motivated by the following properties :

-- $F(s,t)$ is directly related to $\sigma_{\text{tot}}^{I_t=0}(s)$, via the optical theorem

$$\sigma_{\text{tot}}^{I_t=0}(s) = \frac{1}{2} (\sigma_{\text{tot}}^{\pi^+p}(s) + \sigma_{\text{tot}}^{\pi^-p}(s)) = \frac{M}{qW} \text{Im } F(s,0); \quad (2.11)$$

-- $F(s,t)$ has the positivity properties necessary for proving the Froissart bound¹³⁾. Indeed, its partial wave decomposition reads¹³⁾ [E is the nucleon energy; the normalization is such that elastic unitarity implies $f_{\ell\pm} = e^{i\delta_{\ell\pm}} \sin \delta_{\ell\pm}$]

$$F(s,t) = \frac{2\pi}{Mq} \sum_{\ell=0}^{\infty} \left\{ (\ell+1) f_{\ell+}^{(+)}(s) \alpha_{\ell+}(s, \cos \theta) + \ell f_{\ell-}^{(+)}(s) \alpha_{\ell-}(s, \cos \theta) \right\} \quad (2.12)$$

with

$$\alpha_{\ell\pm}(s,z) = 2E_{\pi} \frac{P_{\ell}(z) + P_{\ell\pm 1}(z)}{1+z} + (E+M)P_{\ell}(z) + (E-M)P_{\ell\pm 1}(z). \quad (2.13)$$

One easily checks that, for $z \geq 1$ and physical s ,

$$0 \leq \alpha_{\ell+}(s,z) \leq \alpha_{\ell-}(s,z) \leq \alpha_{\ell+1+}(s,z). \quad (2.14)$$

We should point out that due to the factor $(s-u)$ multiplying $B^{(+)}$, the bound (2.8) is not guaranteed axiomatically. Of course, it is satisfied by any reasonable amplitude [whenever $\alpha_N(0) < \frac{3}{2}$ and $\alpha_\Delta(0) < \frac{3}{2}$]. Thus, we shall not worry about this further.

Finally, we have to evaluate numerically all contributions to Eq. (2.6) except for the s channel, high energy, and near-forward ($t=4$) contribution. Equation (2.6) will then provide an integral constraint for the latter.

The most unambiguous contribution is the one from the s channel at low energies. Using CERN phases ¹⁴⁾, the first integral in Eq. (2.6) evaluated from threshold up to $s=(2.2 \text{ GeV})^2$ gives 0.835×10^{-4} , while using Saclay phases ¹⁵⁾ in the same region, we get 0.864×10^{-4} . The high energy ($\sqrt{s} > 2.2 \text{ GeV}$) backward contribution is less certain but strongly suppressed. Using the Regge models of Refs. 16) and 17), we get -0.04×10^{-4} and -0.012×10^{-4} , respectively (this includes also the high energy contribution from the t channel). At low and unphysical energies in the t channel ($t < 7.5 \text{ GeV}^2$) we used the results of a FESR analysis on hyperbolas in the backward hemisphere ¹⁰⁾. In this analysis, a pole approximation is made. (The parameters are summarized in the Appendix.) The contribution to Eq. (2.6) is -1.12×10^{-4} , the dominating part stemming from the f meson. The sum of the above is :

$$(0.85 - 0.03 - 1.12) \times 10^{-4} = -0.30 \times 10^{-4} \equiv -C, \quad (2.15)$$

its error is essentially determined by the error of the f contribution. We believe that the very small errors of Ref. 10) are too optimistic and unbelievable. Thus, we shall not make any detailed error analysis in the following, but we just mention that even if C had an error of $\sim 50\%$, the errors of our final bounds would only be 20% .

3. BOUNDS ON THE TOTAL CROSS-SECTION

The result of the preceding section can be summarized as

$$C = \int_{s_0}^{\infty} \frac{ds}{(s-M^2+1)^5} \left\{ 2 \left(\frac{E_r}{q} \right)^2 \frac{\partial}{\partial z_s} \Im F(s, t=4) - \frac{3s+M^2-1}{2} \Im F(s, 4) \right\} \quad (3.1)$$

with $s_0 = (2.2 \text{ GeV})^2$ and $C = 0.3 \times 10^{-4}$.

Let us first make some general remarks. As we have already remarked, Eq. (3.1) implies

$$\frac{\partial}{\partial t} \operatorname{Im} F(s, t=4) \underset{s \rightarrow \infty}{\leq} s^2 / \ln s \quad (3.2)$$

which is somewhat stronger than the Jin-Martin bound. Just as the latter implies the Froissart limit, one can show that from Eq. (3.2) follows a slight generalization :

$$\sigma_{tot}(s) \underset{s \rightarrow \infty}{\leq} \frac{\pi}{m_\pi^2} \left(\ln \frac{s}{s_1 (\ln \frac{s}{s_2})^{5/2}} \right)^2. \quad (3.3)$$

Indeed, it is not yet the strongest possible result. Yndurain ⁶⁾ has shown that from normal threshold behaviour of the t channel D wave follows ^{*)}

$$\frac{\partial^2}{\partial t^2} \operatorname{Im} F(s, t=4) \underset{s \rightarrow \infty}{\leq} s^2 / \ln s, \quad (3.4)$$

and from this follows ^{**)}

$$\sigma_{tot}(s) \underset{s \rightarrow \infty}{\leq} \frac{\pi}{m_\pi^2} \left(\ln \frac{s}{s_1 (\ln \frac{s}{s_2})^{7/2}} \right)^2. \quad (3.5)$$

It is amusing that the Serpukhov and ISR pp cross-sections ¹⁸⁾ (these qualitative arguments hold also for pp scattering !) can be fitted by ^{***)}

$$\sigma_{tot}(s) \approx a + b \cdot \left(\ln \frac{s}{s_1 (\ln \frac{s}{s_2})^{7/2}} \right)^2, \quad (3.6)$$

*) In principle, we could also enforce (3.4) by differentiating twice with respect to t in Eq. (2.5). This would, however, induce very large errors.

**) Unfortunately, the exponent of $\ln \ln s$ in Eq. (3.5) is in Ref. 6) wrongly stated to be 7 instead of $7/2$.

***) This is just an eye-ball fit; of course, one has to be cautious in taking such $\ln \ln s$ factors seriously. After all, they can be dropped if one instead makes the constant in front of $(\ln s)^2$ smaller than π/m_π^2 , by an arbitrarily small amount !

with $a = 35.3$ mb, $b = 3$ mb, $s_1 = 0.9$ GeV² and $s_2 = 3$ GeV². Thus, the huge scale parameters s_0 in fits ^{18),19)} to the data with $\sigma_{\text{tot}} \sim b' (\ln s/s_0)^2$ are understood. The abnormally small coefficient b' (which is typically ~ 0.6 mb) has increased by a factor of 5.

After this short digression, let us come to our main problem, i.e., finding upper bounds on an averaged total cross-section

$$\overline{\sigma}_{\text{tot}}(s_1) = \frac{1}{2} \int_{s_1}^{\infty} ds \, w(s) \left(\sigma_{\text{tot}}^{\pi^+p}(s) + \sigma_{\text{tot}}^{\pi^-p}(s) \right), \quad (3.7)$$

where $w(s)$ is some normalized weight function, and s_1 is some (energy)² larger than $(2.2 \text{ GeV})^2$. The upper end of the averaging interval has been put to infinity after we had found that this gave the best bounds, numerically.

Stated mathematically, we are looking for a maximum of $\overline{\sigma}_{\text{tot}}$ which by Eqs. (3.7) and (2.11)-(2.13) is a functional of the partial waves $f_{l\pm}^{(+)}(s)$. The partial waves are constrained by the sum rule (3.1), and by

$$0 \leq \text{Im } f_{l\pm}^{(+)}(s) \leq 1, \quad (3.8)$$

the latter being a consequence of unitarity.

The extremal solution is easily found using, e.g., the Lagrangian multiplier method ²⁰⁾. It is essentially the one well known from saturation of the Froissart limit,

$$\text{Im } f_{l\pm}^{(+)}(s) = \begin{cases} 1 & \text{for } l \leq L_{\pm}(s), \quad s > s_1 \\ 0 & \text{for } l > L_{\pm}(s), \quad s > s_1 \\ 0 & \text{for } s < s_1 \end{cases} \quad (3.9)$$

The angular momentum cut-offs $L_{\pm}(s)$ satisfy $L_+ = L_-$ or $L_+ = L_- + 1$ [due to Eq. (2.14)] and are determined by

$$\frac{q^2 A_{L_{\pm}, \pm}(s)}{w(s)} \leq \text{const} < \frac{q^2 A_{(L_{\pm}+1), \pm}(s)}{w(s)}, \quad (3.10)$$

where

$$A_{\ell,\pm}(s) = \frac{1}{(s-M^2+1)^5} \left\{ 2 \left(\frac{E_\pi}{q} \right)^2 \frac{\partial \alpha_{\ell,\pm}(s,z)}{\partial z} - \frac{3s+M^2-1}{2s} \alpha_{\ell,\pm}(s,z) \right\}_{z=1+\frac{2}{q^2}} \quad (3.11)$$

and the constant in Eq. (3.10) is determined from the sum rule (3.1), the lower limit s_0 of integration replaced by s_1 .

The numerical evaluation is straightforward. We found that the best bounds were obtained with $w(s) \sim s^{-2}$, so we finally took

$$w(s) = \frac{s_1}{s^2} \quad (3.12)$$

The results are shown in Fig. 1. To demonstrate the changes corresponding to different values of the constant C , we show there also results with $C=0.17 \times 10^{-4}$ and $C=0.5 \times 10^{-4}$. Obviously the bounds are not strongly dependent on C . Furthermore, we see a drastic improvement over the results of Ref. 5). [Notice that in the bounds of Ref. 5), the averaging intervals extend from threshold to s_1 . Using intervals starting at higher energies, these authors would get very weak bounds.]

In particular, our bounds can (for $5 \text{ GeV}^2 < s_1 < 100 \text{ GeV}^2$) very well be approximated by a parabola

$$\overline{\sigma}_{tot}(s) \lesssim a + b \left(\ln \frac{s}{s_2} \right)^2 \quad (3.13)$$

with $a \approx 55 \text{ mb}$, $b \approx 16 \text{ mb}$, and $s_2 \approx 1.84 \text{ GeV}^2$. This is of course reminiscent of the Froissart bound, except for the small value of b , which in Froissart's limit is $\pi/m_\pi^2 \approx 63 \text{ mb}$. This shows clearly that asymptopia is not reached at 100 GeV^2 , if asymptopia is defined as the region where the Froissart bound becomes visible. Another (and more appealing ?) interpretation is that our results illustrate Eq. (3.3), and asymptopia should be defined as the region where Eqs. (3.3), respectively (3.5), are relevant. Anyhow, we see that our qualitative discussion at the beginning of this section is borne out by the detailed calculation.

We conclude with the remark that analogous bounds can also be deduced for other meson-baryon cross-sections. For instance, in going over to $K^\pm_p$ scattering, one has only to replace m_π by the kaon mass. Since, however, t channel singularities (especially the coupling of the f' to $N\bar{N}$) are very poorly known in this case, the constant C_K (analogous to C) cannot be calculated as in the πN case, at present.

Just to get an impression of what might be obtained, we have drawn in Fig. 2 the upper bounds for some fixed s_1 [with $w(s) \sim s^{-2}$, as before] as functions of C_K . The arrows indicate the values of C_K which are obtained by inserting into the analogon of Eq. (3.1) the results of a Regge analysis²¹⁾. One sees that bounds of the same order of magnitude as with πN can be expected.

4. BOUNDS ON THE TOTAL INELASTIC CROSS-SECTION, AND OPTIMIZED BOUNDS INVOLVING THE ELASTIC CROSS-SECTION

From the partial waves given in Eq. (3.9), one sees that our extremal solution corresponds to the highly unrealistic case of purely elastic scattering. Thus, one can expect to get a physically more meaningful bound by not searching for a maximum of σ_{tot} , but of $\sigma_{\text{inel}} = \sigma_{\text{tot}} - \sigma_{\text{el}}$. We should remind the reader at this stage that the Froissart bound is not saturated by a black disc, but rather by a "white" one. In contrast, the bounds we are now going to discuss will be saturated by a black disc.

More definitely, we are looking for upper bounds on

$$\overline{\sigma}_{\text{inel}}(s_1) = \frac{1}{2} \int_{s_1}^{\infty} ds w(s) \left[\sigma_{\text{tot}}^{\pi^+ p}(s) + \sigma_{\text{tot}}^{\pi^- p}(s) - \sigma_{\text{el}}^{\pi^+ p}(s) - \sigma_{\text{el}}^{\pi^- p}(s) \right]. \quad (4.1)$$

The real parts of the amplitudes contribute negatively to $\overline{\sigma}_{\text{inel}}$, and using Schwarz' inequality, one finds that the maximum of $\overline{\sigma}_{\text{inel}}$ is reached when $\text{Im } F^{(-)}(s, t) = 0$. Thus, we have

$$\begin{aligned} \overline{\sigma}_{\text{inel}}(s_1) \leq & \int_{s_1}^{\infty} ds w(s) \frac{4\pi}{q^2} \sum_{\ell} \left\{ (\ell+1) \text{Im } f_{\ell+}^{(+)}(s) \cdot \right. \\ & \left. \cdot (1 - \text{Im } f_{\ell+}^{(+)}(s)) + \ell \text{Im } f_{\ell-}^{(+)}(s) \cdot (1 - \text{Im } f_{\ell-}^{(+)}(s)) \right\}. \end{aligned} \quad (4.2)$$

The partial waves are subject to exactly the same constraints as in the last section, and the maximum is easily obtained using the Lagrangian multiplier technique. As all details are very similar to the case of $\pi\pi$ scattering⁸⁾, we refer to this paper for all details. We just mention that asymptotically, the extremal solution has partial waves

$$\operatorname{Im} f_{\ell\pm}^{(+)}(s) \approx \begin{cases} 1/2 & \text{for } \ell \leq L(s) \\ 0 & \text{for } \ell > L(s) \end{cases} \quad (4.3)$$

with $L(s)$ being close to $L_{\pm}(s)$ [see Eq. (3.9)]. Thus, asymptotically, we expect the maximum of $\bar{\sigma}_{\text{inel}}$ to be $1/4$ of the maximum of $\bar{\sigma}_{\text{tot}}$. The result of the actual computation is shown in Fig. 1. From this we see that $2.6 \lesssim \bar{\sigma}_{\text{tot,max}}/\bar{\sigma}_{\text{inel,max}} \lesssim 2.8$, for $5 \text{ GeV}^2 \lesssim s_1 \lesssim 100 \text{ GeV}^2$. Thus asymptopia in the above sense is very far away.

Anyhow, we see that the upper bounds on $\bar{\sigma}_{\text{inel}}$ are (at least at low energies) already of the same order of magnitude as their experimental values, which encourages us to look for further improvements. The direction in which we should search is indicated by looking at the extremal solution. While its average behaviour is quite physical, it still has a too large ratio $\bar{\sigma}_{\text{el}}/\bar{\sigma}_{\text{tot}} \sim 0.37-0.35$, compared to the experimental ratio $\bar{\sigma}_{\text{el}}/\sigma_{\text{tot}} \sim 0.2-0.16$. Thus, further improvement in the comparison of bounds with experimental data can be obtained by using the experimental ratio as an additional constraint. This is straightforward⁸⁾. The result for Kp scattering (with $\bar{\sigma}_{\text{el}}/\bar{\sigma}_{\text{tot}} = 1/5$) is shown in Fig. 2, and in πp scattering the results would be similar.

Such a procedure has, however, the drawback that in principle we would have to show the full dependence of the bounds on $\bar{\sigma}_{\text{el}}/\bar{\sigma}_{\text{tot}}$, since otherwise a change in the experimental ratio would invalidate them. An alternative way which leads to bounds which remain correct even if the experimental ratio changes (but leads to the same improvement over the previous bounds !) consists in searching for a maximum of

$$\left(\bar{\sigma}_{\text{tot}}(s_1) / \bar{\sigma}_{\text{el}}(s_1) \right)^{\beta(s_1)} \cdot \bar{\sigma}_{\text{tot}}(s_1) , \quad (4.4)$$

with some $\beta(s_1) > 0$. The important point is that β can be adjusted in such a way that the optimal solution has any prescribed ratio of $\bar{\sigma}_{\text{el}}/\bar{\sigma}_{\text{tot}}$. If this ratio equals the experimental one, we have already satisfied the constraint. If not, our bounds are still correct (but maybe not quite optimal).

The Lagrangian equations are easily set up, and the solution for the partial waves is (we suppress the dependence on s_1)

$$\text{Im } f_{l\pm}^{(+)}(s) = \begin{cases} \frac{\beta+1}{2\beta} \frac{\bar{\sigma}_{el}}{\bar{\sigma}_{tot}} \left(1 - \lambda \frac{A_{l\pm}(s) \cdot q^2}{w(s)}\right), & \text{if this is between 0 and 1,} \\ 1 & \text{if it is } \geq 1 \\ 0 & \text{if it is } \leq 0 \end{cases} \quad (4.5)$$

The constant λ is a positive Lagrangian multiplier. First, one sees that we can get a solution only for $(\beta+1)/2\beta > 1$, that is for $0 < \beta < 1$. Secondly, one can see that asymptotically the best bound will be obtained with $\beta(s_1) \rightarrow 1$. Indeed, when $s_1 \rightarrow \infty$ we get (up to $\ln \ln s$ terms) the well-known bound ²²⁾

$$\frac{\bar{\sigma}_{tot}(s)}{\bar{\sigma}_{el}(s)} \cdot \bar{\sigma}_{tot}(s) \lesssim \frac{\pi}{m_\pi^2} \left(\ln \frac{s}{s_0}\right)^2. \quad (4.6)$$

This would be violated if the π^-p elastic cross-section continues to decrease in the way found at Serpukhov ²³⁾, i.e., like

$$\bar{\sigma}_{el}^{\pi p}(s) \approx 8.2 p_{lab}^{-0.25} \text{ mb}. \quad (4.7)$$

[p_{lab} is measured in GeV/c; we assume $\sigma_{tot}(s) \gtrsim \text{const}$ at $s \rightarrow \infty$].

The optimal β at finite energies is determined by demanding the ratio $\bar{\sigma}_{el}/\bar{\sigma}_{tot}$ of the maximal solution to be equal to the experimental one. We emphasize again that the validity of the bounds does not depend on this ratio. Using the data of Refs. 23) and 24), we find the values of β shown in Fig. 3. The upper bounds on $(\bar{\sigma}_{tot}/\bar{\sigma}_{el})^\beta \cdot \bar{\sigma}_{tot}$, together with the experimental values, are shown in Fig. 4. We see that the bounds are indeed rather tight, especially at not too high energies. At some very high energy, the extrapolation of the experimental data [obtained with Eq. (4.7)] has to violate the bound. From Fig. 4 we see that this energy is, however, extremely large, showing the limited value of asymptotic bounds when compared to data at finite energies.

Also from this figure we see that we would get a violation at the lowest energy if $C \lesssim 0.28 \times 10^{-4}$. Thus, our bounds represent strong constraints on t channel couplings, if we assume the s channel cross-sections to be known.

5. CONCLUSIONS

Using only one single number obtained from low energy phase shifts, backward amplitudes and t channel discontinuities as input, we have obtained upper bounds on energy-averaged high energy πN cross-sections. The weakest point in our analysis is that we have to use t channel amplitudes in the unphysical gap $4m_\pi^2 < t < 4M^2$ which seem, however, rather well established by now. Otherwise, our results are exact. In particular, we have carefully avoided to use any phenomenological parameter obtained from fixed t dispersion relations [as in Refs. 5) and 9)], since this would, strictly speaking, imply a logical circle in the whole argument ^{*}). Nevertheless, our results show important improvements over those of Refs. 5) and 9). The strongest bounds are obtained for $(\bar{\sigma}_{\text{tot}}/\bar{\sigma}_{\text{el}})^\beta \cdot \bar{\sigma}_{\text{tot}}$, where β is a positive function of the averaging interval as shown in Fig. 3. At the lowest energies considered (just above the phase shift region), the bounds are rather tight, while asymptotically they are even violated with present parametrizations.

As we have already shortly mentioned, our bounds are not the best ones that are in principle obtainable with our method. The main trick of this method consists in using some sum rule of the type (2.5) as a constraint. Stronger results (at least asymptotically) could be obtained by using instead of (2.5) the sum rule

$$\frac{\partial^3}{\partial t^2 \partial R} \Delta(t, R) \Big|_{\substack{t=4 \\ R=(M^2-1)^2}} = 0. \quad (5.1)$$

This involves, however, rather detailed properties of the $\pi\pi \rightarrow N\bar{N}$ amplitudes, especially the threshold behaviour at $t=4$. Thus, we believe that present estimates of these amplitudes are too uncertain to allow any meaningful evaluation of Eq. (5.1). It should, however, be done as soon as the $\pi\pi \rightarrow N\bar{N}$ amplitudes are better known.

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^{*}) We have checked, however, that our input is consistent with fixed t analyticity ¹²⁾.

A P P E N D I X

The partial wave decomposition of $F(s, t)$ in the $\pi\pi \rightarrow N\bar{N}$ channel can be written

$$F(s, t) = 4\pi \sum_J (2J+1) (p \cdot k)^J \left\{ \frac{1}{\sqrt{J(J+1)}} \left[\frac{M}{p^2} P_J'(z_t) + \frac{z_t}{M} P_J'(z_t) \right] f_J^{(+)}(t) + P_J(z_t) \hat{f}_J^{(+)}(t) \right\}. \quad (A.1)$$

The partial waves of Ref. 10) are (in units of GeV)

$$\text{Im } f_2^{(+)}(t) = 6.4 \pi \delta(t - 1.588) - 0.95 \pi \delta(t - 3.6)$$

$$\text{Im } \hat{f}_2^{(+)}(t) = 2.2 \pi \delta(t - 1.588) - 0.8 \pi \delta(t - 5.8)$$

$$\text{Im } f_4^{(+)}(t) = 0.8 \pi \delta(t - 3.8)$$

$$\text{Im } \hat{f}_4^{(+)}(t) = 0.3 \pi \delta(t - 3.8) - 0.13 \pi \delta(t - 6)$$

$$\text{Im } f_6^{(+)}(t) = 0.037 \pi \delta(t - 5.9)$$

$$\text{Im } \hat{f}_6^{(+)}(t) = 0.019 \pi \delta(t - 5.9).$$

The contribution of the f meson [to $\text{Im } f_2^{(+)}$ and $\text{Im } \hat{f}_2^{(+)}$] is by far dominating. While its coupling to $f_2^{(+)}$ is about the same as in earlier analyses ²⁵⁾, the coupling to $\hat{f}_2^{(+)}$ is larger than in most other analyses ²⁵⁾. The latter are, however, inconsistent with our sum rule.

R E F E R E N C E S

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FIGURE CAPTIONS

Figure 1 : Upper bounds on averaged values of σ_{tot} and σ_{inel} , for πp scattering. The energies shown on the abscissa are the lower ends of the averaging intervals, the upper ends being at infinity. The weight function is $\sim 1/s^2$.

———— : $C = 0.3 \times 10^{-4}$ (central value)
 - - - - : $C = 0.17 \times 10^{-4}$
 - . - . - : $C = 0.5 \times 10^{-4}$

Figure 2 : Bounds on averaged total Kp cross-sections, $\sigma_{\text{tot}}^{Kp} = \frac{1}{2}(\sigma_{\text{tot}}^{K^+p} + \sigma_{\text{tot}}^{K^-p})$, versus the sum rule constant C_K . The arrows indicate the values of C_K obtained by inserting the fit of Ref. 21) into Eq. (3.1).

Figure 3 : The optimal exponent $\beta(s_1)$ in Eq. (4.4), obtained with the data of Refs. 22) and 23). The three curves correspond to the same values of C as in Fig. 1.

Figure 4 : Upper bounds on $(\bar{\sigma}_{\text{tot}}/\bar{\sigma}_{\text{el}})^\beta \cdot \bar{\sigma}_{\text{tot}}$, compared with data from Refs. 22) and 23). The averaging procedure is the same as in Fig. 1. Also, the three curves correspond to the same values of C as in Fig. 1.

FIG.1

FIG. 2

FIG. 3

FIG.4