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LARGE MOMENTUM BEHAVIOUR AND ANALYTICITY
IN THE COUPLING CONSTANT

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ABSTRACT

We write down the general solution of the differential equation for a vertex mass insertion which is regular at zero external momenta. The large momentum behaviour of the solution and its analyticity properties in the coupling constant are discussed. Very different results are found for super-renormalizable, renormalizable and not renormalizable theories.

1. INTRODUCTION

In the framework of the Lagrangian quantum field theory some progress has been made in understanding the large momentum behaviour of Green's functions ¹⁾⁻³⁾. The most interesting results can be obtained using a technique introduced by Symanzik ^{4),5)}; a differential equation for the Green's functions is derived :

$$\mathcal{O}_p \Gamma^N(\lambda K_1 \dots \lambda K_{N-1}; m^2, g) = \left[-\lambda^2 \frac{\partial}{\partial \lambda^2} + \beta(g) \frac{\partial}{\partial g} + 2 \frac{-1}{2} N (1 + \gamma(g)) \right] \Gamma^N(\lambda K_1 \dots \lambda K_{N-1}; m^2, g) = \Delta \Gamma^N \quad (1.1)$$

and the large momentum behaviour of the solutions of this equation is studied.

The inhomogeneous term $\Delta \Gamma^N$ is negligible in the large momentum region. The leading term of Γ satisfies asymptotically the homogeneous equation. This technique can be refined ¹⁾ by introducing a Γ_{AS} function; this new function has the property that $(\Gamma_{AS} - \Gamma)/\Gamma$ goes to zero at large momenta and :

$$\mathcal{O}_p \Gamma_{AS}^N = 0 \quad (1.2)$$

It can be proved that if the function $\beta(g)$ has a zero at $g_V > g$ and both Γ_{AS} and $\Delta \Gamma$ are finite at $g = g_V$, Γ is asymptotically invariant under scale and conformal transformations for any value of the coupling constant in an open set.

In this paper we find all the solutions of the differential equation (1.1) which have the properties of being regular when the external momenta are set to zero. Γ are the Green's functions of a massive theory and Γ_{AS} are Green's functions of a zero mass theory : only Γ_{AS} may be irregular at zero momenta.

Four different cases arise in a $g\phi^4$ theory in dimension D . [The dimension of the space may be also a non-integer number ⁶⁾.]

- I) $D < 4$;
- II) $D = 4, \quad g < 0$;
- III) $D = 4, \quad g > 0$;
- IV) $D > 4$.

In the last case the theory is not renormalizable in perturbation theory and Eq. (1.1) cannot be derived. However, we think that it is worth

while to suppose that not renormalizable theories are renormalized outside perturbation theory and that Eq. (1.1) is true. We say that a theory is renormalizable if the solution of the equation of motion depends only on three constants : mass, wave function renormalization and coupling constant. At present, we do not have any sound argument which supports this assumption; the whole treatment of Eq. (1.1) in the case of renormalized theories may be academic.

We impose on the Green's function two boundary conditions : analyticity in the external momenta and matching with perturbation theory in g ; in the first two cases Γ is fixed and it can be written as a functional of $\Delta\Gamma$.

In the third case, the above conditions are not sufficient to fix the form of the function Γ ; ambiguities are present. They are of order $\exp -\frac{1}{g}$ and cannot be discovered in any perturbation expansion for Γ . However, if $\beta(g)$ has a zero at g_V , there exists only one solution of (1.1) which is finite when $g \rightarrow g_V$; one can conjecture that this solution is the right one.

In the last case we found solutions which, for small g , are of order $g^{1/(D-4)}$. In this type of theories, the standard perturbation theory is divergent. We suggest that the appearance of non C^∞ terms in the coupling constant is a genuine feature of a "not renormalizable" interaction and it is the reasoning for the non-existence of perturbative expansions.

In Section 2 we write the equation for the vertex mass insertion and we find all the solutions of the equation which are regular at zero external momenta.

In Section 3 we impose boundary conditions in the coupling constant and finally in Section 4 we study the behaviour of the Green's functions in the large momentum region.

2. THE GENERAL SOLUTION

We consider a theory with a $g\phi^4$ interaction in a D dimensional space (D may also be not integer).

The Fourier transform of the connected one-particle irreducible N point function is denoted by $\Gamma^N(K_1, \dots, K_{N-1})$. The following renormalization conditions are imposed :

$$\begin{cases} \Gamma^2(0) = m^2 \\ \left. \frac{d}{dk^2} \Gamma^2(k^2) \right|_{k^2=0} = 1 \\ \Gamma^4(0,0,0) = g \cdot m^{D-4} \end{cases} \quad (2.1)$$

The theory is renormalizable only if $D=4$, however, the coupling constant g is defined in such a way to be non-dimensional in any dimension. If $D \leq 4$, Eq. (1.1) can be rigorously derived in perturbation theory.

The functions β and γ can be computed in perturbation theory. They are :

$$\begin{aligned} \beta(g) &= \frac{D-4}{2} g + \Gamma\left(3-\frac{D}{2}\right) \frac{3}{32\pi^2} g^2 + O(g^3) \\ \gamma(g) &= O(g^2) \end{aligned} \quad (2.2)$$

The coefficient of the linear term of $\beta(g)$ is negative for super-renormalizable theories, zero for renormalizable theories and positive for not renormalizable theories. All the differences among the three types of interactions can be connected to this property of the $\beta(g)$ function.

Green's functions are analytic at zero external momenta ⁷⁾. We can safely perform i derivatives with respect to λ of Eq. (1.1) and finally put λ equal to zero. We obtain the following equations :

$$\begin{aligned} \left\{ \beta(g) \frac{\partial}{\partial g} + 2 - \frac{N}{2} [1 + \gamma(g)] - \frac{i}{2} \right\} \Gamma_i^N(K_1, \dots, K_{N-1}; m^2, g) = \\ = \Delta \Gamma_i^N(K_1, \dots, K_{N-1}; m^2, g) \end{aligned} \quad (2.3)$$

where we have the positions :

$$\begin{aligned} \Gamma_i^N(K_1, \dots, K_{N-1}; m^2, g) &= \left(\frac{\partial}{\partial \lambda} \right)^i \Gamma^N(\lambda K_1, \dots, \lambda K_{N-1}; m^2, g) \Big|_{\lambda=0} \\ \Delta \Gamma_i^N(K_1, \dots, K_{N-1}; m^2, g) &= \left(\frac{\partial}{\partial \lambda} \right)^i \Delta \Gamma^N(\lambda K_1, \dots, \lambda K_{N-1}; m^2, g) \Big|_{\lambda=0} \end{aligned} \quad (2.4)$$

Equation (2.3) is a simple linear equation in g whose solution is :

$$\begin{aligned} \Gamma_i^N(K_1 \dots K_{N-1}; m^2, g) &= \Gamma_i^N(K_1 \dots K_{N-1}; m^2, G) \times \\ &\times \exp \left\{ \frac{1}{2} (-4 + N + i) \cdot [\rho(g) - \rho(G)] \right\} \left[a(g)/a(G) \right]^{\frac{N}{2}} + \\ &+ \int_G^g d g' / \beta(g') \cdot \Delta \Gamma_i^N(K_1 \dots K_{N-1}; m^2, g') \times \\ &\times \exp \left\{ \frac{1}{2} (-4 + N + i) \cdot [\rho(g) - \rho(g')] \right\} \cdot \left[a(g)/a(g') \right]^{\frac{N}{2}} \end{aligned} \quad (2.5)$$

where G is an arbitrary value of the coupling constant and the functions ρ and a are defined as ¹⁾ :

$$\begin{aligned} \rho(g) &= \int^g \frac{d g'}{\beta(g')} \\ a(g) &= \exp \int^g \frac{d g'}{\beta(g')} \cdot \gamma(g') \end{aligned} \quad (2.6)$$

The function Γ can be reconstructed from the knowledge of all its derivatives; the solution is

$$\begin{aligned} \Gamma^N(K_1 \dots K_{N-1}; \lambda, m^2, g) &= \Gamma^N(K_1 \dots K_{N-1}; \lambda \exp \left\{ \frac{1}{2} \rho(g) - \frac{1}{2} \rho(G) \right\}, m^2, G) \times \\ &\times \exp \left\{ \left(-2 + \frac{N}{2} \right) \cdot [\rho(g) - \rho(G)] \right\} \times \left[a(g)/a(G) \right]^{\frac{N}{2}} + \\ &+ \int_G^g d g' / \beta(g') \Delta \Gamma^N(K_1 \dots K_{N-1}; \lambda \exp \left\{ \frac{1}{2} \rho(g) - \frac{1}{2} \rho(g') \right\}, m^2, g') \times \\ &\times \exp \left\{ \left(-2 + \frac{N}{2} \right) \cdot [\rho(g) - \rho(g')] \right\} \times \left[a(g)/a(g') \right]^{\frac{N}{2}} \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} \Delta \Gamma^N(K_1 \dots K_{N-1}; \lambda, m^2, g) &= \Delta \Gamma^N(\lambda K_1 \dots \lambda K_{N-1}; m^2, g) \\ \Gamma^N(K_1 \dots K_{N-1}; \lambda, m^2, g) &= \Gamma^N(\lambda K_1 \dots \lambda K_{N-1}; m^2, g) \end{aligned} \quad (2.8)$$

Performing a change of variable, Eq. (2.7) can be written as :

$$\Gamma^N(K_1, \dots, K_{N-1}; \lambda, m^2, g) = \Gamma^N(K_1, \dots, K_{N-1}; \lambda \exp\{(\rho(g) - \rho(G))/2\}, m^2, G) \times \\ \times \exp\left\{\left(2 - \frac{N}{2}\right)(\rho(g) - \rho(G))\right\} - 2 \int_1^{\frac{1}{x}} \frac{dx}{x} \exp\left\{\frac{1}{2}\rho(g) - \frac{1}{2}\rho(G)\right\} \quad (2.9)$$

$$\times \Delta \Gamma^N\left(K_1, \dots, K_{N-1}; x\lambda, m^2, \bar{g}\left(g, \frac{1}{x}\right)\right) x^{-4+N} \times \\ \times \left\{a(g)/a\left[\bar{g}\left(g, \frac{1}{x}\right)\right]\right\}^{N/2}$$

where $\bar{g}(g, \lambda)$ is defined as ¹⁾

$$\bar{g}(g, \lambda) = \rho^{-1}[\rho(g) + 2 \log \lambda] \quad (2.10)$$

It is easy to check that (2.9) is a solution of Eq. (1.1). However, it describes the general solution which is regular at external momenta. A first result is achieved. In the next section we will study the effect of imposing boundary conditions in g .

3. BOUNDARY CONDITIONS IN THE COUPLING CONSTANT

In the last section we have integrated Eq. (1.1) by imposing regularity at zero external momenta. The function Γ is fixed from the knowledge of the function $\Delta\Gamma$ at all g and of the function Γ itself at only one value G of the coupling constant.

Let us try to specialize to a particular value of G : we suppose that there exists a coupling constant g_V such that

$$\begin{aligned} \lim_{g \rightarrow g_V} \rho(g) &= +\infty \\ \lim_{g \rightarrow g_V} |\Gamma^N(k_1, \dots, k_{N-1}; m^2, g)| &< \infty \end{aligned} \quad (3.1)$$

If we take the limit $G \rightarrow g_V$ in (2.5), when $N+i \geq 3$, the first expression in the right-hand side goes to zero, and we find the simpler result :

$$\begin{aligned} \Gamma_i^N(k_1, \dots, k_{N-1}; m^2, g) &= \int_{g_V}^g \frac{dg'}{\beta(g')} \exp\left[\frac{-4+N+i}{2}(\rho(g)-\rho(g'))\right] * \\ &\quad \left[a(g)/a(g')\right]^{N/2} \Delta \Gamma_i^N(k_1, \dots, k_{N-1}; m^2, g') \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} \Gamma^N(k_1, \dots, k_{N-1}; \lambda, m^2, g) &= -2 \int_0^1 \frac{dx}{x} x^{-4+N} \\ &\quad \Delta \Gamma^N(k_1, \dots, k_{N-1}; \lambda x, m^2, \bar{g}(g, \frac{1}{x})) \left[a(g)/a(\bar{g}(g, \frac{1}{x}))\right]^{N/2} \end{aligned}$$

The value of the Γ_i^N in the case $N+i \leq 2$ is clearly fixed by the renormalization conditions (2.1). The behaviour of the function $\rho(g)$ can be studied for small coupling constant in perturbation theory. Using (2.2), one finds :

$$\begin{aligned} \exp[\rho(g)] &\propto g^{2/(D-4)} & D \neq 4 \\ \exp[\rho(g)] &\propto \exp[-A/g] & D=4; A>0 \end{aligned} \quad (3.3)$$

In super-renormalizable theories or in renormalizable ones with negative coupling constant G , g_V can be taken equal to zero. No problems arise with the second condition in (3.1); the Green functions in that limit are the same as in free field theory.

In renormalizable theories with positive coupling constant, G cannot be taken equal to zero; moreover, the coefficient of $\Gamma(G)$ in (2.5) is of order $\exp -1/g$: it is a C^∞ term but is not analytic in the coupling constant and cannot be discovered in perturbation theory. The existence of g_V cannot be decided from perturbation arguments.

Also in the $D > 4$ case, G cannot be taken equal to zero; terms proportional to $g^{(N+1)/2-2}$ appear in the right-hand side of Eq. (2.5). They are not even C^∞ in the coupling constant. The appearance of these singular terms may be the origin of the divergences in the standard perturbative approach to not renormalizable interactions. Equation (1.1) may be a useful tool to construct an improved perturbation theory in which terms, which are not analytic in the coupling constant, are computed.

4. ASYMPTOTIC BEHAVIOUR

Some assumptions must be made to study the behaviour of the solution of Eq. (1.1). These assumptions can be proved in perturbation theory, while their use outside perturbation theory is a matter of taste.

For simplicity, we shall consider only non-exceptional momentum configurations. Consistently with Weinberg's theorem, we assume that :

$$\Delta \Gamma^N(k_1, \dots, k_{N-1}; \lambda, m^2, g) \xrightarrow{\lambda \rightarrow \infty} O(\lambda^{2-N} \lg^N(\lambda))$$

$$\lim_{\lambda \rightarrow \infty} \left[\Delta \Gamma^N(k_1, \dots, k_{N-1}; \lambda, m^2, g) \right] / \Gamma^N(k_1, \dots, k_{N-1}; \lambda, m^2, g) = 0 \quad (4.1)$$

Simple power counting assures us that the second term in the right-hand side of (2.9) is asymptotically negligible. The functional form of the leading term in the large momentum region is independent of the coupling constant.

If g_V exists and if, instead of (2.9) we use (3.2), more detailed results are obtained :

$$\Gamma^N(k_1, \dots, k_{N-1}; \lambda, m^2, g) \xrightarrow{\lambda \rightarrow \infty} -2 \lambda^{4-N} [1 + \gamma(g_V)] \times$$

$$\int_0^\lambda \frac{dy}{y} \Delta \Gamma^N(k_1, \dots, k_{N-1}; y, m^2, g_V) y^{-4+N} [1 + \gamma(g_V)] \times \left[Z\left(\frac{\lambda}{y}, g\right) \right]^{N/2} \quad (4.2)$$

where the function Z is defined as :

$$Z(x, g) = \exp \int_g^{\bar{g}(g, \frac{1}{x})} \frac{dg'}{\rho(g')} [\gamma(g') - \gamma(g)] \quad (4.3)$$

and the integral is a function asymptotically smaller than any power of λ .
Let us suppose that there exists also a value g_I of the coupling constant such that

$$\begin{aligned} \lim_{g \rightarrow g_I} \rho(g) &= -\infty \\ \lim_{g \rightarrow g_I} |\Gamma^N(K_1, \dots, K_{N-1}; m^2, g)| &< \infty \end{aligned} \quad (4.4)$$

The function Γ_{AS} of Ref. 1) can simply be written as :

$$\Gamma_{AS}^N(K_1, \dots, K_{N-1}; \lambda, m^2, g) = -2 \int_0^\infty \frac{dx}{x} \left[a(g) / a(\bar{g}(g, \frac{\lambda}{x})) \right]^{\frac{N}{2}} \quad (4.5)$$

$$\Delta \Gamma^N \left(K_1, \dots, K_{N-1}; \lambda x, m^2, \bar{g} \left(g, \frac{\lambda}{x} \right) \right]$$

The function Γ_{AS} satisfies Eq. (1.2); its very existence is connected to the existence of g_I . From Eq. (3.3) g_I can be taken equal to zero for renormalizable theories with positive coupling constant or in not renormalizable theories.

In the other cases, the presence of a coupling constant satisfying Eq. (4.4) cannot be decided in perturbation theory.

R E F E R E N C E S

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