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ISOSPIN RELATIONS OF INCLUSIVE REACTIONS

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A B S T R A C T

A method of obtaining isospin equalities and inequalities of cross-sections and moments of the multiplicity distribution of inclusive reactions is presented. The final relations from which they are obtained do not involve Clebsch-Gordan coefficients. The linear equalities of the most interesting physically inclusive reactions are calculated. The inequalities are also briefly discussed.

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1. - INTRODUCTION

Recently, the inclusive reactions became very popular ¹⁾. Restrictions imposed on them by internal symmetries and particularly isospin invariance, since it is an exact symmetry, have been studied extensively ²⁾⁻¹⁴⁾ in the last two years.

In this paper we discuss a general method of obtaining equalities and also inequalities of inclusive cross-sections, which, of course, can be translated into equalities and inequalities of moments of the multiplicity distribution, by integration with respect to the momenta of the observed particles of the final state. The general relations from which the linear equalities and inequalities are derived do not contain Clebsch-Gordan coefficients. These coefficients were present originally, but they were eliminated from the final relations, because of their orthogonality properties.

In Section 2, the fundamental relations, which are basically a generalization to inclusive reactions of corresponding relations for exclusive reactions ^{15),16)}, are derived and discussed in detail. In Section 3, these relations are used to derive the linear equalities of the cross-sections of the most interesting physically inclusive reactions. Several of these relations, involving two or three non-vanishing isospins, have been calculated before ^{2)-5),7),9)} by different methods, but here this list is completed by considering several new cases. Also equalities of the cross-sections of several of the most interesting physically inclusive reactions involving four non-vanishing isospins are given. Isospin constraints on inclusive reactions involving four non-vanishing isospins have been discussed before ^{8),10)}, in the framework of a method which requires tables of Clebsch-Gordan coefficients. However, only isospin inequalities have been obtained in a model-independent way. The equalities and the inequalities derived in this paper are written in a form more compact than before. In a subsequent paper the equalities are combined with the Mueller formalism and isospin relations among the fragmentation vertices, the pionization vertices, the correlation functions, etc., are obtained.

When the number of non-vanishing isospins of the particles of the initial state and the observed particles of the final state is four or more, the calculation of the isospin equalities by the usual method, which requires tables of Clebsch-Gordan coefficients ²⁾, becomes complicated, while the present approach remains equally simple.

In Section 4, the inequalities obtained by this method are explicitly calculated in one case, to illustrate that it provides also useful inequalities in a simple way.

2. - THE GENERAL CASE

Consider an inclusive process with $n-2$ observed particles in the final state

$$\alpha_1(I_1, I_1^3) + \alpha_2(I_2, I_2^3) \rightarrow \alpha_3(I_3, I_3^3) + \dots + \alpha_n(I_n, I_n^3) + \text{anything} \quad (2.1)$$

where the particles a_i are labelled only by their isospin I_i and its third component I_i^3 . The other quantum numbers of the particles do not enter in the discussion and are omitted. The relations which we shall derive hold for arbitrary n . However, we shall prove them for $n = 4$ for simplicity, and because the generalization to arbitrary n is straightforward. Let I_x and I_x^3 be the isospins of the group of unobserved particles and their third components, respectively. In the calculation of this inclusive cross-section, the following T matrix element enters

$$g(I_1^3, I_2^3, I_3^3, I_4^3) = \sum_{\substack{I_x \\ I_x^3}} |\langle I_1, I_1^3, I_2, I_2^3 | T | I_3, I_3^3, I_4, I_4^3, I_x, I_x^3 \rangle|^2 \quad (2.2)$$

where, for simplicity, the unimportant - for our derivation - summation over a degeneration parameter $\tilde{\eta}^{(2)}$ has been omitted, and the summation over I_x^3 , which reduces to a single term, is explicitly written to comply with the notation of this section.

Since an incoherent sum over the isospins I_x of the group of unobserved particles appears in Eq. (2.2), Shmushkevich's method for exclusive reactions^{15), 16)} can be generalized to inclusive reactions. If $\underline{I}_1 + \underline{I}_2 = \underline{I}$ and $\underline{I}_3 + \underline{I}_4 = \underline{I}_{34}$, we must have $\underline{I}_{34} + \underline{I}_x = \underline{I}$ according to the isospin invariance of the T matrix. Then, Eq. (2.2) gives

$$\begin{aligned}
 g(I_1^3, I_2^3, I_3^3, I_4^3) &= \sum_{\substack{I_x \quad I \quad I_{34} \\ I_x^3 \quad I^3 \quad I_{34}^3}} \sum_{\substack{\bar{I} \quad \bar{I}_{34} \\ \bar{I}^3 \quad \bar{I}_{34}^3}} (I_1 I_2 I, I_1^3 I_2^3 I^3) (I_3 I_4 I_{34}, I_3^3 I_4^3 I_{34}^3) \cdot \\
 & (I_{34} I_x I, I_{34}^3 I_x^3 I^3) (I_1 I_2 \bar{I}, I_1^3 I_2^3 \bar{I}^3) (I_3 I_4 \bar{I}_{34}, I_3^3 I_4^3 \bar{I}_{34}^3) \cdot \\
 & (\bar{I}_{34} I_x \bar{I}, \bar{I}_{34}^3 I_x^3 \bar{I}^3) T(I, I_{34}, I_x) T^*(\bar{I}, \bar{I}_{34}, I_x)
 \end{aligned} \tag{2.3}$$

where $(I_i I_j I_k, I_i^3 I_j^3 I_k^3)$ is a Clebsch-Gordan coefficient, and $T(I, I_{34}, I_x)$ is independent of the third components of all isospins involved. The Clebsch-Gordan coefficients obey the orthogonality relations ¹⁷⁾

$$\sum_{I_i^3, I_j^3} (I_i I_j I, I_i^3 I_j^3 I^3) (I_i I_j \bar{I}, I_i^3 I_j^3 \bar{I}^3) = \delta_{I, \bar{I}} \delta_{I^3, \bar{I}^3} \tag{2.4}$$

$$\sum_{I_j^3, I^3} (I_i I_j I, I_i^3 I_j^3 I^3) (I_k I_j I, I_k^3 I_j^3 I^3) = \frac{2I+1}{2I_i+1} \delta_{I_i, I_k} \delta_{I_i^3, I_k^3} \tag{2.5}$$

From Eqs. (2.3)-(2.5), we get

$$\sum_{I_2^3, I_3^3, I_4^3} g(I_1^3, I_2^3, I_3^3, I_4^3) \equiv g_1(I_1^3) = \sum_{I_x, I, I_{34}} \frac{2I+1}{2I_1+1} |T(I_x, I, I_{34})|^2 \tag{2.6}$$

i.e., $g_1(I_1^3)$ is independent of I_1^3 . The same is true for the functions $g_i(I_i^3)$ defined similarly. Therefore, we get

$$g_i(I_i) = g_i(I_i-1) = \dots = g_i(-I_i) \tag{2.7}$$

Summing $g(I_1^3, I_2^3, I_3^3, I_4^3)$ over $I_1^3, \dots, I_4^3$ with the constraint

$$I_3^3 + I_4^3 \equiv I_{34}^3 : \text{ fixed, or } I_1^3 + I_2^3 \equiv I_{12}^3 : \text{ fixed} \tag{2.8}$$

and using Eqs. (2.4) and (2.5), we get respectively

$$\sum_{\substack{I_1^3, I_2^3, I_3^3, I_4^3 \\ I_3^3 + I_4^3 = I_{34}^3}} g(I_1^3, I_2^3, I_3^3, I_4^3) \equiv g_{34}(I_{34}^3) = \sum_{I_x, I, I_{34} \geq |I_{34}^3|} \frac{2I+1}{2I_{34}+1} |T(I_x, I, I_{34})|^2 \quad (2.9)$$

$$\sum_{\substack{I_1^3, I_2^3, I_3^3, I_4^3 \\ I_1^3 + I_2^3 = I_{12}^3}} g(I_1^3, I_2^3, I_3^3, I_4^3) \equiv g_{12}(I_{12}^3) = \sum_{I_x, I_{34}, I \geq |I_{12}^3|} |T(I_x, I, I_{34})|^2 \quad (2.10)$$

In the above equations, both isospins, whose sum of the third components is fixed, belong to particles of the initial state, or to particles of the final state. A similar equation can be obtained if we sum over the third components of all isospins with the constraint that the difference of the third components of the isospin of a particle in the initial state and a particle in the final state is held fixed. Suppose, for example, that we sum over the third components of all isospins with the constraint

$$I_1^3 - I_4^3 \equiv I_{14}^3 : \text{fixed} \quad (2.11)$$

For convenience, we couple the isospins I_3, I_3^3 and I_x, I_x^3 to get I_{3x}, I_{3x}^3 and then the isospins I_{3x}, I_{3x}^3 and I_4, I_4^3 to get I, I^3 . Therefore, we obtain from (2.2)

$$\sum_{\substack{I_1^3, I_2^3, I_3^3, I_4^3 \\ I_1^3 - I_4^3 = I_{14}^3}} g(I_1^3, I_2^3, I_3^3, I_4^3) \equiv g_{14}(I_{14}^3) = \sum_{\substack{I_1^3, I_2^3, I_3^3, I_4^3 \\ I_1^3 - I_4^3 = I_{14}^3}} \sum_{\substack{I_x, I, I_{3x}, \bar{I}, \bar{I}_{3x} \\ I_x^3, I^3, I_{3x}^3, \bar{I}^3, \bar{I}_{3x}^3}} (I_1 I_2 I, I_1^3 I_2^3 I^3) \cdot$$

$$(I_3 I_x I_{3x}, I_3^3 I_x^3 I_{3x}^3) (I_{3x} I_4 I, I_{3x}^3 I_4^3 I^3) (I_1 I_2 \bar{I}, I_1^3 I_2^3 \bar{I}^3) (I_3 I_x \bar{I}_{3x}, I_3^3 I_x^3 \bar{I}_{3x}^3) \cdot$$

$$(\bar{I}_{3x} I_4 \bar{I}, \bar{I}_{3x}^3 I_4^3 \bar{I}^3) T(I, I_{3x}, I_x) T^*(\bar{I}, \bar{I}_{3x}, I_x) =$$

$$(2.12)$$

$$\begin{aligned}
 &= \sum_{\substack{I_1^3, I_2^3, I_4^3 \\ I_1^3 - I_4^3 = I_{14}^3}} \sum_{\substack{I_X, I, I_{3X}, \bar{I} \\ I_{3X}^3}} (I_1 I_2 I, I_1^3 I_2^3, I_1^3 + I_2^3) (I_{3X} I_4 I, I_{3X}^3 I_4^3, I_1^3 + I_2^3) (I_1 I_2 \bar{I}, I_1^3 I_2^3, I_1^3 + I_2^3) \cdot \\
 &\cdot (I_{3X} I_4 \bar{I}, I_{3X}^3 I_4^3, I_1^3 + I_2^3) T(I, I_{3X}, I_X) T^*(\bar{I}, I_{3X}, I_X)
 \end{aligned} \tag{2.12}$$

where, to get the last equality, the sum with respect to I_{3X}^3 and I_X^3 was performed. Also, for simplicity, the sum with respect to I^3 and $\bar{I}^3$ was performed.

From the definition of the Racah W coefficients for the coupling schemes

$$\begin{aligned}
 (I_1 + I_2) + I_{3X} &\rightarrow I + I_{3X} \rightarrow I_4 \\
 I_1 + (I_2 + I_{3X}) &\rightarrow I_1 + K \rightarrow I_4
 \end{aligned} \tag{2.13}$$

we get ¹⁷⁾

$$\begin{aligned}
 (I_1 I_2 I, I_1^3 I_2^3 I^3) (I I_{3X} I_4, I^3 I_{3X}^3 I_4^3) &= \sum_K \left[(2I+1)(2K+1) \right]^{\frac{1}{2}} (I_2 I_{3X} K, I_2^3 I_{3X}^3 K^3) \cdot \\
 (I_1 K I_4, I_1^3 K^3 I_4^3) W(I_1 I_2 I_4 I_{3X}, I K)
 \end{aligned} \tag{2.14}$$

Using the relation ¹⁷⁾

$$(I I_{3X} I_4, I^3 I_{3X}^3 I_4^3) = (-1)^{I_{3X} + I_{3X}^3} \left(\frac{2I_4 + 1}{2I + 1} \right)^{\frac{1}{2}} (I_{3X} I_4 I, -I_{3X}^3 I_4^3 I^3) \tag{2.15}$$

Eq. (2.14) becomes

$$(I_1 I_2 I, I_1^3 I_2^3 I^3) (I_{3X} I_4 I, -I_{3X}^3 I_4^3 I^3) = (-1)^{I_{3X} + I_{3X}^3} \sum_K (2I+1) \left(\frac{2K+1}{2I_4+1} \right)^{\frac{1}{2}} \cdot$$

$$(I_2 I_{3X} K, I_2^3 I_{3X}^3 K^3) (I_1 K I_4, I_1^3 K^3 I_4^3) W(I_1 I_2 I_4 I_{3X}, I K) \tag{2.16}$$

Substituting Eq. (2.16) and a similar one with I replaced by $\bar{I}$ in (2.12) and performing the summation with respect to I_2^3 and I_{3x}^3 , we get

$$g_{14}(I_{14}^3) = \sum_{\substack{I_1^3, I_4^3 \\ I_1^3 - I_4^3 = I_{14}^3}} \sum_{I_x, I, I_{3x}, \bar{I}, K} \frac{(2I+1)(2\bar{I}+1)(2K+1)}{2I_4+1} (I_1 K I_4, I_1^3 K^3 I_4^3)^2$$

$$W(I_1 I_2 I_4 I_{3x}, I K) W(I_1 I_2 I_4 I_{3x}, \bar{I} K) T(I, I_{3x}, I_x) \cdot T^*(\bar{I}, I_{3x}, I_x) \quad (2.17)$$

The presence of the Clebsch-Gordan coefficient $(I_1 K I_4; I_1^3 K^3 I_4^3)$ in the above expression implies that

$$K \geq |K^3| = |I_{14}^3| \quad (2.18)$$

Using Eq. (2.5) the summation with respect to I_1^3 and I_4^3 with I_{14}^3 fixed can be easily performed giving

$$g_{14}(I_{14}^3) = \sum_{I_x, I_{3x}, K \geq |I_{14}^3|} \left| \sum_I (2I+1) W(I_1 I_2 I_4 I_{3x}, I K) T(I, I_{3x}, I_x) \right|^2 \quad (2.19)$$

Since the Racah coefficients do not depend on the third components of the isospins involved, the above function g_{14} depends on the third components of the isospins of the external particles only through I_{14}^3 . The case of a constraint like the one of Eq. (2.11) was not considered explicitly in Ref. 16).

From Eqs. (2.9), (2.10) and (2.19), one easily obtains inequalities and in some cases equalities for the functions $g_{ij}(I_{ij}^3)$. Consider for example Eq. (2.19). We have $K = I_1 - I_4 = I_3 + I_x - I_2$, and since the process is inclusive, we get

$$K_{\min} = |I_1 - I_4| \quad (2.20)$$

$$K_{\max} = I_1 + I_4 \quad (2.21)$$

From Eq. (2.19) we easily see that the contribution of each K to g_{14} is positive definite, and also that if $|I_{14}^3| \leq K_{\min}$ the expression (2.19) gets contributions from all possible values of K . Therefore,

$$\begin{aligned} g_{14}(K_{\max}) &\leq g_{14}(K_{\max}-1) \leq \dots \leq g_{14}(K_{\min}) = \\ &= g_{14}(K_{\min}-1) = \dots = g_{14}\left(0 \text{ or } \frac{1}{2}\right) \end{aligned} \quad (2.22)$$

Similar expressions can be derived from Eqs. (2.9) or (2.10). The above results are easily generalized to inclusive reactions whose final states contain an arbitrary number of observed particles with non-vanishing isospins.

The $(n-2)$ particle inclusive cross-sections

$$\sigma(I_1^3, I_2^3, I_3^3, \dots, I_n^3) \equiv \frac{d\sigma(\alpha_1 + \alpha_2 \rightarrow \alpha_3 + \alpha_4 + \dots + \alpha_n + \text{any thing})}{dp_3 dp_4 \dots dp_n} \quad (2.23)$$

where $dp_i = (d^3p/2E_i)$, and to simplify the notation, the functions σ are labelled only by the third components of the isospins, are expressed linearly in terms of the functions $g(I_1^3, I_2^3, I_3^3, \dots, I_n^3)$. Therefore the functions g and σ satisfy the same equalities and inequalities. We summarize our previous results for the functions σ . We have

$$\sum_{I_1^3, \dots, I_K^3, I_{K+2}^3, \dots, I_n^3} \sigma(I_1^3, \dots, I_K^3, I_{K+1}^3, I_{K+2}^3, \dots, I_n^3) = \sum_{I_1^3, \dots, I_K^3, I_{K+2}^3, \dots, I_n^3} \sigma(I_1^3, \dots, I_K^3, I_{K+1}'^3, I_{K+2}^3, \dots, I_n^3) \quad (2.24)$$

where I_{k+1}^3 and $I_{k+1}'^3$ are two different values of the third component of the isospin of particle $k+1$. Equation (2.24) gives a relation beyond charge independence only if $I_{k+1} \geq 1$. Also we have

$$\begin{aligned} \sum_{\substack{I_1^3, \dots, I_n^3 \\ \{I_i^3\} = I_{\max}}} \sigma(I_1^3, \dots, I_n^3) &\leq \sum_{\substack{I_1^3, \dots, I_n^3 \\ \{I_i^3\} = I_{\max}-1}} \sigma(I_1^3, \dots, I_n^3) \leq \dots \leq \sum_{\substack{I_1^3, \dots, I_n^3 \\ \{I_i^3\} = I_{\min}}} \sigma(I_1^3, \dots, I_n^3) = \\ \sum_{\substack{I_1^3, \dots, I_n^3 \\ \{I_i^3\} = I_{\min}-1}} \sigma(I_1^3, \dots, I_n^3) &= \dots = \sum_{\substack{I_1^3, \dots, I_n^3 \\ \{I_i^3\} = 0 \text{ or } \frac{1}{2}}} \sigma(I_1^3, \dots, I_n^3) \end{aligned} \quad (2.25)$$

The symbol $\{I_i^3\}$ of relations (2.25) is defined as follows. From the isospins $I_1, I_2, \dots, I_n$, a subset of r isospins, which are labelled by $\underline{I}_i$, is selected. Then

$$\{I_i^3\} = \sum_i \epsilon_i I_i^3 \quad (2.26)$$

where the sum runs over the elements of the above chosen subset, and ϵ_i is 1 (or -1) if the isospin $\underline{I}_i$ belongs to a particle of the initial (or the final) state. The $I_{\max}$ and $I_{\min}$ appearing in Eq. (2.25) are given by

$$I_{\min} = \min \left(\sum_i \underline{I}_i \right) \quad (2.27)$$

$$I_{\max} = \sum_i I_i \quad (2.28)$$

Relations (2.25) are valid for arbitrary r , and for given r they hold for any choice of the isospins of the subset. Equalities are obtained from them only if $I_{\min} \geq 1$.

We observe the following.

- a) In Eqs. (2.24) and (2.25), which give identical relations for inclusive cross-sections of particles with the same isospins but arbitrary otherwise, Clebsch-Gordan coefficients do not appear.
- b) The equalities and inequalities of inclusive cross-sections obtained from (2.24) and (2.25) can be integrated with respect to the momenta of the observed particles of the final state, to give relations of the moments of the multiplicity distribution.
- c) In Eq. (2.24), the isospin $\underline{I}_k$ may belong to a particle of the initial state or of the final state. Also the isospins of the subset $\{\underline{I}_i\}$ of Eq. (2.25) may belong to particles of the initial state or to particles of the final state or some may belong to particles of the initial state and some to particles of the final state.

- d) In Eqs. (2.24) and (2.25) some of the isospins may be zero, in which case their third components may be omitted from the σ functions.
- e) If in a relation derived from Eqs. (2.24) or (2.25) for the processes (2.1) we make the replacement

$$I_2^3 \rightarrow -I_\ell^3, \quad I_\ell^3 \rightarrow -I_2^3 \quad (2.29)$$

where a_ℓ is a particle of the final state, we get the corresponding relation of the processes

$$\begin{aligned} \alpha_1(I_1, I_1^3) + \alpha'_\ell(I_\ell, -I_\ell^3) &\rightarrow \alpha_3(I_3, I_3^3) + \dots + \alpha_{\ell-1}(I_{\ell-1}, I_{\ell-1}^3) + \\ + \alpha'_2(I_2, -I_2^3) + \alpha_{\ell+1}(I_{\ell+1}, I_{\ell+1}^3) + \dots + \alpha_n(I_n, I_n^3) + \text{anything} \end{aligned} \quad (2.30)$$

where $a'_i(I_i, -I_i^3)$, $i=2, \ell$ can be the particle $a_i(I_i, -I_i^3)$ or the anti-particle of $a_i(I_i, I_i^3)$. This is obtained by applying Eqs. (2.24) and (2.25) to processes (2.30).

- f) If in a relation derived from Eqs. (2.24) or (2.25), we make the replacement

$$\sigma(I_1^3, I_2^3, \dots, I_n^3) \rightarrow \sum_{I_{n+1}^3} \sigma(I_1^3, I_2^3, \dots, I_n^3, I_{n+1}^3) \quad (2.31)$$

we get a corresponding relation of the processes

$$\begin{aligned} \alpha_1(I_1, I_1^3) + \alpha_2(I_2, I_2^3) &\rightarrow \alpha_3(I_3, I_3^3) + \dots + \alpha_n(I_n, I_n^3) + \\ + \alpha_{n+1}(I_{n+1}, I_{n+1}^3) + \text{anything} \end{aligned} \quad (2.32)$$

This is obtained by applying Eqs. (2.24) and (2.25) to processes (2.32).

- g) In the calculation of the inclusive cross-section for the process (2.1) the following expression appears

$$\sum_{I_x} |T_1 + T_2 + \dots + T_{p(I_x)}|^2 \quad (2.33)$$

where $T_1, \dots, T_{\rho(I_x)}$ are the linearly independent T matrix elements corresponding to the isospin I_x of the unobserved particles, multiplied by some Clebsch-Gordan coefficients. Certain summations in (2.33) have been omitted for simplicity. Since each T_i is characterized by a modulus and a phase, and for each value of I_x one of the phases may be chosen arbitrarily, the number of linearly independent terms in the summation (2.33) is

$$\sum_{I_x} [2\rho(I_x) - 1] \quad (2.34)$$

If process (2.1) is characterized by R independent reactions, i.e., not related by isospin symmetry, then the number of equalities relating these reactions is

$$R - \sum_{I_x} [2\rho(I_x) - 1] \quad (2.35)$$

where $\rho(I_x)$ is the number of times the isospin I_x appears in the product

$$I_1 \times I_2 \times \dots \times I_n \quad (2.36)$$

If the expression (2.35) is zero or negative, no equality exists. This number of equalities of the most common inclusive reactions is given in Table I.

h) If for some of the isospins I_x

$$\rho(I_x) \geq 3 \quad (2.37)$$

some of the equalities are non-linear in the cross-sections. Such equalities are not given by Eqs. (2.24) and (2.25). The physical reactions with the lowest isospins in which this phenomenon occurs, are those involving the isospins 1, 1, 1. In Table V we give explicitly this non-linear relation.

3. - ISOSPIN EQUALITIES OF THE MOST INTERESTING PHYSICALLY INCLUSIVE REACTIONS

In this section, Eqs. (2.24) and (2.25) will be used to derive isospin equalities of the cross-sections of the most interesting physically inclusive reactions. In the notation of Eq. (2.23), the expression $\sigma(I_1^3, I_2^3, I_3^3, \dots, I_n^3)$ represents the cross-section of the process (2.1), i.e., the first two labels of σ are the third isospin components of the particles of the initial state, and the rest are the third isospin components of the particles of the final state. In this notation $\sigma(I_1^3, -I_\ell^3, I_3^3, \dots, I_{\ell-1}^3, -I_2^3, I_{\ell+1}^3, \dots, I_n^3)$ represents the cross-section of the process (2.30). As we argued before, given an isospin relation of the process (2.1), a corresponding isospin relation of the process (2.30), in our notation, can be obtained from it by making everywhere the replacement

$$\sigma(I_1^3, I_2^3, I_3^3, \dots, I_\ell^3, \dots, I_n^3) \rightarrow \sigma(I_1^3, -I_\ell^3, I_3^3, \dots, I_{\ell-1}^3, -I_2^3, I_{\ell+1}^3, \dots, I_n^3) \quad (3.1)$$

It should be pointed out that all terms of the relation which is obtained by the replacement (3.1) refer to processes (2.30). Since the isospin equalities are unaffected by the presence of isosinglets in the final state, such particles will not be shown explicitly, or at most one of them will be shown when it is necessary. Of course, a particle of the initial state may be replaced by an isosinglet of the final state. In several occasions, the relation

$$\sigma(I_1^3, I_2^3, \dots, I_n^3) = \sigma(-I_1^3, -I_2^3, \dots, -I_n^3) \quad (3.2)$$

which follows from isospin invariance will be employed to write the equalities in a form more suitable physically.

We shall consider cases involving several combinations of non-vanishing isospins of particles of the initial state and observed particles of the final state. The list of equalities is complete in all cases except the cases of isospins $(1, 1, \frac{3}{2})$ and $(\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2})$ in which one non-linear equality is not given and the case of isospins $(\frac{1}{2}, 1, 1, 1)$ in which two such equalities are not given.

1) Isospins involved : $\frac{1}{2}, 1$. Number of equalities : 1

From Eqs. (2.24) and (3.2) we obtain

$$\sigma\left(\frac{1}{2}, 1\right) + \sigma\left(\frac{1}{2}, -1\right) = 2\sigma\left(\frac{1}{2}, \underline{0}\right) \quad (3.3)$$

A bold-faced zero is used throughout the paper to indicate the zero value of the third component I^3 of the isospin $I = 1$, while the number 0 is used to indicate a vanishing isospin I . The above equality refers to the processes $\frac{1}{2} + 1 \rightarrow 0 + X$, and its explicit form is

$$\sigma\left(\frac{1}{2}, 1 \rightarrow 0, X\right) + \sigma\left(\frac{1}{2}, -1 \rightarrow 0, X\right) = 2\sigma\left(\frac{1}{2}, \underline{0} \rightarrow 0, X\right) \quad (3.4)$$

The corresponding equalities for the processes $\frac{1}{2} + 0 \rightarrow 1 + X$, $1 + 0 \rightarrow \frac{1}{2} + X$, $0 + 0 \rightarrow \frac{1}{2} + 1 + X$, etc., can be easily derived from it by the replacement (3.1). For example, for the processes $\frac{1}{2} + 0 \rightarrow 1 + X$ it gives

$$\sigma\left(\frac{1}{2}, 0 \rightarrow -1, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow 1, X\right) = 2\sigma\left(\frac{1}{2}, 0 \rightarrow \underline{0}, X\right) \quad (3.5)$$

We shall consider reactions whose initial states have isospins 0, $\frac{1}{2}$, 1 and $\frac{3}{2}$, and final states with one or two observed particles having again isospins 0, $\frac{1}{2}$, 1 and $\frac{3}{2}$. In Table II we give some examples for each of these states. Of course, for reactions of the type $I_1 + I_2 \rightarrow I_3 + X$ and $I_1 + I_2 \rightarrow I_3 + I_4 + X$ any state $[I_1, I_2]$ of the Table can be combined with any state $[I_3, X]$ or $[I_3, I_4, X]$ in writing explicitly the isospin relations. For example, in relation (3.5) the initial state may be any of the states $[\frac{1}{2}, 0]$, say Nd, and the final state any of the states $[1, X]$, say πX , in which case it becomes

$$\sigma(p d \rightarrow \pi^- X) + \sigma(p d \rightarrow \pi^+ X) = 2\sigma(p d \rightarrow \pi^0 X) \quad (3.6)$$

In Table II the Pomeron and also unstable particles appear in the initial state. We shall consider such possibilities because the isospin relations are also useful when some of the particles involved are exchanged particles in another process, having the inclusive reaction as a subprocess. In this sense, the $I = 1$ object of Table II may also be identified with the weak current, for which the assumption is made that it has isospin $I = 1$.

2) Isospins involved : 1,1. Number of equalities : 2

Equations (2.24) and (3.2) give

$$\sigma(1,1) + \sigma(1,0) + \sigma(1,-1) = 2\sigma(0,1) + \sigma(0,0) \quad (3.7)$$

$$\sigma(1,1) + \sigma(0,1) + \sigma(1,-1) = 2\sigma(1,0) + \sigma(0,0) \quad (3.8)$$

from which we obtain the simpler relations

$$\sigma(1,1) + \sigma(1,-1) = \sigma(0,1) + \sigma(0,0) \quad (3.9)$$

$$\sigma(1,0) = \sigma(0,1) \quad (3.10)$$

The above equalities apply to the processes $1+1 \rightarrow 0+X$, $1+0 \rightarrow 1+X$, $0+0 \rightarrow 1+1+X$, etc.

3) Isospins involved : $\frac{1}{2}, \frac{3}{2}$. Number of equalities : 2

In the present case we have $I_{\min} = 1$, and therefore an equality is obtained from (2.25). From (2.24) and (2.25) we get, respectively,

$$\sigma\left(\frac{1}{2}, \frac{3}{2}\right) + \sigma\left(-\frac{1}{2}, \frac{3}{2}\right) = \sigma\left(\frac{1}{2}, \frac{1}{2}\right) + \sigma\left(-\frac{1}{2}, \frac{1}{2}\right) \quad (3.11)$$

$$\sigma\left(-\frac{1}{2}, \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}\right) = \sigma\left(\frac{1}{2}, -\frac{1}{2}\right) + \sigma\left(-\frac{1}{2}, \frac{1}{2}\right) \quad (3.12)$$

from which we obtain the simpler expressions

$$\sigma\left(\frac{1}{2}, \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}\right) = 2\sigma\left(\frac{1}{2}, \frac{1}{2}\right) \quad (3.13)$$

$$\sigma\left(\frac{1}{2}, -\frac{3}{2}\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}\right) = 2\sigma\left(\frac{1}{2}, -\frac{1}{2}\right) \quad (3.14)$$

The $I_3 = \frac{1}{2}$ in Eqs. (3.11)-(3.14) may come from an $I = \frac{1}{2}$ or $\frac{3}{2}$. To avoid confusion, a prime is added to all $I_3 = \frac{1}{2}$ coming from an $I = \frac{3}{2}$. Equations (3.11)-(3.14) apply to processes of the type $\frac{1}{2} + 0 \rightarrow \frac{3}{2} + X$, $\frac{1}{2} + \frac{3}{2} \rightarrow 0 + X$, $\frac{3}{2} + 0 \rightarrow \frac{1}{2} + X$, $0 + 0 \rightarrow \frac{1}{2} + \frac{3}{2} + X$, etc.

4) Isospins involved : $1, \frac{3}{2}$. Number of equalities : 3

The equalities are the following :

$$\sigma\left(1, \frac{3}{2}\right) + \sigma\left(1, \frac{1}{2}\right) + \sigma\left(1, -\frac{1}{2}\right) + \sigma\left(1, -\frac{3}{2}\right) = 2\sigma\left(0, \frac{3}{2}\right) + 2\sigma\left(0, \frac{1}{2}\right) \quad (3.15)$$

$$\sigma\left(1, \frac{3}{2}\right) + \sigma\left(0, \frac{3}{2}\right) + \sigma\left(-1, \frac{3}{2}\right) = \sigma\left(1, \frac{1}{2}\right) + \sigma\left(0, \frac{1}{2}\right) + \sigma\left(1, -\frac{1}{2}\right) \quad (3.16)$$

$$\sigma\left(1, \frac{3}{2}\right) + 3\sigma\left(1, -\frac{1}{2}\right) = \sigma\left(1, -\frac{3}{2}\right) + 3\sigma\left(1, \frac{1}{2}\right) \quad (3.17)$$

The first two were obtained from Eqs. (2.24) and (3.2), and the last one from a corresponding equality of the processes involving the isospins $\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ as indicated below.

5) Isospins involved : $\frac{1}{2}, \frac{1}{2}, 1$. Number of equalities : 1

From Eq. (2.24), we get

$$\begin{aligned} \sigma\left(\frac{1}{2}, \frac{1}{2}, 1\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, 1\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}, -1\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, -1\right) &= 2\sigma\left(\frac{1}{2}, \frac{1}{2}, 0\right) + \\ &+ 2\sigma\left(\frac{1}{2}, -\frac{1}{2}, 0\right) \end{aligned} \quad (3.18)$$

In Table III, we give explicitly the equalities obtained from (3.18) for the processes $\frac{1}{2} + \frac{1}{2} \rightarrow 1 + X$, $\frac{1}{2} + 1 \rightarrow \frac{1}{2} + X$, $\frac{1}{2} + 0 \rightarrow \frac{1}{2} + 1 + X$ and $1 + 0 \rightarrow \frac{1}{2} + \frac{1}{2} + X$, because of their physical interest. Examples of such processes are given in Table II.

6) Isospins involved : $\frac{1}{2}, 1, 1$. Number of equalities : 2

Both equalities of this case are obtained from Eqs. (2.24). They can be replaced by the following simpler expressions

$$\begin{aligned} \sigma\left(\frac{1}{2}, 1, 1\right) + \sigma\left(\frac{1}{2}, 1, -1\right) + \sigma\left(\frac{1}{2}, -1, 1\right) + \sigma\left(\frac{1}{2}, -1, -1\right) &= \sigma\left(\frac{1}{2}, 0, 1\right) + \\ &+ \sigma\left(\frac{1}{2}, 0, -1\right) + 2\sigma\left(\frac{1}{2}, 0, 0\right) \end{aligned} \quad (3.19)$$

$$\sigma\left(\frac{1}{2}, 1, \underline{0}\right) + \sigma\left(\frac{1}{2}, -1, \underline{0}\right) = \sigma\left(\frac{1}{2}, \underline{0}, 1\right) + \sigma\left(\frac{1}{2}, \underline{0}, -1\right) \quad (3.20)$$

In Table IV, we write explicitly the equalities obtained from (3.19) and (3.20) for the processes $\frac{1}{2}+1 \rightarrow 1+X$, $\frac{1}{2}+0 \rightarrow 1+1+X$, $1+0 \rightarrow \frac{1}{2}+1+X$ and $1+1 \rightarrow \frac{1}{2}+X$. Examples of such processes are given in Table II.

7) Isospins involved : 1, 1, 1. Number of equalities : 4

Three equalities are obtained from Eqs. (2.24). They are long but they can be replaced by the following simpler expressions

$$\begin{aligned} \sigma(1, 1, 1) + \sigma(1, 1, -1) + \sigma(1, -1, 1) + \sigma(1, -1, -1) = & \sigma(1, \underline{0}, \underline{0}) + \\ & + \sigma(\underline{0}, 1, \underline{0}) + \sigma(\underline{0}, \underline{0}, 1) + \sigma(\underline{0}, \underline{0}, \underline{0}) \end{aligned} \quad (3.21)$$

$$\sigma(1, \underline{0}, 1) + \sigma(1, \underline{0}, -1) + \sigma(1, \underline{0}, \underline{0}) = \sigma(\underline{0}, 1, 1) + \sigma(\underline{0}, 1, -1) + \sigma(\underline{0}, 1, \underline{0}) \quad (3.22)$$

$$\sigma(1, 1, \underline{0}) + \sigma(1, -1, \underline{0}) + \sigma(1, \underline{0}, \underline{0}) = \sigma(\underline{0}, 1, 1) + \sigma(\underline{0}, 1, -1) + \sigma(\underline{0}, \underline{0}, 1) \quad (3.23)$$

In addition, there is one more equality non-linear in the cross-sections, which, of course, cannot be obtained from Eqs. (2.24) and (2.25). This complicated equality is given in Table V, as an example of a non-linear relation. It was obtained by the usual procedure ²⁾ which requires tables of Clebsch-Gordan coefficients.

As the isospins of the particles increase and also as the number of particles with non-vanishing isospins increase, the inequalities derived from isospin invariance involve more and more terms. To write these relations in a more compact form, we introduce the notation

$$\begin{aligned} \sigma(I_1^3, I_2^3, \dots, I_{i-1}^3, \pm I_i^3, I_{i+1}^3, \dots, I_n^3) \equiv & \sigma(I_1^3, I_2^3, \dots, I_{i-1}^3, I_i^3, I_{i+1}^3, \dots, I_n^3) + \\ & + \sigma(I_1^3, I_2^3, \dots, I_{i-1}^3, -I_i^3, I_{i+1}^3, \dots, I_n^3) \end{aligned} \quad (3.24)$$

8) Isospins involved : $\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$. Number of equalities : 3

The equalities are the following

$$\sigma\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, \pm \frac{3}{2}\right) = \sigma\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, \pm \frac{1}{2}\right) \quad (3.25)$$

$$\sigma\left(\frac{1}{2}, \pm \frac{1}{2}, -\frac{1}{2}\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, \frac{3}{2}\right) = 2\sigma\left(\frac{1}{2}, \pm \frac{1}{2}, \frac{1}{2}\right) \quad (3.26)$$

$$\sigma\left(\pm \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) + \sigma\left(\pm \frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right) = 2\sigma\left(\pm \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \quad (3.27)$$

where the notation of Eq. (3.24) was used for simplicity. The first of the above equalities was obtained from Eqs. (2.24) and (3.2) and the other two from Eqs. (2.25) and (3.2). The meaning of $I_3 = \frac{1}{2}$ is explained after Eq. (3.14). Two of the above equalities can be replaced by

$$3\sigma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right) = 3\sigma\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right) \quad (3.28)$$

$$3\sigma\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right) = 3\sigma\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}\right) \quad (3.29)$$

In all terms of Eq. (3.28) the two isospin $\frac{1}{2}$ particles are in a pure isospin 1 state, and may be considered as an isospin 1 particle. Thus, Eq. (3.28) gives also an equality for the case involving the isospins 1, $\frac{3}{2}$. Equality (3.17) was obtained in this way.

9) Isospins involved : $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1$. Number of equalities : 1

From Eq. (2.24) we obtain

$$\begin{aligned} \sigma\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}, 1\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, \pm \frac{1}{2}, 1\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}, -1\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, \pm \frac{1}{2}, -1\right) = \\ = 2\sigma\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}, 0\right) + 2\sigma\left(\frac{1}{2}, -\frac{1}{2}, \pm \frac{1}{2}, 0\right) \end{aligned} \quad (3.30)$$

where for simplicity the notation of Eq. (3.24) was used. In Table VI, we give explicitly the equalities obtained from the above expression for the processes $\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + 1 + X$ and $\frac{1}{2} + 1 \rightarrow \frac{1}{2} + \frac{1}{2} + X$, because of their physical importance, and in Table II we give some examples of such processes.

10) Isospins involved : $\frac{1}{2}, \frac{1}{2}, 1, 1$. Number of equalities : 2

The two equalities which are obtained from Eq. (2.24) can be replaced by the following simpler expressions

$$\begin{aligned} \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 1, 1\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, -1, -1\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 1, -1\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, -1, 1\right) = \\ = \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 0, 1\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 0, -1\right) + 2\sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 0, 0\right) \end{aligned} \quad (3.31)$$

$$\sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 1, 0\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, -1, 0\right) = \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 0, 1\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 0, -1\right) \quad (3.32)$$

where for simplicity the notation of Eq. (3.24) was used. In Table VII, the above equalities are written explicitly for the processes $\frac{1}{2} + \frac{1}{2} \rightarrow 1 + 1 + X$, $\frac{1}{2} + 1 \rightarrow \frac{1}{2} + 1 + X$ and $1 + 1 \rightarrow \frac{1}{2} + \frac{1}{2} + X$, examples of which are obtained from Table II.

11) Isospins involved : $\frac{1}{2}, 1, 1, 1$. Number of equalities : 5

Three of these equalities are obtained from Eq. (2.24). They can be replaced by the following simpler expressions :

$$\begin{aligned} \sigma\left(\frac{1}{2}, \pm 1, 0, 0\right) + \sigma\left(\frac{1}{2}, 0, \pm 1, 0\right) + \sigma\left(\frac{1}{2}, 0, 0, \pm 1\right) + 2\sigma\left(\frac{1}{2}, 0, 0, 0\right) = \\ = \sigma\left(\pm \frac{1}{2}, 1, 1, 1\right) + \sigma\left(\pm \frac{1}{2}, 1, 1, -1\right) + \sigma\left(\pm \frac{1}{2}, 1, -1, 1\right) + \sigma\left(\pm \frac{1}{2}, 1, -1, -1\right) \end{aligned} \quad (3.33)$$

$$\begin{aligned} \sigma\left(\pm \frac{1}{2}, 1, 0, 1\right) + \sigma\left(\pm \frac{1}{2}, 1, 0, -1\right) + \sigma\left(\frac{1}{2}, \pm 1, 0, 0\right) = \sigma\left(\pm \frac{1}{2}, 0, 1, 1\right) + \\ + \sigma\left(\pm \frac{1}{2}, 0, -1, 1\right) + \sigma\left(\frac{1}{2}, 0, \pm 1, 0\right) \end{aligned} \quad (3.34)$$

$$\begin{aligned} \sigma\left(\pm \frac{1}{2}, 1, 1, 0\right) + \sigma\left(\pm \frac{1}{2}, 1, -1, 0\right) + \sigma\left(\frac{1}{2}, \pm 1, 0, 0\right) = \sigma\left(\pm \frac{1}{2}, 0, 1, 1\right) + \\ + \sigma\left(\pm \frac{1}{2}, 0, 1, -1\right) + \sigma\left(\frac{1}{2}, 0, 0, \pm 1\right) \end{aligned} \quad (3.35)$$

where the notation of Eq. (3.24) was used. The other two are non-linear in the inclusive cross-sections. In Table VIII we give explicitly the equalities obtained from the above expressions for the processes $\frac{1}{2} + 1 \rightarrow 1 + 1 + X$ and $1 + 1 \rightarrow \frac{1}{2} + 1 + X$.

Additional isospin equalities for processes involving the isospin combinations $(\frac{1}{2}, 1, \frac{3}{2})$, $(1, 1, \frac{3}{2})$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ and $(\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2})$ are given in Tables IX, X, XI and XII, respectively,

4. - ISOSPIN INEQUALITIES

Relations (2.25) give equalities of inclusive cross-sections in relatively few cases, but always they give inequalities. The main advantage of these equations is that they do not contain Clebsch-Gordan coefficients. This makes the calculation of inequalities very simple even though the number of observed particles in the final state is large. In this paper, we shall give the inequalities derived from (2.25) only for one case to illustrate the method. A systematic investigation of the inequalities of inclusive cross-sections and moments of the multiplicity distribution will be given elsewhere.

Consider processes involving the isospins $\frac{1}{2}$, $\frac{1}{2}$, 1. For the subset $\frac{1}{2}$, $\frac{1}{2}$ we have $I_{\max} = 1$. Then the relations (2.25) for $\{I_i^3\} = I_{\max} = 1$ and $\{I_i^3\} = I_{\max} - 1 = 0$, combined with Eq. (3.18) give

$$\sigma\left(\frac{1}{2}, -\frac{1}{2}, 1\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, -1\right) \geq \sigma\left(\frac{1}{2}, \frac{1}{2}, 0\right) \quad (4.1)$$

Similarly we get for the subset $\frac{1}{2}$, 1

$$3\sigma\left(\frac{1}{2}, \frac{1}{2}, 1\right) + 3\sigma\left(\frac{1}{2}, -\frac{1}{2}, -1\right) \geq \sigma\left(\frac{1}{2}, \frac{1}{2}, -1\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, 1\right) \quad (4.2)$$

and for the set of all isospins

$$\sigma\left(\frac{1}{2}, \frac{1}{2}, 1\right) + \frac{1}{3}\sigma\left(\frac{1}{2}, \frac{1}{2}, -1\right) \geq \sigma\left(\frac{1}{2}, \frac{1}{2}, 0\right) \quad (4.3)$$

$$\sigma\left(\frac{1}{2}, \frac{1}{2}, 0\right) \geq \sigma\left(\frac{1}{2}, \frac{1}{2}, -1\right) - \sigma\left(\frac{1}{2}, -\frac{1}{2}, -1\right) - \sigma\left(\frac{1}{2}, -\frac{1}{2}, 1\right) \quad (4.4)$$

The above inequalities are valid for processes of the type $\frac{1}{2} + \frac{1}{2} \rightarrow 1 + X$, $\frac{1}{2} + 1 \rightarrow \frac{1}{2} + X$, etc. For example, Eqs. (4.1) give for these cases

$$\sigma\left(\frac{1}{2}, -\frac{1}{2} \rightarrow 1, X\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2} \rightarrow -1, X\right) \geq \sigma\left(\frac{1}{2}, \frac{1}{2} \rightarrow 0, X\right) \quad (4.5)$$

$$\sigma\left(\frac{1}{2}, -1 \rightarrow \frac{1}{2}, X\right) + \sigma\left(\frac{1}{2}, 1 \rightarrow \frac{1}{2}, X\right) \geq \sigma\left(\frac{1}{2}, 0 \rightarrow -\frac{1}{2}, X\right) \quad (4.6)$$

The isospin inequalities of the process $\frac{1}{2} + \frac{1}{2} \rightarrow 1 + X$ have been derived before, using tables of Clebsch-Gordan coefficients. Two of the inequalities obtained are identical to our relations (4.1) and (4.3) for this special case, while the third is stronger than our expression (4.4).

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Non-Vanishing Isospins

Number of
Isospin Equalities

$\frac{1}{2}$	0
1	1
$\frac{3}{2}$	1
$\frac{1}{2}, \frac{1}{2}$	0
$\frac{1}{2}, 1$	1
1, 1	2
$\frac{1}{2}, \frac{3}{2}$	2
1, $\frac{3}{2}$	3
$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	0
$\frac{1}{2}, \frac{1}{2}, 1$	1
$\frac{1}{2}, 1, 1$	2
1, 1, 1	4
$\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$	3
$\frac{1}{2}, 1, \frac{3}{2}$	4
1, 1, $\frac{3}{2}$	6
$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	0
$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1$	1
$\frac{1}{2}, \frac{1}{2}, 1, 1$	2
$\frac{1}{2}, 1, 1, 1$	5
1, 1, 1, 1	8
$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}$	4
$\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}$	6
$\frac{1}{2}, 1, 1, \frac{3}{2}$	9
1, 1, 1, $\frac{3}{2}$	15
$\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}$	9
$\frac{1}{2}, 1, \frac{3}{2}, \frac{3}{2}$	15

- TABLE I -

$$\begin{aligned}
 [0,0] &= [PP, Pd, P\eta, P\Lambda, P\omega, \eta d, \Lambda d, \eta\eta, \eta\Lambda, \eta\omega, \Lambda\Lambda, \Lambda\omega, \dots] \\
 [\frac{1}{2},0] &= [NP, Nd, N\eta, N\Lambda, N\omega, KP, Kd, K\eta, K\Lambda, K\omega, \equiv P, \equiv d, \equiv \eta, \\
 &\quad \equiv \Lambda, \equiv \omega, K^*P, K^*d, K^*\eta, K^*\Lambda, K^*\omega, \dots] \\
 [1,0] &= [\pi P, \pi d, \pi\eta, \pi\Lambda, \pi\omega, \Sigma P, \Sigma d, \Sigma\eta, \Sigma\Lambda, \Sigma\omega, \gamma P, \gamma d, \\
 &\quad \gamma\eta, \gamma\Lambda, \gamma\omega, \rho P, \rho\eta, \rho\Lambda, \rho\omega, \dots] \\
 [\frac{3}{2},0] &= [\Delta P, \Delta d, \Delta\eta, \Delta\Lambda, \Delta\omega, \dots] \\
 [\frac{1}{2},\frac{1}{2}] &= [NN, NK, N\equiv, NK^*, KK, K\equiv, KK^*, \equiv\equiv, \equiv K^*, \dots] \\
 [\frac{1}{2},1] &= [N\pi, N\Sigma, N\gamma, N\rho, K\pi, K\Sigma, K\gamma, K\rho, \equiv\pi, \equiv\Sigma, \equiv\gamma, K^*\pi, K^*\Sigma, K^*\gamma, \dots] \\
 [\frac{1}{2},\frac{3}{2}] &= [N\Delta, K\Delta, \equiv\Delta, K^*\Delta, \dots] \\
 [1,1] &= [\pi\pi, \pi\Sigma, \pi\gamma, \pi\rho, \Sigma\Sigma, \Sigma\gamma, \Sigma\rho, \gamma\rho, \dots] \\
 [1,\frac{3}{2}] &= [\pi\Delta, \Sigma\Delta, \gamma\Delta, \dots] \\
 [0,X] &= [\eta X, \Lambda X, \omega X, \dots] \\
 [\frac{1}{2},X] &= [NX, KX, \equiv X, K^*X, \dots] \\
 [1,X] &= [\pi X, \Sigma X, \gamma X, \rho X, \dots] \\
 [\frac{3}{2},X] &= [\Delta X] \\
 [\frac{1}{2},0,X] &= [N\eta X, N\Lambda X, N\omega X, K\eta X, K\Lambda X, K\omega X, \equiv\eta X, \equiv\Lambda X, \equiv\omega X, K^*\eta X, K^*\Lambda X, \dots] \\
 [1,0,X] &= [\pi\eta X, \pi\Lambda X, \pi\omega X, \Sigma\eta X, \Sigma\Lambda X, \Sigma\omega X, \gamma\eta X, \gamma\Lambda X, \gamma\omega X, \rho\eta X, \rho\Lambda X, \dots] \\
 [\frac{3}{2},0,X] &= [\Delta\eta X, \Delta\Lambda X, \Delta\omega X, \dots] \\
 [\frac{1}{2},\frac{1}{2},X] &= [NNX, NKX, N\equiv X, NK^*X, KKX, K\equiv X, KK^*X, \equiv\equiv X, \equiv K^*X, K^*K^*X, \dots] \\
 [\frac{1}{2},1,X] &= [N\pi X, N\Sigma X, \gamma NX, \rho NX, K\pi X, K\Sigma X, \gamma KX, \rho KX, \equiv\pi X, \\
 &\quad \equiv\Sigma X, \gamma\equiv X, \rho\equiv X, K^*\pi X, K^*\Sigma X, \gamma K^*X, \rho K^*X, \dots] \\
 [\frac{1}{2},\frac{3}{2},X] &= [N\Delta X, K\Delta X, \equiv\Delta X, K^*\Delta X, \dots] \\
 [1,1,X] &= [\pi\pi X, \pi\Sigma X, \gamma\pi X, \rho\pi X, \Sigma\Sigma X, \gamma\Sigma X, \rho\Sigma X, \gamma\gamma X, \gamma\rho X, \rho\rho X, \dots] \\
 [1,\frac{3}{2},X] &= [\pi\Delta X, \Sigma\Delta X, \gamma\Delta X, \rho\Delta X, \dots]
 \end{aligned}$$

$$\begin{aligned} & \sigma\left(\frac{1}{2}, \frac{1}{2} \rightarrow 1, X\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2} \rightarrow 1, X\right) + \sigma\left(\frac{1}{2}, \frac{1}{2} \rightarrow -1, X\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2} \rightarrow -1, X\right) = \\ & = 2\sigma\left(\frac{1}{2}, \frac{1}{2} \rightarrow 0, X\right) + 2\sigma\left(\frac{1}{2}, -\frac{1}{2} \rightarrow 0, X\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(\frac{1}{2}, -1 \rightarrow -\frac{1}{2}, X\right) + \sigma\left(\frac{1}{2}, -1 \rightarrow \frac{1}{2}, X\right) + \sigma\left(\frac{1}{2}, 1 \rightarrow -\frac{1}{2}, X\right) + \sigma\left(\frac{1}{2}, 1 \rightarrow \frac{1}{2}, X\right) = \\ & = 2\sigma\left(\frac{1}{2}, 0 \rightarrow -\frac{1}{2}, X\right) + 2\sigma\left(\frac{1}{2}, 0 \rightarrow \frac{1}{2}, X\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(\frac{1}{2}, 0 \rightarrow -\frac{1}{2}, 1, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow \frac{1}{2}, 1, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow -\frac{1}{2}, -1, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow \frac{1}{2}, -1, X\right) = \\ & = 2\sigma\left(\frac{1}{2}, 0 \rightarrow -\frac{1}{2}, 0, X\right) + 2\sigma\left(\frac{1}{2}, 0 \rightarrow \frac{1}{2}, 0, X\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(1, 0 \rightarrow \frac{1}{2}, \frac{1}{2}, X\right) + \sigma\left(1, 0 \rightarrow \frac{1}{2}, -\frac{1}{2}, X\right) + \sigma\left(1, 0 \rightarrow -\frac{1}{2}, -\frac{1}{2}, X\right) + \sigma\left(1, 0 \rightarrow -\frac{1}{2}, \frac{1}{2}, X\right) = \\ & = 2\sigma\left(0, 0 \rightarrow \frac{1}{2}, \frac{1}{2}, X\right) + 2\sigma\left(0, 0 \rightarrow \frac{1}{2}, -\frac{1}{2}, X\right) \end{aligned}$$

$$\begin{aligned} \sigma\left(\frac{1}{2}, 1 \rightarrow 1, X\right) + \sigma\left(\frac{1}{2}, 1 \rightarrow -1, X\right) + \sigma\left(\frac{1}{2}, -1 \rightarrow 1, X\right) + \sigma\left(\frac{1}{2}, -1 \rightarrow -1, X\right) = \\ = \sigma\left(\frac{1}{2}, 0 \rightarrow 1, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow -1, X\right) + 2\sigma\left(\frac{1}{2}, 0 \rightarrow 0, X\right) \end{aligned}$$

$$\sigma\left(\frac{1}{2}, 1 \rightarrow 0, X\right) + \sigma\left(\frac{1}{2}, -1 \rightarrow 0, X\right) = \sigma\left(\frac{1}{2}, 0 \rightarrow 1, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow -1, X\right)$$

$$\begin{aligned} \sigma\left(\frac{1}{2}, 0 \rightarrow 1, -1, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow -1, -1, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow 1, 1, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow -1, 1, X\right) = \\ = \sigma\left(\frac{1}{2}, 0 \rightarrow 1, 0, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow -1, 0, X\right) + 2\sigma\left(\frac{1}{2}, 0 \rightarrow 0, 0, X\right) \end{aligned}$$

$$\sigma\left(\frac{1}{2}, 0 \rightarrow 0, -1, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow 0, 1, X\right) = \sigma\left(\frac{1}{2}, 0 \rightarrow 1, 0, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow -1, 0, X\right)$$

$$\begin{aligned} \sigma\left(-1, 0 \rightarrow \frac{1}{2}, -1, X\right) + \sigma\left(-1, 0 \rightarrow \frac{1}{2}, 1, X\right) + \sigma\left(1, 0 \rightarrow \frac{1}{2}, -1, X\right) + \sigma\left(1, 0 \rightarrow \frac{1}{2}, 1, X\right) = \\ = \sigma\left(0, 0 \rightarrow \frac{1}{2}, -1, X\right) + \sigma\left(0, 0 \rightarrow \frac{1}{2}, 1, X\right) + 2\sigma\left(0, 0 \rightarrow \frac{1}{2}, 0, X\right) \end{aligned}$$

$$\sigma\left(-1, 0 \rightarrow \frac{1}{2}, 0, X\right) + \sigma\left(1, 0 \rightarrow \frac{1}{2}, 0, X\right) = \sigma\left(0, 0 \rightarrow \frac{1}{2}, -1, X\right) + \sigma\left(0, 0 \rightarrow \frac{1}{2}, 1, X\right)$$

$$\begin{aligned} \sigma\left(1, -1 \rightarrow \frac{1}{2}, X\right) + \sigma\left(-1, -1 \rightarrow \frac{1}{2}, X\right) + \sigma\left(1, 1 \rightarrow \frac{1}{2}, X\right) + \sigma\left(-1, 1 \rightarrow \frac{1}{2}, X\right) = \\ = \sigma\left(1, 0 \rightarrow \frac{1}{2}, X\right) + \sigma\left(-1, 0 \rightarrow \frac{1}{2}, X\right) + 2\sigma\left(0, 0 \rightarrow \frac{1}{2}, X\right) \end{aligned}$$

$$\sigma\left(-1, 0 \rightarrow \frac{1}{2}, X\right) + \sigma\left(1, 0 \rightarrow \frac{1}{2}, X\right) = \sigma\left(0, -1 \rightarrow \frac{1}{2}, X\right) + \sigma\left(0, 1 \rightarrow \frac{1}{2}, X\right)$$

$$A_1 A_2^2 + A_3 A_4^2 + A_5 A_6^2 = A_1 A_2 A_5 + 2 A_2 A_4 A_6$$

$$A_1 = \frac{1}{6} (\sigma_1 + 10 \sigma_2 - 5 \sigma_3)$$

$$A_2 = \frac{1}{6\sqrt{3}} (5 \sigma_1 + 3 \sigma_2 - 24 \sigma_3 - 6 \sigma_4 - 12 \sigma_6 - 18 \sigma_7 + 6 \sigma_{10} + 18 \sigma_{11} + 12 \sigma_{12} + 45 \sigma_{13} - 27 \sigma_{14})$$

$$A_3 = \frac{1}{2} (-\sigma_1 - 2 \sigma_2 + \sigma_3 - 6 \sigma_4 - 4 \sigma_6 + 2 \sigma_{10} + 8 \sigma_{11} + 4 \sigma_{12})$$

$$A_4 = \frac{\sqrt{5}}{2} (\sigma_1 + \sigma_2 - 2 \sigma_3 + 3 \sigma_{13} - 3 \sigma_{14})$$

$$A_5 = \frac{1}{3} (2 \sigma_1 + 2 \sigma_2 - 10 \sigma_3 + 18 \sigma_{13} - 9 \sigma_{14})$$

$$A_6 = \frac{1}{2} \sqrt{\frac{5}{3}} (-\sigma_4 - 2 \sigma_5 + \sigma_{10} + 2 \sigma_{11})$$

$$\sigma_1 = \sigma(1, 1, 1)$$

$$\sigma_6 = \sigma(1, 0, -1)$$

$$\sigma_{11} = \sigma(0, 1, 0)$$

$$\sigma_2 = \sigma(1, 1, -1)$$

$$\sigma_7 = \sigma(1, -1, 1)$$

$$\sigma_{12} = \sigma(0, 1, -1)$$

$$\sigma_3 = \sigma(1, 1, 0)$$

$$\sigma_8 = \sigma(1, -1, 0)$$

$$\sigma_{13} = \sigma(0, 0, 1)$$

$$\sigma_4 = \sigma(1, 0, 1)$$

$$\sigma_9 = \sigma(1, -1, -1)$$

$$\sigma_{14} = \sigma(0, 0, 1)$$

$$\sigma_5 = \sigma(1, 0, 0)$$

$$\sigma_{10} = \sigma(0, 1, 1)$$

$$\begin{aligned} & \sigma\left(\frac{1}{2}, \frac{1}{2} \rightarrow \pm \frac{1}{2}, 1, X\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2} \rightarrow \pm \frac{1}{2}, 1, X\right) + \sigma\left(\frac{1}{2}, \frac{1}{2} \rightarrow \pm \frac{1}{2}, -1, X\right) + \\ & + \sigma\left(\frac{1}{2}, -\frac{1}{2} \rightarrow \pm \frac{1}{2}, -1, X\right) = 2\sigma\left(\frac{1}{2}, \frac{1}{2} \rightarrow \pm \frac{1}{2}, 0, X\right) + 2\sigma\left(\frac{1}{2}, -\frac{1}{2} \rightarrow \pm \frac{1}{2}, 0, X\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(\frac{1}{2}, -1 \rightarrow -\frac{1}{2}, \pm \frac{1}{2}, X\right) + \sigma\left(\frac{1}{2}, -1 \rightarrow \frac{1}{2}, \pm \frac{1}{2}, X\right) + \sigma\left(\frac{1}{2}, 1 \rightarrow -\frac{1}{2}, \pm \frac{1}{2}, X\right) + \\ & + \sigma\left(\frac{1}{2}, 1 \rightarrow \frac{1}{2}, \pm \frac{1}{2}, X\right) = 2\sigma\left(\frac{1}{2}, 0 \rightarrow -\frac{1}{2}, \pm \frac{1}{2}, X\right) + 2\sigma\left(\frac{1}{2}, 0 \rightarrow \frac{1}{2}, \pm \frac{1}{2}, X\right) \end{aligned}$$

$$\sigma\left(\frac{1}{2}, \pm \frac{1}{2} \rightarrow 1, 1, X\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2} \rightarrow -1, -1, X\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2} \rightarrow 1, -1, X\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2} \rightarrow -1, 1, X\right) =$$

$$= \sigma\left(\frac{1}{2}, \pm \frac{1}{2} \rightarrow 0, 1, X\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2} \rightarrow 0, -1, X\right) + 2\sigma\left(\frac{1}{2}, \pm \frac{1}{2} \rightarrow 0, 0, X\right)$$

$$\sigma\left(\frac{1}{2}, \pm \frac{1}{2} \rightarrow 1, 0, X\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2} \rightarrow -1, 0, X\right) = \sigma\left(\frac{1}{2}, \pm \frac{1}{2} \rightarrow 0, 1, X\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2} \rightarrow 0, -1, X\right)$$

$$\sigma\left(\frac{1}{2}, -1 \rightarrow \pm \frac{1}{2}, 1, X\right) + \sigma\left(\frac{1}{2}, 1 \rightarrow \pm \frac{1}{2}, -1, X\right) + \sigma\left(\frac{1}{2}, 1 \rightarrow \pm \frac{1}{2}, 1, X\right) + \sigma\left(\frac{1}{2}, -1 \rightarrow \pm \frac{1}{2}, -1, X\right) =$$

$$= \sigma\left(\frac{1}{2}, -1 \rightarrow \pm \frac{1}{2}, 0, X\right) + \sigma\left(\frac{1}{2}, 1 \rightarrow \pm \frac{1}{2}, 0, X\right) + 2\sigma\left(\frac{1}{2}, 0 \rightarrow \pm \frac{1}{2}, 0, X\right)$$

$$\sigma\left(\frac{1}{2}, 0 \rightarrow \pm \frac{1}{2}, 1, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow \pm \frac{1}{2}, -1, X\right) = \sigma\left(\frac{1}{2}, -1 \rightarrow \pm \frac{1}{2}, 0, X\right) + \sigma\left(\frac{1}{2}, 1 \rightarrow \pm \frac{1}{2}, 0, X\right)$$

$$\sigma\left(1, 1 \rightarrow \frac{1}{2}, \pm \frac{1}{2}, X\right) + \sigma\left(1, 1 \rightarrow -\frac{1}{2}, \pm \frac{1}{2}, X\right) + \sigma\left(1, -1 \rightarrow \frac{1}{2}, \pm \frac{1}{2}, X\right) + \sigma\left(1, -1 \rightarrow -\frac{1}{2}, \pm \frac{1}{2}, X\right) =$$

$$= \sigma\left(0, 1 \rightarrow \frac{1}{2}, \pm \frac{1}{2}, X\right) + \sigma\left(0, 1 \rightarrow -\frac{1}{2}, \pm \frac{1}{2}, X\right) + 2\sigma\left(0, 0 \rightarrow \frac{1}{2}, \pm \frac{1}{2}, X\right)$$

$$\sigma\left(1, 0 \rightarrow \frac{1}{2}, \pm \frac{1}{2}, X\right) + \sigma\left(1, 0 \rightarrow -\frac{1}{2}, \pm \frac{1}{2}, X\right) = \sigma\left(0, 1 \rightarrow \frac{1}{2}, \pm \frac{1}{2}, X\right) + \sigma\left(0, 1 \rightarrow -\frac{1}{2}, \pm \frac{1}{2}, X\right)$$

$$\begin{aligned} & \sigma\left(\frac{1}{2}, \pm 1 \rightarrow 0, 0, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow \pm 1, 0, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow 0, \pm 1, X\right) + 2\sigma\left(\frac{1}{2}, 0 \rightarrow 0, 0, X\right) = \\ & = \sigma\left(\pm \frac{1}{2}, 1 \rightarrow 1, 1, X\right) + \sigma\left(\pm \frac{1}{2}, 1 \rightarrow 1, -1, X\right) + \sigma\left(\pm \frac{1}{2}, 1 \rightarrow -1, 1, X\right) + \sigma\left(\pm \frac{1}{2}, 1 \rightarrow -1, -1, X\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(\pm \frac{1}{2}, 1 \rightarrow 0, 1, X\right) + \sigma\left(\pm \frac{1}{2}, 1 \rightarrow 0, -1, X\right) + \sigma\left(\frac{1}{2}, \pm 1 \rightarrow 0, 0, X\right) = \sigma\left(\pm \frac{1}{2}, 0 \rightarrow 1, 1, X\right) + \\ & + \sigma\left(\pm \frac{1}{2}, 0 \rightarrow 1, -1, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow \pm 1, 0, X\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(\pm \frac{1}{2}, 1 \rightarrow 1, 0, X\right) + \sigma\left(\pm \frac{1}{2}, 1 \rightarrow -1, 0, X\right) + \sigma\left(\frac{1}{2}, \pm 1 \rightarrow 0, 0, X\right) = \sigma\left(\pm \frac{1}{2}, 0 \rightarrow 1, 1, X\right) + \\ & + \sigma\left(\pm \frac{1}{2}, 0 \rightarrow 1, -1, X\right) + \sigma\left(\frac{1}{2}, 0 \rightarrow 0, \pm 1, X\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(0, \pm 1 \rightarrow \frac{1}{2}, 0, X\right) + \sigma\left(\pm 1, 0 \rightarrow \frac{1}{2}, 0, X\right) + \sigma\left(0, 0 \rightarrow \frac{1}{2}, \pm 1, X\right) + 2\sigma\left(0, 0 \rightarrow \frac{1}{2}, 0, X\right) = \\ & = \sigma\left(-1, 1 \rightarrow \pm \frac{1}{2}, 1, X\right) + \sigma\left(-1, 1 \rightarrow \pm \frac{1}{2}, -1, X\right) + \sigma\left(1, 1 \rightarrow \pm \frac{1}{2}, 1, X\right) + \sigma\left(1, 1 \rightarrow \pm \frac{1}{2}, -1, X\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(0, 1 \rightarrow \pm \frac{1}{2}, 1, X\right) + \sigma\left(0, 1 \rightarrow \pm \frac{1}{2}, -1, X\right) + \sigma\left(0, \pm 1 \rightarrow \frac{1}{2}, 0, X\right) = \sigma\left(1, 0 \rightarrow \pm \frac{1}{2}, -1, X\right) + \\ & + \sigma\left(1, 0 \rightarrow \pm \frac{1}{2}, 1, X\right) + \sigma\left(\pm 1, 0 \rightarrow \frac{1}{2}, 0, X\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(1, -1 \rightarrow \pm \frac{1}{2}, 0, X\right) + \sigma\left(1, 1 \rightarrow \pm \frac{1}{2}, 0, X\right) + \sigma\left(0, \pm 1 \rightarrow \frac{1}{2}, 0, X\right) = \sigma\left(1, 0 \rightarrow \pm \frac{1}{2}, -1, X\right) + \\ & + \sigma\left(1, 0 \rightarrow \pm \frac{1}{2}, 1, X\right) + \sigma\left(0, 0 \rightarrow \frac{1}{2}, \pm 1, X\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(\frac{1}{2}, 1, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, 1, \pm \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -1, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, -1, \pm \frac{1}{2}\right) = \\ & = 2\sigma\left(\frac{1}{2}, 0, \pm \frac{3}{2}\right) + 2\sigma\left(\frac{1}{2}, 0, \pm \frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(\frac{1}{2}, 1, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, 0, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, -1, \pm \frac{3}{2}\right) = \sigma\left(\frac{1}{2}, 1, \pm \frac{1}{2}\right) + \\ & + \sigma\left(\frac{1}{2}, 0, \pm \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -1, \pm \frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(\frac{1}{2}, 1, -\frac{1}{2}\right) + \sigma\left(\frac{1}{2}, 0, -\frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -1, -\frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -1, \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, 0, \frac{3}{2}\right) + \\ & + \sigma\left(\frac{1}{2}, 1, \frac{3}{2}\right) = 2\sigma\left(\frac{1}{2}, 1, \frac{1}{2}\right) + 2\sigma\left(\frac{1}{2}, 0, \frac{1}{2}\right) + 2\sigma\left(\frac{1}{2}, -1, \frac{1}{2}\right) \end{aligned}$$

$$\sigma\left(\pm \frac{1}{2}, 1, \frac{3}{2}\right) + 3\sigma\left(\pm \frac{1}{2}, 1, -\frac{1}{2}\right) = \sigma\left(\pm \frac{1}{2}, 1, -\frac{3}{2}\right) + 3\sigma\left(\pm \frac{1}{2}, 1, \frac{1}{2}\right)$$

$$\begin{aligned} \sigma(1, 1, \pm \frac{3}{2}) + \sigma(1, 1, \pm \frac{1}{2}) + \sigma(1, -1, \pm \frac{3}{2}) + \sigma(1, -1, \pm \frac{1}{2}) &= \sigma(0, 1, \pm \frac{3}{2}) + \\ &+ \sigma(0, 1, \pm \frac{1}{2}) + \sigma(0, 0, \pm \frac{3}{2}) + \sigma(0, 0, \pm \frac{1}{2}) \end{aligned}$$

$$\sigma(1, 0, \pm \frac{3}{2}) + \sigma(1, 0, \pm \frac{1}{2}) = \sigma(0, 1, \pm \frac{3}{2}) + \sigma(0, 1, \pm \frac{1}{2})$$

$$\begin{aligned} \sigma(1, \pm 1, \frac{3}{2}) + \sigma(1, \pm 1, -\frac{3}{2}) + \sigma(1, 0, \pm \frac{3}{2}) + \sigma(0, \pm 1, \frac{3}{2}) + \sigma(0, 0, \frac{3}{2}) &= \\ = \sigma(1, \pm 1, \frac{1}{2}) + \sigma(1, \pm 1, -\frac{1}{2}) + \sigma(1, 0, \pm \frac{1}{2}) + \sigma(0, \pm 1, \frac{1}{2}) + \sigma(0, 0, \frac{1}{2}) \end{aligned}$$

$$\begin{aligned} \sigma(1, \pm 1, \frac{3}{2}) + 3\sigma(1, \pm 1, -\frac{1}{2}) + \sigma(1, 0, \frac{3}{2}) + 3\sigma(1, 0, -\frac{1}{2}) &= \sigma(1, \pm 1, -\frac{3}{2}) + \\ &+ 3\sigma(1, \pm 1, \frac{1}{2}) + \sigma(1, 0, -\frac{3}{2}) + 3\sigma(1, 0, \frac{1}{2}) \end{aligned}$$

$$\begin{aligned} \sigma(\pm 1, 1, \frac{3}{2}) + 3\sigma(\pm 1, 1, -\frac{1}{2}) + \sigma(0, 1, \frac{3}{2}) + 3\sigma(0, 1, -\frac{1}{2}) &= \sigma(\pm 1, 1, -\frac{3}{2}) + \\ &+ 3\sigma(\pm 1, 1, \frac{1}{2}) + \sigma(0, 1, -\frac{3}{2}) + 3\sigma(0, 1, \frac{1}{2}) \end{aligned}$$

$$\begin{aligned} & \sigma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \pm \frac{3}{2}\right) = \\ & = \sigma\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \pm \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \pm \frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}, -\frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, \pm \frac{1}{2}, -\frac{1}{2}\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}, \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, \pm \frac{1}{2}, \frac{3}{2}\right) = \\ & = 2\sigma\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}, \frac{1}{2}\right) + 2\sigma\left(\frac{1}{2}, -\frac{1}{2}, \pm \frac{1}{2}, \frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}, -\frac{1}{2}\right) + \sigma\left(-\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}, -\frac{1}{2}\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}, \frac{3}{2}\right) + \sigma\left(-\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}, \frac{3}{2}\right) = \\ & = 2\sigma\left(\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}, \frac{1}{2}\right) + 2\sigma\left(-\frac{1}{2}, \frac{1}{2}, \pm \frac{1}{2}, \frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned} & \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \sigma\left(-\frac{1}{2}, \pm \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}\right) + \sigma\left(-\frac{1}{2}, \pm \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}\right) = \\ & = 2\sigma\left(\frac{1}{2}, \pm \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) + 2\sigma\left(-\frac{1}{2}, \pm \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned}
& \sigma\left(\frac{1}{2}, \frac{1}{2}, 1, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, 1, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}, 1, \pm \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, 1, \pm \frac{1}{2}\right) + \\
& + \sigma\left(\frac{1}{2}, \frac{1}{2}, -1, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, -1, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}, -1, \pm \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, -1, \pm \frac{1}{2}\right) = \\
& = 2\sigma\left(\frac{1}{2}, \frac{1}{2}, 0, \pm \frac{3}{2}\right) + 2\sigma\left(\frac{1}{2}, -\frac{1}{2}, 0, \pm \frac{3}{2}\right) + 2\sigma\left(\frac{1}{2}, \frac{1}{2}, 0, \pm \frac{1}{2}\right) + 2\sigma\left(\frac{1}{2}, -\frac{1}{2}, 0, \pm \frac{1}{2}\right) \\
& \sigma\left(\frac{1}{2}, \frac{1}{2}, 1, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, 1, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}, 0, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, 0, \pm \frac{3}{2}\right) + \\
& + \sigma\left(\frac{1}{2}, \frac{1}{2}, -1, \pm \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, -1, \pm \frac{3}{2}\right) = \sigma\left(\frac{1}{2}, \frac{1}{2}, 1, \pm \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, 1, \pm \frac{1}{2}\right) + \\
& + \sigma\left(\frac{1}{2}, \frac{1}{2}, 0, \pm \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, 0, \pm \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, \frac{1}{2}, -1, \pm \frac{1}{2}\right) + \sigma\left(\frac{1}{2}, -\frac{1}{2}, -1, \pm \frac{1}{2}\right) \\
& \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 1, -\frac{1}{2}\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 0, -\frac{1}{2}\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, -1, -\frac{1}{2}\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, -1, \frac{3}{2}\right) + \\
& + \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 0, \frac{3}{2}\right) + \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 1, \frac{3}{2}\right) = 2\sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 1, \frac{1}{2}\right) + 2\sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 0, \frac{1}{2}\right) + \\
& + 2\sigma\left(\frac{1}{2}, \pm \frac{1}{2}, -1, \frac{1}{2}\right) \\
& \sigma\left(\pm \frac{1}{2}, \frac{1}{2}, 1, -\frac{1}{2}\right) + \sigma\left(\pm \frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}\right) + \sigma\left(\pm \frac{1}{2}, \frac{1}{2}, -1, -\frac{1}{2}\right) + \sigma\left(\pm \frac{1}{2}, \frac{1}{2}, -1, \frac{3}{2}\right) + \\
& + \sigma\left(\pm \frac{1}{2}, \frac{1}{2}, 0, \frac{3}{2}\right) + \sigma\left(\pm \frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}\right) = 2\sigma\left(\pm \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}\right) + 2\sigma\left(\pm \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}\right) + \\
& + 2\sigma\left(\pm \frac{1}{2}, \frac{1}{2}, -1, \frac{1}{2}\right) \\
& \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 1, \frac{3}{2}\right) + \sigma\left(-\frac{1}{2}, \pm \frac{1}{2}, 1, \frac{3}{2}\right) + 3\sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 1, -\frac{1}{2}\right) + 3\sigma\left(-\frac{1}{2}, \pm \frac{1}{2}, 1, -\frac{1}{2}\right) = \\
& = \sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 1, -\frac{3}{2}\right) + \sigma\left(-\frac{1}{2}, \pm \frac{1}{2}, 1, -\frac{3}{2}\right) + 3\sigma\left(\frac{1}{2}, \pm \frac{1}{2}, 1, \frac{1}{2}\right) + 3\sigma\left(-\frac{1}{2}, \pm \frac{1}{2}, 1, \frac{1}{2}\right)
\end{aligned}$$

TABLE CAPTIONS

TABLE I Number of isospin equalities for some of the most interesting physically inclusive reactions. The reactions are specified by the isospins of the particles of the initial state, and the isospins of the observed particles of the final state. Since the presence of a number of isosinglets, and the interchange of particles of the initial state with particles of the final state does not affect the number of equalities, only the non-vanishing isospins are indicated, and without specifying if they belong to particles of the initial state or the final state.

TABLE II Some initial and final states to which the isospin equalities apply. For simplicity states containing $\bar{N}$, $\bar{K}$, $\bar{\Xi}$, $\bar{\Sigma}$, $\bar{\Lambda}$, ... have been omitted. The isospin relations apply also to such states obtained by making the interchanges $N \rightleftharpoons \bar{N}$, $K \rightleftharpoons \bar{K}$, $\Xi \rightleftharpoons \bar{\Xi}$, $\Sigma \rightleftharpoons \bar{\Sigma}$, $\Lambda \rightleftharpoons \bar{\Lambda}$, ... anywhere in the Table.

TABLE III Isospin equalities of cross-sections of processes $\frac{1}{2} + \frac{1}{2} \rightarrow 1 + X$, $\frac{1}{2} + 1 \rightarrow \frac{1}{2} + X$, $\frac{1}{2} + 0 \rightarrow \frac{1}{2} + 1 + X$ and $1 + 0 \rightarrow \frac{1}{2} + \frac{1}{2} + X$. The meaning of $\underline{0}$ is explained after Eq. (3.3).

TABLE IV Isospin equalities of cross-sections of processes $\frac{1}{2} + 1 \rightarrow 1 + X$, $\frac{1}{2} + 0 \rightarrow 1 + 1 + X$, $1 + 0 \rightarrow \frac{1}{2} + 1 + X$ and $1 + 1 \rightarrow \frac{1}{2} + X$. In each case there are two isospin equalities.

TABLE V Non-linear isospin equality of inclusive cross-sections of the case involving the isospins 1, 1, 1.

TABLE VI Isospin equalities of cross-sections of processes $\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + 1 + X$ and $\frac{1}{2} + 1 \rightarrow \frac{1}{2} + \frac{1}{2} + X$. The notation of Eq. (3.24) is used for simplicity.

TABLE VII Isospin equalities of cross-sections of processes $\frac{1}{2} + \frac{1}{2} \rightarrow 1 + 1$, $\frac{1}{2} + 1 \rightarrow \frac{1}{2} + 1 + X$ and $1 + 1 \rightarrow \frac{1}{2} + \frac{1}{2} + X$. In each case there are two isospin equalities. The notation of Eq. (3.24) is used for simplicity.

TABLE VIII Isospin equalities of cross-sections of processes $\frac{1}{2} + 1 \rightarrow 1 + 1 + X$ and $1 + 1 \rightarrow \frac{1}{2} + 1 + X$. In each case there are three isospin equalities. The notation of Eq. (3.24) is used for simplicity.

TABLE IX Isospin equalities of cross-sections of inclusive processes whose non-vanishing isospins of the particles of the initial state and the observed particles of the final state are $\frac{1}{2}, 1, \frac{3}{2}$. The first two were obtained from Eqs. (2.24), the third from Eq. (2.25) and the fourth from Eq. (3.17) by the rule (2.31). The meaning of $I_3 = \frac{1}{2}$ is explained after Eq. (3.14). The notation of Eq. (3.24) is used for simplicity. The equalities apply to processes $\frac{1}{2} + 1 \rightarrow \frac{3}{2} + X$, $\frac{1}{2} + \frac{3}{2} \rightarrow 1 + X$, $1 + \frac{3}{2} \rightarrow \frac{1}{2} + X$, $\frac{1}{2} + 0 \rightarrow 1 + \frac{3}{2} + X$, $1 + 0 \rightarrow \frac{1}{2} + \frac{3}{2} + X$, etc.

TABLE X Linear isospin equalities of cross-sections of inclusive processes whose non-vanishing isospins of the particles of the initial state and the observed particles of the final state are $1, 1, \frac{3}{2}$. In this case there is in addition one non-linear equality. The first three are simplified versions of equalities derived from Eq. (2.24). The last two were obtained from Eq. (3.17) by the rule of Eq. (2.31). The notation of Eq. (3.24) is used for simplicity. The equalities apply to processes $1 + 0 \rightarrow 1 + \frac{3}{2} + X$, $\frac{3}{2} + 0 \rightarrow 1 + 1 + X$, $1 + 1 \rightarrow \frac{3}{2} + X$, $1 + \frac{3}{2} \rightarrow 1 + X$, etc.

TABLE XI Isospin equalities of cross-sections of inclusive processes whose non-vanishing isospins of the particles of the initial state and the observed particles of the final state are $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}$. The first was derived from Eq. (2.24) and the other three from Eq. (2.25). The notation of Eq. (3.24) is used, and the meaning of $I_3 = \frac{1}{2}$ is explained after Eq. (3.14). The equalities apply to processes $\frac{1}{2} + \frac{1}{2} \rightarrow \frac{1}{2} + \frac{3}{2} + X$, $\frac{1}{2} + \frac{3}{2} \rightarrow \frac{1}{2} + \frac{1}{2} + X$, etc.

TABLE XII Linear isospin equalities of cross-sections of inclusive processes whose non-vanishing isospins of the particles of the initial state and the observed particles of the final state are $\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}$. The first two were derived from Eq. (2.24), the next two from Eq. (2.25) and the last one from Eq. (3.17) by the rule of Eq. (2.31). In addition there is one non-linear equality. The notation of Eq. (3.24) is used and the meaning of $I_3 = \frac{1}{2}$ is explained after Eq. (3.14). The equalities apply to processes $\frac{1}{2} + \frac{1}{2} \rightarrow 1 + \frac{3}{2} + X$, $\frac{1}{2} + 1 \rightarrow \frac{1}{2} + \frac{3}{2} + X$, etc.

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