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Archives

FACTORIZATION OF DISCONTINUITIES OF MULTI-REGGE AMPLITUDES

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A B S T R A C T

It is shown that an arbitrary discontinuity of an arbitrary multi-Regge amplitude factorizes in a large class of models which encompasses the dual resonance model. It is argued that multiple discontinuities also factorize and that the factorization of discontinuities is expected to hold on very general grounds as a result of the singularity structure of multiparticle amplitudes. These results allow the full amplitude as well as its discontinuities to be written as a product of Regge propagators, single-, double- and triple-Regge vertices and their discontinuities.

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1. INTRODUCTION

The idea of a Regge pole as an exchanged object in a well-defined state leads naturally to the assumption that multi-Regge amplitudes should factorize into products of Regge pole propagators and vertices. Here we shall argue that multi-Regge amplitudes have the remarkable property that any discontinuity or multiple discontinuity also factorizes. Therefore, both the full amplitude and its discontinuities can be written as a product of Regge propagators, single-, double- and triple-Regge vertices and their discontinuities. In a previous paper ¹⁾ we have identified these basic objects and given the rules for writing down the behaviour of an arbitrary multi-Regge amplitude. Here we explicitly verify these results for discontinuities in a large class of models which are characterized as integrals over the dual resonance model.

Factorization is most naturally expressed as the factorization of the residues of Regge poles in the signatured multiple partial wave amplitude. One then expects that the multi-Regge contribution to the signatured amplitude factorizes. However, since the full amplitude is a sum of signatured amplitudes, it is by no means obvious that its multi-Regge behaviour should factorize ²⁾. Nevertheless, the full amplitude has been shown to factorize in certain models ³⁾⁻⁵⁾. This result crucially depends on taking into proper account the singularities in energy variables which are dependent due to non-linear Gram-determinant constraints. Furthermore, the required singularity structure of multiparticle amplitudes (specifically, the absence of simultaneous discontinuities in overlapping energy variables) was shown to be an essential ingredient in the factorization of the full amplitude ⁶⁾.

Factorization of discontinuities, like factorization of the full amplitude, is by no means an obvious consequence of factorization of the signatured amplitude, since they are the difference of two amplitudes. We shall see that the singularity structure of multiparticle amplitudes again plays an essential role. We shall find that discontinuities factorize when all the remaining energies are positive and above their cuts. Since the signatured amplitude also factorizes when all energies are positive and above their cuts, we conclude that the signatured amplitude with one subenergy below its cut does not factorize. Thus factorization does not hold throughout the complex energy planes as is the case for the four-particle amplitude. It only holds for certain boundary values and differences of boundary values.

In Section 2 we compute the behaviour of the ordinary dual resonance model (DRM) for an arbitrary multi-Regge limit. We prove factorization for a single term for all the energy variables positive and above their cuts.

In Section 3 we identify the basic vertices and prove the factorization of an arbitrary discontinuity of an arbitrary multi-Regge amplitude in the DRM. We also prove the factorization of multiple discontinuities in the simplest non-trivial case - the $2 \rightarrow 4$ amplitude.

In Section 4 we note that the DRM results can be trivially generalized to a large class of models which are essentially integrals over the DRM amplitudes. This class of models is expected to contain the usual ladder models for Reggeons and Gribov's hybrid model. We note that the essential feature of these models is the absence of simultaneous discontinuities in overlapping energy variables and conjecture that they may be the most general form for amplitudes satisfying this property and the factorization of the full amplitude. We conclude with some general comments and a restatement of the rules for writing down multi-Regge amplitudes.

2. MULTI-REGGE LIMIT OF THE DRM

In this section we compute an arbitrary multi-Regge limit of a single generalized beta function term. We obtain a representation for the multi-Regge limit as a multiple Mellin-Barnes type integral since this form is particularly convenient for analytic continuation to positive energies from negative energies where the term was originally defined. We then show factorization for all energies positive and above their cuts.

In order to establish our notation we consider a typical multi-Regge amplitude (Fig. 1). The energy variables $s_{ij\dots k}$ are labeled by the Reggeons of momentum transfer t_i which they overlap. Thus, in the multi-Regge limit

$$s_{ij\dots k} \propto s_i s_j \dots s_k . \quad (2.1)$$

Actually, energies which overlap more than two Reggeons are kinematically dependent

$$\frac{s_{ij\dots kl} s_{j\dots k}}{s_{ij\dots k} s_{j\dots kl}} = \phi_{ij\dots kl} = 1. \quad (2.2)$$

Thus the amplitude can be expressed in terms of the s_i and s_{jk} , or alternatively,

$$\eta_{jk} = \frac{s_{jk}}{s_j s_k}. \quad (2.3)$$

An amplitude with N internal Reggeons has $N+3$ external lines independent of the number D of double-Regge vertices or the number $\frac{1}{2}(N-1-D)$ of triple Regge vertices. The independent variables are N momentum transfers t_i , N energies s_i , and $N-1$ Toller angle variables η_{ij} , one for each double-Regge vertex and two for each triple-Regge vertex (there being one constraint among the three that can be defined ^{7),8)}), for a total of $3(N+3) - 10$.

In order to study a multi-Regge limit with N internal Reggeons we start with the Chan representation ⁹⁾ for the generalized beta-function B_{N+3} ,

$$B_{N+3} = \int \frac{\prod_{P'} du_{P'}}{J} \prod_P u_P^{-\alpha_P-1}, \quad (2.4)$$

where the product P is over all partitions of the external lines into groups of adjacent lines and the product P' is over a set of N independent integration variables. The Regge trajectory α_P corresponding to u_P is a function of the total energy of the lines in the partition P . The Jacobian J need not concern us since it turns out to be unity in the multi-Regge limit. The dependent variables are given by

$$u_P = 1 - \prod_{\bar{P}} u_{\bar{P}}, \quad (2.5)$$

where the product $\bar{P}$ is over all partitions dual to P , e.g., all partitions which contain some of the lines of P in addition to some lines not in P .

We choose the independent integration variables to be those corresponding to the momentum transfers of the Reggeons and label them u_i . The remaining variables then correspond to the energy variables $s_{ij\dots k}$ and

we label them $u_{ij\dots k}$. From (2.5) we then have

$$u_{ij\dots k} = 1 - u_i u_j \dots u_k \prod_{\tilde{P}} u_{\tilde{P}}. \quad (2.6)$$

We have explicitly indicated the independent u_i amongst the set $\tilde{P}$. These consist of precisely those corresponding to the Reggeons which $s_{ij\dots k}$ overlaps. The remaining partitions $\tilde{P}$ correspond to other energy variables.

As usual we first calculate the Regge limit with all $s_{ij\dots k}$ going to minus infinity where (2.4) is defined. Clearly $s_{ij\dots k} \rightarrow \infty$ causes $u_{ij\dots k} \approx 1$ to dominate. All the $u_{ij\dots k}$ go to unity if the u_i go to zero. Thus it is convenient to make the change of variables

$$u_i = -\gamma_i/s_i. \quad (2.7)$$

Then

$$u_{ij\dots k} \approx 1 - (-\gamma_i/s_i)(-\gamma_j/s_j) \dots (-\gamma_k/s_k)$$

and

$$u_{ij\dots k}^{-\alpha_{ij\dots k}-1} \approx \exp \left[-\gamma_i \gamma_j \dots \gamma_k \left(\frac{-s_{ij\dots k}}{(-s_i)(-s_j) \dots (-s_k)} \right) \right]. \quad (2.8)$$

Using (2.8) we find

$$\begin{aligned} B_{N+3} &\sim (-s_1)^{\alpha_1} (-s_2)^{\alpha_2} \dots (-s_N)^{\alpha_N} \\ &\times \int_0^\infty d\gamma_1 \dots \int_0^\infty d\gamma_N \gamma_1^{-\alpha_1-1} \dots \gamma_N^{-\alpha_N-1} \\ &\times \exp \left[-\gamma_1 \dots \gamma_N - \sum \gamma_i \gamma_j \dots \gamma_k \left(\frac{-s_{ij\dots k}}{(-s_i)(-s_j) \dots (-s_k)} \right) \right], \end{aligned} \quad (2.9)$$

where the sum is over all paths through the diagram which start at one Reggeon and end at another, e.g., all $s_{ij\dots k}$ corresponding to groups of adjacent lines, for example, s_{125} , s_{1256} and $s_{1256789}$ in Fig. 1.

In order to obtain a form convenient for analytic continuation we use the identity

$$e^x = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn \Gamma(-n) (-x)^n, \quad (2.10)$$

which can be trivially verified by closing the contour to the right, on all the exponentials in (2.9) containing more than one y_i . Then the y_i integrals can be done to give

$$\begin{aligned} B_{N+3} &\sim (-s_1)^{\alpha_1} (-s_2)^{\alpha_2} \dots (-s_N)^{\alpha_N} \\ &\times \prod_{i,j,\dots,k} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn_{ij\dots k} \Gamma(-n_{ij\dots k}) (-s_{ij\dots k})^{n_{ij\dots k}} \\ &\times \prod_{\ell=1}^N (-s_\ell)^{-\sum_{p,q,\dots,r} n_{pq\dots r}} \Gamma(-\alpha_\ell + \sum_{p,q,\dots,r} n_{pq\dots r}), \end{aligned} \quad (2.11)$$

where the product is over all paths through the diagram starting at one Reggeon and ending at another and the sum is over all $n_{pq\dots r}$ which contain ℓ . The contours lie to the left of the poles in the first set of gamma functions and to the right of the poles in the second set. We note that the arguments of the gamma functions in (2.11) are just the negative of the powers of energy variables.

We now collect together powers of the $s_{ij\dots k}$ to form the combinations η_{ij} and $\phi_{ij\dots k}$ [Eqs. (2.2) and (2.3)]. Then (2.11) becomes after a change to a new set of variables $n_{ij\dots k}$,

$$\begin{aligned} B_{N+3} &\sim (-s_1)^{\alpha_1} (-s_2)^{\alpha_2} \dots (-s_N)^{\alpha_N} \\ &\times \prod_{i,j,\dots,k} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn_{ij\dots k} \Gamma(-\eta_{ij})^{n_{ij}} \Gamma(-\phi_{ij\dots k})^{n_{ij\dots k}} \\ &\times \prod_{\ell} \Gamma(-\alpha_\ell + \sum_p \eta_{\ell p} + \sum_p \eta_{p\ell} - \sum_{rs} \eta_{r\ell s}) \\ &\times \prod_{i,j,\dots,k} \Gamma(-n_{ij\dots k} + \sum_p n_{ij\dots kp} + \sum_p n_{pij\dots k} - \sum_{rs} n_{rij\dots ks}), \end{aligned} \quad (2.12)$$

where again the products are over all paths through the diagram starting at one Reggeon and ending at another and the sums over p , r and s are over adjacent Reggeons and thus contain at most two terms. We note that each n_{ij} will occur in three of the gamma functions, namely those beginning with

$-\alpha_i$, $-\alpha_j$, and $-n_{ij}$, and that each $n_{ij\dots kl}$ will occur in four of the gamma functions, namely those beginning with $-n_{j\dots k}$ (or $-\alpha_j$ if only one subscript), $-n_{ij\dots k}$, $-n_{j\dots kl}$, and $-n_{ij\dots kl}$. The $n_{ij\dots k}$ contour is to be drawn such that poles arising from gamma functions with $n_{ij\dots k}$ negative lie to the right and poles arising from gamma functions with $n_{ij\dots k}$ positive lie to the left.

Using (2.12) we can easily identify the Regge propagator and vertices. Taking the Regge propagator to be

$$\text{wavy line} = \Gamma(-\alpha)(-s)^\alpha,$$

we find from B_4 that the single-Regge vertex is unity. From the double Regge limit of B_5 ,

$$B_5 \sim (-s_1)^{\alpha_1} (-s_2)^{\alpha_2} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\eta_{12} \Gamma(-\alpha_1 + \eta_{12}) \Gamma(-\alpha_2 + \eta_{12}) \Gamma(-\eta_{12}) (-\eta_{12})^{\eta_{12}}, \quad (2.13)$$

we find the double-Regge vertex

$$\begin{aligned} \text{Diagram: } t_1 \text{ --- wavy line --- } t_2 &= V(\alpha_1, \alpha_2; \eta_{12}) \\ &= \frac{1}{\Gamma(-\alpha_1) \Gamma(-\alpha_2) 2\pi i} \int_{-i\infty}^{i\infty} d\eta_{12} \Gamma(-\alpha_1 + \eta_{12}) \Gamma(-\alpha_2 + \eta_{12}) \Gamma(-\eta_{12}) (-\eta_{12})^{\eta_{12}}. \end{aligned} \quad (2.14)$$

Similarly, from the triple Regge limit of B_6 we obtain the triple-Regge vertex

$$\begin{aligned} \text{Diagram: } t_1 \text{ --- wavy line --- } t_2 \text{ --- wavy line --- } t_3 &= V(\alpha_1, \alpha_2, \alpha_3; \eta_{12}, \eta_{23}, \eta_{31}) \\ &= \frac{1}{\Gamma(-\alpha_1) \Gamma(-\alpha_2) \Gamma(-\alpha_3) (2\pi i)^3} \iiint_{-i\infty}^{i\infty} d\eta_{12} d\eta_{23} d\eta_{31} \\ &\quad \times \Gamma(-\alpha_1 + \eta_{12} + \eta_{31}) \Gamma(-\alpha_2 + \eta_{12} + \eta_{23}) \Gamma(-\alpha_3 + \eta_{23} + \eta_{31}) \\ &\quad \times \Gamma(-\eta_{12}) \Gamma(-\eta_{23}) \Gamma(-\eta_{31}) (-\eta_{12})^{\eta_{12}} (-\eta_{23})^{\eta_{23}} (-\eta_{31})^{\eta_{31}}. \end{aligned} \quad (2.15)$$

Equations (2.14) and (2.15) agree with previous calculations, see, for example Ref. 10).

We now prove the factorization of B_{N+3} when all the energies are positive and above their cuts. We do this by induction by showing that (2.12) can be expressed as a double- or triple-Regge vertex times the amplitude with $N-1$ or $N-2$ Reggeons. Consider the path(s) through the multi-Regge amplitude which passes through the largest number of Reggeons. If there are several such paths, for example, $(1,2,5,6,7,8,9)$, $(1,2,5,6,7,8,10)$, or $(3,4,5,6,7,11,12)$ in Fig. 1, select one of them. This path will by necessity start and end at Reggeons attached at one end to single-Regge vertices. Let us label the Reggeons along this path, $1, 2, \dots, k$ ($k \leq N$), see Fig. 2. We consider the most general case where the vertices between Reggeons 1 and 2 and $k-1$ and k are triple Regge vertices; the case of double-Regge vertices can be easily obtained by taking the residues of the poles at $\alpha_0 = 0$ and/or $\alpha_\ell = 0$ at the end. In this general case, the paths $(0, \dots, k)$, $(1, \dots, \ell)$ and $(0, \dots, \ell)$ are of the same length as $(1, \dots, k)$ and must be treated on the same footing.

We first examine the dependence on $n_{1 \dots k}$ in Eq. (2.12). It is easily seen to be

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn_{1 \dots k} \phi_{1 \dots k}^{n_{1 \dots k}} \\ & \times \Gamma(-n_{1 \dots k}) \Gamma(-n_{1 \dots k-1} + n_{1 \dots k} + n_{1 \dots \ell}) \Gamma(-n_{2 \dots k} + n_{0 \dots k} + n_{1 \dots k}) \\ & \times \Gamma(-n_{2 \dots k-1} + n_{0 \dots k-1} + n_{1 \dots k-1} + n_{2 \dots k} + n_{2 \dots \ell} - n_{0 \dots k} - n_{0 \dots \ell} - n_{1 \dots k} - n_{1 \dots \ell}). \end{aligned} \quad (2.16)$$

We recognize (2.16) as the integral representation of the hypergeometric function ¹¹⁾

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn \Gamma(a+n) \Gamma(b+n) \Gamma(c-a-b-n) \Gamma(-n) z^n \\ & = \frac{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; 1-z) \\ & = \frac{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)} F(z). \end{aligned} \quad (2.17)$$

Since $\phi_{1\dots k} = 1$ and $F(1) = 1$, Eq. (2.16) is easily evaluated as the ratio of gamma functions

$$\begin{aligned} & \frac{\Gamma(-n_{1\dots k-1} + n_{1\dots l}) \Gamma(-n_{2\dots k} + n_{0\dots k})}{\Gamma(-n_{2\dots k-1} + n_{0\dots k-1} + n_{2\dots l} - n_{0\dots l})} \\ & \times \Gamma(-n_{2\dots k-1} + n_{0\dots k-1} + n_{1\dots k-1} + n_{2\dots l} - n_{0\dots l} - n_{1\dots l}) \\ & \times \Gamma(-n_{2\dots k-1} + n_{0\dots k-1} + n_{2\dots k} + n_{2\dots l} - n_{0\dots k} - n_{0\dots l}) . \end{aligned} \quad (2.18)$$

The $n_{1\dots l}$ integral can be done in the same manner. It occurs in two factors of (2.18) and in two further factors from (2.12)

$$\Gamma(-n_{1\dots l}) \Gamma(-n_{2\dots l} + n_{0\dots l} + n_{1\dots l}) . \quad (2.19)$$

The result of this integral and the remaining factors in (2.18) yield

$$\begin{aligned} & \frac{\Gamma(-n_{1\dots k-1}) \Gamma(-n_{2\dots k} + n_{0\dots k}) \Gamma(-n_{2\dots l} + n_{0\dots l}) \Gamma(-n_{2\dots k-1} + n_{0\dots k-1} + n_{1\dots k-1})}{\Gamma(-n_{2\dots k-1} + n_{0\dots k-1})} \\ & \times \Gamma(-n_{2\dots k-1} + n_{0\dots k-1} + n_{2\dots k} + n_{2\dots l} - n_{0\dots k} - n_{0\dots l}) . \end{aligned} \quad (2.20)$$

The integrals over the other two variables corresponding to longest chains, $n_{0\dots k}$ and $n_{0\dots l}$, can be similarly done. Their further dependence in (2.12) occurs as

$$\Gamma(-n_{0\dots k}) \Gamma(-n_{0\dots l}) \Gamma(-n_{0\dots k-1} + n_{0\dots k} + n_{0\dots l}) , \quad (2.21)$$

Comparing the result to the original four-fold integral, we have

$$\begin{aligned} & \frac{1}{(2\pi i)^4} \iiint_{-i\infty}^{i\infty} dn_{0\dots k} dn_{0\dots l} dn_{1\dots k} dn_{1\dots l} \phi_{0\dots k}^{n_{0\dots k}} \phi_{0\dots l}^{n_{0\dots l}} \phi_{1\dots k}^{n_{1\dots k}} \phi_{1\dots l}^{n_{1\dots l}} \\ & \times \Gamma(-n_{0\dots k}) \Gamma(-n_{0\dots l}) \Gamma(-n_{1\dots k}) \Gamma(-n_{1\dots l}) \Gamma(-n_{0\dots k-1} + n_{0\dots k} + n_{0\dots l}) \\ & \times \Gamma(-n_{1\dots k-1} + n_{1\dots k} + n_{1\dots l}) \Gamma(-n_{2\dots k} + n_{0\dots k} + n_{1\dots k}) \Gamma(-n_{2\dots l} + n_{0\dots l} + n_{1\dots l}) \\ & \times \Gamma(-n_{2\dots k-1} + n_{0\dots k-1} + n_{1\dots k-1} + n_{2\dots k} + n_{2\dots l} - n_{0\dots k} - n_{0\dots l} - n_{1\dots k} - n_{1\dots l}) \end{aligned}$$

$$= \frac{\Gamma(-n_{0\dots k-1})\Gamma(-n_{1\dots k-1})\Gamma(-n_{2\dots k-1}+n_{0\dots k-1}+n_{1\dots k-1})}{\Gamma(-n_{2\dots k-1})} \quad (2.22)$$

$$\times \Gamma(-n_{2\dots k})\Gamma(-n_{2\dots l})\Gamma(-n_{2\dots k-1}+n_{2\dots k}+n_{2\dots l}) ,$$

Examining the right-hand side of (2.22) we see that the last three gamma functions are the appropriate ones for the longest chains in an amplitude with Reggeons 0 and 1 factorized off. As we successively do all the integrals over $n_{ij\dots k}$ involving Reggeons 0 and 1 we build up the full amplitude with Reggeons 0 and 1 factorized off. In the next step we do the $n_{0\dots k-1}$, $n_{1\dots k-1}$, $n_{0\dots l-1}$, and $n_{1\dots l-1}$ integrals. The full dependence on these variables arises from the factor

$$\Gamma(-n_{0\dots k-2}+n_{0\dots k-1}+n_{0\dots l-1})\Gamma(-n_{1\dots k-2}+n_{1\dots k-1}+n_{1\dots l-1}) \quad (2.23a)$$

$$\times \Gamma(-n_{2\dots k-2}+n_{0\dots k-2}+n_{1\dots k-2}+n_{2\dots k-1}+n_{2\dots l-1}-n_{0\dots k-1}-n_{0\dots l-1}-n_{1\dots k-1}-n_{1\dots l-1})$$

from (2.12), the factor

$$\Gamma(-n_{0\dots k-1})\Gamma(-n_{1\dots k-1})\Gamma(-n_{2\dots k-1}+n_{0\dots k-1}+n_{1\dots k-1}) \quad (2.23b)$$

from (2.22), and the factor

$$\Gamma(-n_{0\dots l-1})\Gamma(-n_{1\dots l-1})\Gamma(-n_{2\dots l-1}+n_{0\dots l-1}+n_{1\dots l-1}) \quad (2.23c)$$

which arises by applying the same analysis leading to (2.22) to the chains $n_{0\dots l-1\dots}, n_{1\dots l-1\dots}$ passing through Reggeon $l-1$. This is independent of the former computation since all terms in (2.22) contain Reggeons $k-2$ and $k-1$, whereas all the latter chains contain Reggeons $k-2$ and $l-1$. We see that the three groups of gamma functions in (2.23) produce again the integrand of (2.22) with $k \rightarrow k-1$ and $l \rightarrow l-1$. Taking into account the factors $\Gamma^{-1}(-n_{2\dots k-1})$ and $\Gamma^{-1}(-n_{2\dots l-1})$ originally present we then obtain

$$\frac{\Gamma(-n_{0\dots k-2})\Gamma(-n_{1\dots k-2})\Gamma(-n_{2\dots k-2}+n_{0\dots k-2}+n_{1\dots k-2})}{\Gamma(-n_{2\dots k-2})} \quad (2.24)$$

$$\times \Gamma(-n_{2\dots k-2}+n_{2\dots k-1}+n_{2\dots \ell-1}) .$$

We see that the last factor is another term necessary for the amplitude with Reggeons 0 and 1 factorized off and the first factor is the same as that in (2.22) with $k-1 \rightarrow k-2$. The process can thus clearly be iterated. In so doing we build up the full amplitude with Reggeons 0 and 1 removed times

$$\frac{\Gamma(-n_{02})\Gamma(-n_{12})\Gamma(-\alpha_2+n_{02}+n_{12})}{\Gamma(-\alpha_2)} \quad (2.25)$$

$$\times \Gamma(-n_{01})\Gamma(-\alpha_0+n_{01}+n_{02})\Gamma(-\alpha_1+n_{01}+n_{12})$$

where the first factor arises from (2.22) with $k=3$ and the second from the analog of (2.23a). But (2.25) is just the integrand for the triple-Regge vertex (2.15). Therefore,

$$B_{N+3} = \Gamma(-\alpha_0)\Gamma(-\alpha_1)V(\alpha_0,\alpha_1,\alpha_2;\eta_{01},\eta_{12},\eta_{20})B_{N-1} \quad (2.26)$$

as required.

We have therefore proven factorization for an arbitrary multi-Regge amplitude when all the energies are positive and above their cuts. (Factorization for the special case of only double-Regge vertices was proven in the original paper of Bardakçi and Ruegg¹²⁾.) We emphasize that the crucial fact which allows the factorization proof to go through is $\phi_{ij\dots kl}=1$ so that we can take $F(\phi)=1$ in (2.17). All the $\phi_{ij\dots k}$ will also be unity for all $s_{ij\dots kl}$ negative so factorization will also hold there. However, factorization will not generally hold if some $s_{ij\dots kl}$ are negative and some positive or some above their cuts and some below. For example, if $s_{ij\dots k}$ and $s_{j\dots kl}$ are negative and $s_{ij\dots kl}$ and $s_{j\dots k}$ are positive and above their cuts $\phi_{ij\dots kl}=e^{-2\pi i}$ or if

$s_{ij...kl}$ is positive and below its cut while the other three are positive and above their cuts $\phi_{ij...kl} = e^{2\pi i}$. It is clear from (2.17) that the amplitude has a cut in $\phi_{ij...kl}$ from 0 to $-\infty$ so that it does not have the same value at $\phi_{ij...kl} = 1$ and $\phi_{ij...kl} = e^{\pm 2\pi i}$. The actual value must be obtained by analytic continuation from all $s_{ij...k} \rightarrow -\infty$, e.g., $\phi_{ij...kl} = 1$, where the amplitude was originally defined in (2.4). In the next section we show, however, that certain discontinuities actually also factorize.

3. FACTORIZATION OF DISCONTINUITIES

In this Section we show the factorization of single and multiple discontinuities in the DRM. We first consider single discontinuities, for example $\text{Disc}_{s_{45678}}$ shown in Fig. 1. By the discontinuity in an energy variable we mean what is obtained by circling the normal threshold in that variable, i.e., what gives the sum over intermediate states in that channel. Due to the non-linear constraints (2.2) the singularities in dependent energy variables will propagate into the independent variables. Thus the discontinuity in a dependent variable should actually be thought of as being obtained by taking the independent variables around the cut due to thresholds in that variable but not the cuts due to thresholds of any of the other dependent or independent variables. We can clearly keep proper track of the singularities simply by not explicitly applying the constraints (2.2) on our amplitude (2.12) until the very end since it exhibits explicitly the origin of singularities from normal thresholds in the various energies.

We first identify the basic vertices by studying B_4 , B_5 and B_6 . From $\text{Disc}_{s_{B_4}}$ we obtain the cut Regge propagator

$$\text{---}\text{AAAA}\text{---} = \frac{2\pi i}{\Gamma(\alpha+1)} s^\alpha. \quad (3.1)$$

From $\text{Disc}_{s_{12}B_5}$, using Eqs. (2.3), (2.13) and (3.1), we obtain

$$\begin{aligned} \text{---}\text{AAAA}\text{---}_{t_1} \text{---}\text{AAAA}\text{---}_{t_2} &= \frac{\Gamma(\alpha_1+1)}{2\pi i} \frac{\Gamma(\alpha_2+1)}{2\pi i} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\eta_{12} \\ &\times \Gamma(-\alpha_1+\eta_{12}) \Gamma(-\alpha_2+\eta_{12}) \Gamma(-\eta_{12}) \eta_{12}^{n_{12}} e^{i\pi(2\eta_{12}-\alpha_1-\alpha_2)} 2i \sin\pi(-\eta_{12}). \end{aligned} \quad (3.2)$$

Similarly from $\text{Disc}_{s_1} B_5$, we have

$$\begin{aligned}
 \text{Diagram: } t_1 \text{ (horizontal wavy line), } t_2 \text{ (vertical wavy line), } t_3 \text{ (diagonal wavy line)} &= \frac{\Gamma(d_1+1)}{2\pi i} \frac{1}{\Gamma(-d_2)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn_{12} \Gamma(-d_1+n_{12}) \Gamma(-d_2+n_{12}) \\
 &\times \Gamma(-n_{12}) \eta_{12}^{n_{12}} 2i \sin \pi(-d_1+n_{12}), \quad (3.3)
 \end{aligned}$$

From $\text{Disc}_{s_{12}} B_6$, we obtain

$$\begin{aligned}
 \text{Diagram: } t_1 \text{ (horizontal wavy line), } t_2 \text{ (diagonal wavy line), } t_3 \text{ (vertical wavy line)} &= \frac{\Gamma(d_1+1)}{2\pi i} \frac{\Gamma(d_2+1)}{2\pi i} \frac{1}{\Gamma(-d_3)} \left(\frac{1}{2\pi i}\right)^3 \iiint_{-i\infty}^{i\infty} dn_{12} dn_{23} dn_{31} \\
 &\times \Gamma(-d_1+n_{12}+n_{31}) \Gamma(-d_2+n_{12}+n_{23}) \Gamma(-d_3+n_{23}+n_{31}) \Gamma(-n_{12}) \Gamma(-n_{23}) \Gamma(-n_{31}) \\
 &\times \eta_{12}^{n_{12}} \eta_{23}^{n_{23}} \eta_{31}^{n_{31}} e^{i\pi(2n_{12}+n_{23}+n_{31}-d_1-d_2)} 2i \sin \pi(-n_{12}), \quad (3.4)
 \end{aligned}$$

and from $\text{Disc}_{s_1} B_6$,

$$\begin{aligned}
 \text{Diagram: } t_1 \text{ (horizontal wavy line), } t_2 \text{ (diagonal wavy line), } t_3 \text{ (vertical wavy line)} &= \frac{\Gamma(d_1+1)}{2\pi i} \frac{1}{\Gamma(-d_2)} \frac{1}{\Gamma(-d_3)} \left(\frac{1}{2\pi i}\right)^3 \iiint_{-i\infty}^{i\infty} dn_{12} dn_{23} dn_{31} \\
 &\times \Gamma(-d_1+n_{12}+n_{31}) \Gamma(-d_2+n_{12}+n_{23}) \Gamma(-d_3+n_{23}+n_{31}) \Gamma(-n_{12}) \Gamma(-n_{23}) \Gamma(-n_{31}) \\
 &\times \eta_{12}^{n_{12}} \eta_{23}^{n_{23}} \eta_{31}^{n_{31}} e^{i\pi n_{23}} 2i \sin \pi(-d_1+n_{12}+n_{31}), \quad (3.5)
 \end{aligned}$$

In order to illustrate our procedure and develop the basic identities needed, we first consider the simplest case where factorization needs to be proved, the linear triple-Regge limit of B_6 (Fig. 3). From (2.12) we have

$$B_6 \sim (-s_1)^{d_1} (-s_2)^{d_2} (-s_3)^{d_3} \left(\frac{1}{2\pi i}\right)^3 \iiint_{-i\infty}^{i\infty} dn_{12} dn_{23} dn_{123} \times$$

$$\begin{aligned}
 & \times \left(-\frac{s_{12}}{s_1 s_2}\right)^{n_{12}} \left(-\frac{s_{23}}{s_2 s_3}\right)^{n_{23}} \left(\frac{s_{123} s_2}{s_{12} s_{23}}\right)^{n_{123}} \\
 & \times \Gamma(-d_1 + n_{12}) \Gamma(-d_2 + n_{12} + n_{23} - n_{123}) \Gamma(-d_3 + n_{23}) \\
 & \times \Gamma(-n_{12} + n_{123}) \Gamma(-n_{23} + n_{123}) \Gamma(-n_{123}) ,
 \end{aligned} \tag{3.6}$$

Factorization for $\phi_{123}=1$ is easily checked using (2.17)

$$\frac{1}{2\pi i} \int dn_{123} \dots = \frac{\Gamma(-n_{12}) \Gamma(-n_{23}) \Gamma(-d_2 + n_{12}) \Gamma(-d_2 + n_{23})}{\Gamma(-d_2)}$$

We now compute $\text{Disc}_{s_{123}} B_6$. We see this is proportional to $(1/2i) [\bar{F}(\phi_{123}=1) - \bar{F}(\phi_{123}=e^{2\pi i})]$ since s_{123} rotates through $e^{2\pi i}$ in going below its cut. Since $F(\phi)$ has a cut from 0 to $-\infty$ we must obtain the appropriate analytic continuation of (2.17) through this cut. This can be done by closing the n contour to the left in (2.17). We thus obtain ¹³⁾

$$\begin{aligned}
 & \frac{1}{2\pi i} \int dn \Gamma(a+n) \Gamma(b+n) \Gamma(c-a-b-n) \Gamma(-n) z^n \\
 & = \Gamma(a) \Gamma(c-b) \Gamma(b-a) z^{-a} {}_2F_1(a, c-b; a-b+1; z^{-1}) \\
 & + (a \leftrightarrow b) .
 \end{aligned} \tag{3.7}$$

The hypergeometric functions on the right are now entire on the ring $|z|=1-\epsilon$ and using ¹⁴⁾

$${}_2F_1(a, c-b; a-b+1; 1) = \frac{\Gamma(a-b+1) \Gamma(-c+1)}{\Gamma(-b+1) \Gamma(-c+a+1)}$$

we have

$$F(e^{n2\pi i}) = \frac{\sin \pi(c-a) \sin \pi b e^{-n2\pi i a} - \sin \pi(c-b) \sin \pi a e^{-n2\pi i b}}{\sin \pi c \sin \pi(b-a)} . \tag{3.8}$$

After straightforward algebra we then obtain

$$\frac{1}{2i} [F(1) - F(e^{2\pi i})] = \frac{e^{i\pi(c-a-b)} \sin \pi a \sin \pi b}{\sin \pi c} , \tag{3.9}$$

and from (2.17) and (3.6)

$$\text{Disc}_{S_{123}} B_6 \sim s_1^{\alpha_1} s_2^{\alpha_2} s_3^{\alpha_3} \left(\frac{1}{2\pi i} \right)^2 \int_{-i\infty}^{i\infty} dn_{12} dn_{23} \eta_{12}^{n_{12}} \eta_{23}^{n_{23}} \quad (3.10)$$

$$\times \frac{\Gamma(-d_1+n_{12})\Gamma(-d_2+n_{12})\Gamma(-n_{12})}{\Gamma(-d_2)} \frac{\Gamma(-d_2+n_{23})\Gamma(-d_3+n_{23})\Gamma(-n_{23})}{\Gamma(-d_2)} \\ \times 2i \left[e^{-i\pi(d_1+d_2+d_3)} e^{i\pi n_{12}} e^{i\pi n_{23}} e^{i\pi(-d_2+n_{12}+n_{23})} \frac{\sin\pi(-n_{12})\sin\pi(-n_{23})}{\sin\pi(-d_2)} \right].$$

Using

$$\Gamma(-d)\Gamma(d+1) = \frac{\pi}{\sin\pi(-d)},$$

$\text{Disc}_{S_{123}} B_6$ is easily seen to factorize into the product of two vertices (3.2) and three propagators (3.1).

In a similar manner we compute $\text{Disc}_{S_{12}} B_6$. From (3.6) we see this is proportional to $(1/2i) [\mathbb{F}(1) - e^{2\pi i n_{12}} \mathbb{F}(e^{-2\pi i})]$. Using (3.8) we find

$$\frac{1}{2i} [e^{i\pi a} F(1) - e^{-i\pi a} F(e^{-2\pi i})] = \frac{e^{i\pi b} \sin\pi a \sin\pi(c-b)}{\sin\pi c}, \quad (3.11)$$

and hence

$$\text{Disc}_{S_{12}} B_6 \sim s_1^{\alpha_1} s_2^{\alpha_2} (-s_3)^{\alpha_3} \left(\frac{1}{2\pi i} \right)^2 \int_{-i\infty}^{i\infty} dn_{12} dn_{23} \eta_{12}^{n_{12}} \eta_{23}^{n_{23}} \\ \times \frac{\Gamma(-d_1+n_{12})\Gamma(-d_2+n_{12})\Gamma(-n_{12})}{\Gamma(-d_2)} \frac{\Gamma(-d_2+n_{23})\Gamma(-d_3+n_{23})\Gamma(-n_{23})}{\Gamma(-d_2)} \\ \times 2i e^{-i\pi(d_1+d_2)} e^{2i\pi n_{12}} e^{i\pi n_{23}} e^{-i\pi n_{23}} \frac{\sin\pi(-n_{12})\sin\pi(-d_2+n_{23})}{\sin\pi(-d_2)} \quad (3.12)$$

which clearly factorizes into a vertex (3.2), a vertex (3.3), two-cut propagators and one uncut propagator.

For $\text{Disc}_{S_2} B_6$ we need

$$\frac{1}{2i} [e^{i\pi(c-a-b)} F(1) - e^{-i\pi(c-a-b)} F(e^{2\pi i})] = \frac{\sin\pi(c-a)\sin\pi(c-b)}{\sin\pi c}, \quad (3.13)$$

Factorization is easily checked. Finally $\text{Disc}_{s_1} B_6$ is easily seen to factorize since we can immediately set $\phi_{123} = 1$ and use the factorized form of (3.6).

Armed with the basic identities (3.9), (3.11) and (3.13) we can now prove factorization for an arbitrary multi-Regge amplitude. First consider $\text{Disc}_{s_0 \dots l}$ of the amplitude of Fig. 2. Repeating the steps of section 2 we can do the $n_{1 \dots k}$, $n_{1 \dots l}$, and $n_{0 \dots k}$ integrals. We then have $1/2i [\bar{F}(\phi_{0 \dots l} = 1) - \bar{F}(\phi_{0 \dots l} = e^{2\pi i})]$ where in (2.17),

$$a = -n_{2 \dots l}, b = -n_{0 \dots k-1}, c = -n_{2 \dots k-1}.$$

By (3.9) this leads to an extra factor

$$e^{i\pi(-n_{2 \dots k-1} + n_{2 \dots l} + n_{0 \dots k-1})} \frac{\sin \pi(-n_{2 \dots l}) \sin \pi(-n_{0 \dots k-1})}{\sin \pi(-n_{2 \dots k-1})} \quad (3.14)$$

in (2.22). We next do the integrals over $n_{1 \dots k-1}$, $n_{1 \dots l-1}$, $n_{0 \dots l-1}$ and finally $n_{0 \dots k-1}$. We can rewrite the $n_{0 \dots k-1}$ dependence in (3.14) as $(1/2i) [\bar{F}(\phi_{0 \dots k-1} = 1) - \bar{F}(\phi_{0 \dots k-1} = e^{2\pi i})]$ and apply (3.9) again. This introduces an extra factor

$$e^{i\pi(-n_{2 \dots k-2} + n_{2 \dots k-1} + n_{0 \dots k-2})} \frac{\sin \pi(-n_{2 \dots k-1}) \sin \pi(-n_{0 \dots k-2})}{\sin \pi(-n_{2 \dots k-2})}$$

which combined with (3.14) gives the extra factor in (2.24)

$$e^{i\pi(-n_{2 \dots k-2} + n_{2 \dots l} + n_{0 \dots k-2})} \frac{\sin \pi(-n_{2 \dots l}) \sin \pi(-n_{0 \dots k-2})}{\sin \pi(-n_{2 \dots k-2})}.$$

The procedure therefore iterates and we continue until $k-2=2$ where we have

$$\frac{e^{i\pi(-\alpha_2 + n_{02})} \sin \pi(-n_{02})}{\sin \pi(-\alpha_2)} \times \frac{1}{2i} [1 - e^{2i\pi n_{2 \dots l}}] \quad (3.15)$$

The first factor in (3.15) when combined with the factor $e^{-i\pi(\alpha_0 + \alpha_1)} \times e^{i\pi(n_{01} + n_{12} + n_{02})}$ from the $s_0, s_1, n_{01}, n_{12}, n_{02}$ phases gives precisely the discontinuity of the triple-Regge vertex in (3.4). The second factor gives the discontinuity of the remaining B_{N-1} ,

$$\text{Disc}_{s_{0 \dots l}} B_{N+3} = \frac{2\pi i}{\Gamma(\alpha_0+1)} \frac{2\pi i}{\Gamma(\alpha_2+1)} \left[\text{Disc}_{s_{02}} V(\alpha_0, d_1, d_2; n_{01}, n_{12}, n_{02}) \right] \times \text{Disc}_{s_{2 \dots l}} B_{N-1}. \quad (3.16)$$

The procedure clearly iterates so $\text{Disc}_{s_0 \dots \ell_{N+3}}^{B_{N+3}}$ is the product of appropriate discontinuities of individual vertices.

The quantities $\text{Disc}_{s_2 \dots \ell_{N+3}}^{B_{N+3}}$ and $\text{Disc}_{s_2 \dots k-1}^{B_{N+3}}$ can be computed in a similar manner by repeated use of (3.11) and (3.18), respectively. They factorize into products of the appropriate cut vertices and propagators. This shows factorization for all discontinuities which cut through the triple-(or double-) Regge vertices adjacent to the ends of the longest path through the amplitude. For discontinuities which do not cut these vertices we can first repeatedly apply (2.26) to reduce the problem to such a situation. We have therefore shown factorization for an arbitrary discontinuity of an arbitrary multi-Regge amplitude ^{*}). (Factorization of the total energy discontinuity for the special case of only double-Regge vertices has been shown before ^{5), 15), 16)} as has factorization of the total energy discontinuity of B_8 in the di-triple Regge limit ¹⁵⁾).

We now discuss multiple discontinuities. As usual we define the basic vertices by studying B_5 and B_6 . We easily obtain

$$\begin{aligned}
 & \text{Diagram: A horizontal line with a wavy line above it, labeled } t_1 \text{ and } t_2. \end{aligned}
 = \frac{\Gamma(d_1+1)}{2\pi i} \frac{\Gamma(d_2+1)}{2\pi i} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dn_{12} \Gamma(-d_1+n_{12}) \Gamma(-d_2+n_{12})$$

$$\times \Gamma(-n_{12}) \eta_{12}^{n_{12}} e^{i\pi(-d_2+n_{12})} 2i \sin \pi(-d_1+n_{12}) 2i \sin \pi(-n_{12}), \quad (3.17)$$

$$\begin{aligned}
 & \text{Diagram: A horizontal line with wavy lines above and below it, labeled } t_1, t_2, \text{ and } t_3. \end{aligned}
 = \frac{\Gamma(d_1+1)}{2\pi i} \frac{\Gamma(d_2+1)}{2\pi i} \frac{1}{\Gamma(-d_3)} \left(\frac{1}{2\pi i} \right)^3 \iiint_{-i\infty}^{i\infty} dn_{12} dn_{23} dn_{31}$$

$$\times \left[e^{i\pi(-d_2+n_{12})} 2i \sin \pi(-d_1+n_{12}+n_{31}) 2i \sin \pi(-n_{12}) \right]$$

$$\times \left\{ \eta_{12}^{n_{12}} \eta_{23}^{n_{23}} \eta_{31}^{n_{31}} \Gamma(-d_1+n_{12}+n_{31}) \Gamma(-d_2+n_{12}+n_{23}) \Gamma(-d_3+n_{23}+n_{31}) \right.$$

$$\left. \times \Gamma(-n_{12}) \Gamma(-n_{23}) \Gamma(-n_{31}) \right\}, \quad (3.18)$$

^{*}) We recall that double-Regge vertices can be treated by taking the residue of a pole at $\alpha = 0$ in a triple-Regge vertex.

$$= \prod_{i=1,2,3} \frac{\Gamma(d_i+1)}{2\pi i} \left(\frac{1}{2\pi i}\right)^3 \iiint_{-i\infty}^{i\infty} dn_{12} dn_{23} dn_{31} \left\{ \dots \right\} \\ \times \left[e^{i\pi(-d_1-d_2-d_3+n_{12}+n_{31})} 2i \sin\pi(-d_1+n_{12}+n_{31}) \right. \\ \left. \times \sin\pi(-n_{12}) \sin\pi(-n_{31}) \right], \quad (3.19)$$

$$= \prod_{i=1,2,3} \frac{\Gamma(d_i+1)}{2\pi i} \left(\frac{1}{2\pi i}\right)^3 \iiint_{-i\infty}^{i\infty} dn_{12} dn_{23} dn_{31} \left\{ \dots \right\} \\ \times \left[e^{i\pi(-d_1-d_2-d_3+2n_{12}+2n_{31}+n_{23})} 2i \sin\pi(-n_{12}) 2i \sin\pi(-n_{31}) \right], \quad (3.20)$$

$$= \prod_{i=1,2,3} \frac{\Gamma(d_i+1)}{2\pi i} \left(\frac{1}{2\pi i}\right)^3 \iiint_{-i\infty}^{i\infty} dn_{12} dn_{23} dn_{31} \left\{ \dots \right\} \\ \times \left[e^{i\pi(-d_1-d_2-d_3+2n_{12}+2n_{23}+2n_{31})} 2i \sin\pi(-n_{12}) \right. \\ \left. \times 2i \sin\pi(-n_{23}) 2i \sin\pi(-n_{31}) \right], \quad (3.21)$$

where in (3.19) to (3.21) we have only given the factor in the integrand which differs from (3.18). We note that vertices like that in Fig. 4 which contain cutting lines which cross each other are zero since they correspond to discontinuities in overlapping variables. Such discontinuities are absent in the DRM since discontinuities are asymptotic representations of poles and are expected to be absent quite generally in the physical region due to the Steinmann relation.

To illustrate the factorization of multiple discontinuities in a simple case we consider the multi-Regge limit of Fig. 3. Let us consider first the double discontinuity shown in Fig. 5a, $\text{Disc}_{s_2} \text{Disc}_{s_{123} B_6}$. One easily computes the phase of the integrand of Equation (3.6):

$$e^{-i\pi(d_1+d_2+d_3)} e^{i\pi(n_{12}+n_{23})} \left\{ [F(1) - F(e^{2\pi i})] \right. \\ \left. - e^{2\pi i d_2} e^{-2\pi i(n_{12}+n_{23})} [F(e^{2\pi i}) - F(e^{4\pi i})] \right\}. \quad (3.22)$$

In order to evaluate this we need the quantity $(1/2i) [\bar{F}(e^{2\pi i}) - F(e^{4\pi i})]$ which is straightforwardly obtained from (3.8),

$$\frac{1}{2i} [F(e^{2\pi i}) - F(e^{4\pi i})] = (e^{i\pi(c-a-3b)} + e^{i\pi(c-3a-b)} - e^{i\pi(-c-a-b)}) \times \frac{\sin\pi a \sin\pi b}{\sin\pi c} \quad (3.23)$$

We then find for (3.22),

$$(2i)^3 \left[e^{i\pi(-d_1-d_3+n_{12}+n_{23})} \sin\pi(-d_2+n_{12}) \sin\pi(-d_2+n_{23}) \frac{\sin\pi(-n_{12}) \sin\pi(-n_{23})}{\sin\pi(-d_2)} \right],$$

which when inserted in (3.10) in place of the last factor clearly gives the product of two vertices (3.17) as required. Let us also compute the triple discontinuity of Fig. 5b. This is equal to (3.22) minus the amplitude below the s_{12} cut which is proportional to

$$e^{-i\pi(d_1+d_2+d_3)} e^{i\pi(3n_{12}+n_{23})} \left\{ [F(e^{-2\pi i}) - F(1)] - e^{2\pi i(d_2-n_{12}-n_{23})} [F(1) - F(e^{2\pi i})] \right\} \quad (3.24)$$

Since

$$\left\{ \frac{1}{2i} [F(e^{-2\pi i}) - F(1)] \right\}^* = \frac{1}{2i} [F(1) - F(e^{2\pi i})]$$

and from (3.9) the phase of the latter quantity is $e^{i\pi(-d_2+n_{12}+n_{23})}$, (3.24) vanishes. Therefore, $\text{Disc}_{s_2} \text{Disc}_{s_{123}} B_6 = \text{Disc}_{s_{12}} \text{Disc}_{s_2} \text{Disc}_{s_{123}} B_6$ and we conclude that vertices and propagators are unchanged under further identical cuttings - see Fig. 6. We have computed all multiple discontinuities of B_6 using Eqs. (3.9), (3.11), (3.13) and (3.23) and verified that they factorize when all the remaining energies are positive and above their cuts. It seems quite clear that we can generalize this to an arbitrary multi-Regge amplitude by repeated use of these identities as is the case for the single discontinuity but we have not carried this out in detail. One can also

compute the multiple discontinuities of the six-particle amplitude and verify factorization in a model independent way using the structure of this amplitude obtained in Refs. 1), 6).

4. DISCUSSION

In the preceding section we have argued that an arbitrary multiple discontinuity of an arbitrary multi-Regge limit of the DRM factorizes when all the remaining energies are positive and above their cuts. This result can clearly be trivially generalized to a much wider class of models which have the behavior for a given multi-Regge limit

$$A_{N+3} \sim \prod_{ij} \int dz_{ij} \pi f(t_i, t_j, t_k; z_{ij}, z_{jk}, z_{ki}) \times B_{N+3}(\{d_i\}, \{s_i\}, \{z_{ij} \eta_{ij}\}, \{\phi_{ij\dots k}\}) . \quad (4.1)$$

This is just the DRM with a smearing function for each vertex ^{*)}. As long as the z_{ij} integrals do not create any further singularities in the η_{ij} or $\phi_{ij\dots k}$, the analysis of Sections 2 and 3 applies directly to (4.1). All the results follow with

$$V(t_1, t_2, t_3; \eta_{12}, \eta_{23}, \eta_{31}) = \iiint dz_{12} dz_{23} dz_{31} f(t_1, t_2, t_3; z_{12}, z_{23}, z_{31}) \times V^{\text{DRM}}(t_1, t_2, t_3; z_{12} \eta_{12}, z_{23} \eta_{23}, z_{31} \eta_{31}) , \quad (4.2)$$

and similarly for the cut vertices.

Representations of the form (4.2) were introduced by the Cambridge group in their original study of the double and triple-Regge vertices ¹⁷⁾. They found that the models they studied (ladder model, Gribov's model, and Van Hove-Durand model) indeed had this form. Halliday ⁴⁾ has found that the ladder model also has the form (4.1) for the limit in Fig. 3 and thus our treatment of factorization of discontinuities applies to it. Indeed the factorization of discontinuities is implicitly contained in Ref. 4). Therefore, we believe that the ladder model for Reggeons will generally satisfy (4.1).

^{*)} We have written the form for triple-Regge vertices in (4.1). For double Regge vertices we have factors $f(t_i, t_j; z_{ij})$.

There are rather general reasons to expect that (4.1) holds independently of any models ⁶⁾. For the limit of Fig. 3 we have shown that the absence of simultaneous singularities in overlapping energies in the physical region (Steinman relation) and factorization of the full amplitude imply (4.1). The representation (4.1) clearly satisfies the Steinman relation since the DRM amplitude does. We suspect it is the most general form which satisfies the Steinmann relation and factorization of the full amplitude. If this is so, these two properties then imply the factorization of all discontinuities. While this gives a partial understanding of these remarkable factorization properties, one has the feeling that there is a much deeper reason for them. Perhaps they have their origin in the factorization of amplitudes of definite (complex) helicity as suggested by White ¹⁸⁾.

A couple of comments should be made about the discussion in the above paragraph. First, in applying the Steinman relations we have assumed that powers $(-s)^\alpha$ simply represent cuts in the s plane, e.g., $(-s)^\alpha$ holds throughout the s plane. If singularities other than normal thresholds contribute this may well not be true, since, for example, the nature of anomalous singularities depends on the relative phase of several variables, see, e.g., 19). Such singularities may arise at the collision of Regge poles with Reggeon triangle singularities ²⁰⁾. On the other hand, we do expect (4.1) to hold even when there are non-planar couplings between the Reggeons, see, e.g., 21). In such a case the amplitude must first be decomposed into pieces with only right-hand cuts in energies. Each piece then is expected to have the behaviour (4.1) with the possible absence of non-sense wrong-signature zeroes due to the presence of multiplicative fixed poles ²²⁾.

We thus believe that the factorization properties discussed here hold quite generally for multi-Regge amplitudes. Multi-Regge amplitudes and their discontinuities can thus be expressed as products of basic propagators and vertices. Let us summarize here the rules ¹⁾ for doing this. In order to write them in a somewhat more transparent form it is useful to rewrite (4.2) using (2.15) as follows

$$\begin{aligned}
 V(t_1, t_2, t_3; n_{12}, n_{23}, n_{31}) &= \frac{1}{\Gamma(-d_1)\Gamma(-d_2)\Gamma(-d_3)} \frac{1}{(2\pi i)^3} \\
 &\times \int_{-i\infty}^{i\infty} \int \int dn_{12} dn_{23} dn_{31} \Gamma(-d_1 + n_{12} + n_{31}) \Gamma(-d_2 + n_{12} + n_{23}) \Gamma(-d_3 + n_{31} + n_{23}) \\
 &\times \Gamma(-n_{12}) \Gamma(-n_{23}) \Gamma(-n_{31}) (-\eta_{12})^{n_{12}} (-\eta_{23})^{n_{23}} (-\eta_{31})^{n_{31}} \\
 &\times \beta(n_{12}, n_{23}, n_{31}; t_1, t_2, t_3) ,
 \end{aligned} \tag{4.3}$$

where

$$\beta(n_{12}, n_{23}, n_{31}; t_1, t_2, t_3) = \iiint dz_{12} dz_{23} dz_{31} f(t_1, t_2, t_3; z_{12}, z_{23}, z_{31}) z_{12}^{\eta_{12}} z_{23}^{\eta_{23}} z_{31}^{\eta_{31}}, \quad (4.4)$$

and, similarly, for the double-Regge vertex.

The condition that the z_{ij} integrals generate no further singularities in the η_{ij} is equivalent to the condition that β have no singularities in the n_{ij} ^{*)}. If the n_{ij} contours are closed to the left we obtain ¹⁰⁾

$$V(t_1, t_2; \eta_{12}) = (-\eta_{12})^{d_1} V_1(t_1, t_2; \eta_{12}) + (-\eta_{12})^{d_2} V_2(t_1, t_2; \eta_{12}), \quad (4.5)$$

$$\begin{aligned} V(t_1, t_2, t_3; \eta_{12}, \eta_{23}, \eta_{31}) &= (-\eta_{12})^{d_2} (-\eta_{31})^{d_3} V_{23}(t_1, t_2, t_3; \eta_{12}, \eta_{23}, \eta_{31}) \\ &\quad + (123 \rightarrow 231) + (123 \rightarrow 312) \\ &\quad + (-\eta_{12})^{\frac{1}{2}(d_1+d_2-d_3)} (-\eta_{23})^{\frac{1}{2}(d_2+d_3-d_1)} (-\eta_{31})^{\frac{1}{2}(d_3+d_1-d_2)} \\ &\quad \times V_{123}(t_1, t_2, t_3; \eta_{12}, \eta_{23}, \eta_{31}), \end{aligned} \quad (4.6)$$

where

$$V_i = \frac{1}{\Gamma(-d_i)\Gamma(-d_j)} \sum_{m=0}^{\infty} \Gamma(-d_i+m)\Gamma(-d_j+d_i-m) \frac{\eta_{ij}^{-m}}{m!} \times \beta(d_i-m; t_i, t_j),$$

*) Actually, singularities to the right of the n_{ij} contours which correspond to multiplicative nonsense wrong-signature fixed poles are allowed ²²⁾.

$$V_{ij} = \frac{1}{\Gamma(-d_i)\Gamma(-d_j)\Gamma(-d_k)} \sum_{m,n,p=0}^{\infty} \Gamma(-d_i+m+p)\Gamma(-d_j+n+p)\Gamma(-d_k+d_i+d_j-m-n-2p) \\ \times \frac{1}{m!n!p!} \eta_{ki}^{-m} \eta_{jk}^{-n} \left(\frac{\eta_{ki}\eta_{jk}}{\eta_{ij}} \right)^{-p} \beta(p, d_j-n-p, d_i-m-p; t_i, t_j, t_k),$$

$$V_{ijk} = \frac{1}{2\Gamma(-d_i)\Gamma(-d_j)\Gamma(-d_k)} \sum_{m,n,p=0}^{\infty} \Gamma\left[\frac{1}{2}(d_i-d_j-d_k-m+n+p)\right] \\ \times \Gamma\left[\frac{1}{2}(d_j-d_k-d_i-n+p+m)\right] \Gamma\left[\frac{1}{2}(d_k-d_i-d_j-p+m+n)\right] \\ \times \frac{(-1)^{m+n+p} e^{-i\pi(m+n+p)}}{m!n!p!} \left(\frac{\eta_{ki}\eta_{ij}}{\eta_{jk}} \right)^{-m/2} \left(\frac{\eta_{ij}\eta_{jk}}{\eta_{ki}} \right)^{-n/2} \left(\frac{\eta_{jk}\eta_{ki}}{\eta_{ij}} \right)^{-p/2} \\ \times \beta\left[\frac{1}{2}(d_i+d_j-d_k-m-n+p), \frac{1}{2}(d_j+d_k-d_i-n-p+m), \frac{1}{2}(d_k+d_i-d_j-p-m+n); t_i, t_j, t_k\right]. \quad (4.7)$$

Similar expressions are easily derived for the cut vertices. We thus obtain the basic propagators and vertices shown in Fig. 7 (where the relations like Fig. 6 are always to be remembered ^{*}). It is amusing to note that some of the vertices are in fact equal [f) and (n); (j), (ℓ) and (m)] although this is not immediately evident from the integral representations of Section 3. A cut coming out of a vertex selects just one of the terms V and further cuts cannot change it.

The explicit forms (4.7) are appropriate for amplitudes with all scalar particles and thus all trajectories of natural parity. They also have the behaviour at $t=0$ corresponding to Toller quantum number $M=0$ trajectories. To treat other cases, the type of analysis discussed here should be applied to appropriate invariant amplitudes ²²⁾. For an explicit example of factorization with double-Regge vertices involving change of parity sequence, see Ref. 23).

We note that factorization has not actually been shown in the DRM [or thus for (4.1)] for signaturized Reggeons in the full amplitude for an arbitrary multi-Regge limit or for the case where the uncut Reggeons are signaturized. It has been shown only for the full amplitude with only

^{*}) We have corrected some misprints in Ref. 1) and changed our definitions slightly.

double-Regge vertices ⁵⁾ and for all cuttings of Fig. 3 ¹⁾. We certainly expect it to hold, however. We also expect our rules can be generalized to the case where there are several additional particles of each vertex by combining the techniques of Ref. 24) with those given here.

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FIGURE CAPTIONS

Figure 1 :

Example of a multi-Regge amplitude.

Figure 2 :

Diagram showing the longest path through an arbitrary multi-Regge amplitude.

Figure 3 :

Linear triple-Regge limit.

Figure 4 :

A vanishing cut vertex.

Figure 5 :

- a) A double discontinuity of B_6 .
- b) A triple discontinuity of B_6 .

Figure 6 :

Equality of propagator and vertices under several identical cuttings.

Figure 7 :

Basic propagators and vertices. $\xi_i = e^{-i\pi\alpha_i} + \tau_i$,

$$\xi_{ij} = e^{-i\pi(\alpha_i + \alpha_j)} + \tau_i \tau_j, \quad \xi_{ijk} = e^{-i\pi(\alpha_i + \alpha_j + \alpha_k)} + \tau_i \tau_j \tau_k,$$

where τ_i is the signature of $\alpha_i(t_i)$. The dependence of the functions $\bar{V}_i = \eta_{ij}^{\alpha_i} V_i$, $\bar{V}_{ij} = \eta_{ki}^{\alpha_i} \eta_{jk}^{\alpha_j} V_{ij}$, etc.

on the t_i and η_{ij} have been suppressed. In the expansion (4.7) for $\bar{V}_{ijk}$ the following factors must be inserted:

for (i); $[1 + (-1)^{m+n+p} \tau_i \tau_j \tau_k]$,

for (k)

$$\frac{\sin \frac{\pi}{2}(\alpha_i + \alpha_j - \alpha_k - m - n + p)}{\sin \frac{\pi}{2}(\alpha_i + \alpha_j - \alpha_k)},$$

Figure 1

Figure 2

Figure 3

Figure 4

a)

b)

Figure 5

$$\text{---} \text{wavy line} \text{---} = \text{---} \text{wavy line} \text{---} = \text{---} \text{wavy line} \text{---} = \dots$$

$$\text{---} \text{wavy line} \text{---} \text{---} \text{dashed line} \text{---} = \text{---} \text{wavy line} \text{---} \text{---} \text{dashed line} \text{---} = \dots$$

$$\text{---} \text{wavy line} \text{---} \text{---} \text{dashed line} \text{---} = \text{---} \text{wavy line} \text{---} \text{---} \text{dashed line} \text{---} = \text{---} \text{wavy line} \text{---} \text{---} \text{dashed line} \text{---} = \dots$$

Figure 6

$$a) \quad t_i \text{ (wavy line) } = \xi_i \Gamma(-\alpha_i) s^{\alpha_i}$$

$$b) \quad t_i \text{ (double wavy line) } = \frac{2\pi i}{\Gamma(\alpha_i + 1)} s^{\alpha_i}$$

$$c) \quad t_i \text{ (wavy line with two outgoing lines) } = R(t_i)$$

$$d) \quad t_i \text{ (double wavy line with two outgoing lines) } = R(t_i)$$

$$e) \quad t_i \text{ (wavy line with a vertical line) } t_j = R(t_i, t_j; \eta_{ij}) = \xi_j^{-1} \xi_{ji} \bar{v}_i + \xi_i^{-1} \xi_{ij} \bar{v}_j$$

$$f) \quad t_i \text{ (double wavy line with a vertical line and a dashed line) } t_j = \frac{\sin \pi(\alpha_i - \alpha_j)}{\sin \pi \alpha_i} \bar{v}_j$$

$$g) \quad t_i \text{ (double wavy line with a vertical line) } t_j = -\frac{e^{i\pi(\alpha_i - \alpha_j)}}{2i \sin \pi \alpha_j} \bar{v}_i - \frac{e^{i\pi(\alpha_j - \alpha_i)}}{2i \sin \pi \alpha_i} \bar{v}_j$$

$$h) \quad t_i \text{ (double wavy line with a vertical line and a dashed line) } t_j = \frac{\sin \pi(\alpha_i - \alpha_j)}{\sin \pi \alpha_i} \bar{v}_j$$

Figure 7 - Part 1

i) = $R(t_i, t_j, t_k; \eta_{ij}, \eta_{jk}, \eta_{ki}) =$
 $\xi_k^{-1} \xi_{kij} \bar{v}_{ij} + \xi_i^{-1} \xi_{ijk} \bar{v}_{jk} + \xi_j^{-1} \xi_{jki} \bar{v}_{ki} +$
 $+ \xi_i^{-1} \xi_j^{-1} \xi_k^{-1} e^{-i\frac{\pi}{2}(\alpha_i + \alpha_j + \alpha_k)} \left(1 + \tau_i e^{i\pi\alpha_i} + \tau_j e^{i\pi\alpha_j} + \tau_k e^{i\pi\alpha_k} \right) \bar{v}_{ijk}$

j) = $\frac{\sin\pi(\alpha_i - \alpha_j - \alpha_k)}{\sin\pi\alpha_i} \bar{v}_{jk}$

k) = $-\frac{e^{i\pi(\alpha_j + \alpha_k - \alpha_i)}}{2i\sin\pi\alpha_i} \bar{v}_{jk} - \frac{e^{i\pi(\alpha_k + \alpha_i - \alpha_j)}}{2i\sin\pi\alpha_j} \bar{v}_{ki} - \frac{\sin\frac{\pi}{2}(\alpha_i + \alpha_j - \alpha_k)}{2i\sin\pi\alpha_i \sin\pi\alpha_j} \bar{v}_{ijk}$

l) = $\frac{\sin\pi(\alpha_i - \alpha_j - \alpha_k)}{\sin\pi\alpha_i} \bar{v}_{jk}$

m) = $\frac{\sin\pi(\alpha_i - \alpha_j - \alpha_k)}{\sin\pi\alpha_i} \bar{v}_{jk}$

n) = $-\frac{e^{i\pi(\alpha_j + \alpha_k - \alpha_i)}}{2i\sin\pi\alpha_i} \bar{v}_{jk} + \frac{e^{i\frac{\pi}{2}(\alpha_i - \alpha_j - \alpha_k)} \sin\frac{\pi}{2}(\alpha_i + \alpha_j - \alpha_k) \sin\frac{\pi}{2}(\alpha_k + \alpha_i - \alpha_j)}{2i\sin\pi\alpha_i \sin\pi\alpha_j \sin\pi\alpha_k} \bar{v}_{ijk}$

o) = $\frac{\sin\frac{\pi}{2}(\alpha_i + \alpha_j - \alpha_k) \sin\frac{\pi}{2}(\alpha_j + \alpha_k - \alpha_i) \sin\frac{\pi}{2}(\alpha_k + \alpha_i - \alpha_j)}{\sin\pi\alpha_i \sin\pi\alpha_j \sin\pi\alpha_k} \bar{v}_{ijk}$

Figure 7 - Part 2