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QUARK DYNAMICS AND STRONG MESON DECAYS

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A B S T R A C T

The strong decays of meson resonances are treated in a dynamical quark model. The model is formulated in the framework of general field theory and therefore all calculations are fully relativistic covariant. Spectrum and wave functions are derived from a Bethe-Salpeter equation describing the binding of heavy quarks by a smooth, very strong interaction. The mesonic vertices are calculated with help of these BS amplitudes in triangle graph approximation, thereby guaranteeing a symmetric treatment of all mesons involved in the process. A particular spin dependence of the interaction has the consequence that the super-strong quark binding forces lead to mesonic forces of moderate strength only (saturation of quark forces !).

The applications refer to the decays of vector mesons, tensor mesons, scalar mesons into pseudoscalars and vectors and are extended to resonances in the R region. In particular, we include the decays into two pions of the radial excited $\rho'(1600)$. The helicity structure of the decays of the axial vector meson is discussed.

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1. - OUTLINE OF THE MODEL

Strong decays of mesons and baryons and their coupling constants offer good tests of symmetry schemes and dynamical models for the hadrons. In this paper we propose and discuss a model for the decays of the meson resonances resulting from the dynamical pattern of a relativistic quark field theory ¹⁾. This scheme is based on the assumption of heavy quarks. They have to be bound by very strong smooth forces in order to lead to a meson spectrum with excitations lying on linear Regge trajectories ²⁾. In such a dynamical model, the interaction between mesons, measured by the meson coupling constants, results from these superstrong forces. Therefore, the main dynamical problem is to explain how these superstrong forces lead to the relatively moderate mesonic forces, usually called the strong interaction. A model which shows this feature we call saturating. We remark that saturation of superstrong forces has also to be assumed in order to explain the absence of other bound states than those of $q\bar{q}$ and qqq ³⁾. It is not known whether these two kinds of saturation properties are related to each other.

In quark field theory, meson bound states are described by Bethe-Salpeter amplitudes which are solutions of the fermion-antifermion Bethe-Salpeter equation ⁴⁾. There are different types of spin structure of the interaction kernel which yield a meson spectrum with quark-spin-singlets and triplets familiar from the non-relativistic quark model ^{5), 1)}. However, the Bethe-Salpeter amplitudes resulting from these different interactions differ in their form with respect to the relativistic Dirac spinor structure.

Guided by the successful selection rules derivable from the non-relativistic quark model ⁶⁾ or the quark duality diagrams ⁷⁾, we calculate the hadronic vertices in tree graph approximation with help of these Bethe-Salpeter amplitudes $\chi(k, P)$ ⁸⁾

$$\mathcal{M} = \text{diagram} + G, \quad P_1 + P_2 + P_3 = 0.$$

The diagram shows a vertex with three external lines. The leftmost line is a double line with momentum P_1 and an arrow pointing right. The top-right line is a single line with momentum P_2 and an arrow pointing right. The bottom-right line is a single line with momentum P_3 and an arrow pointing left. The internal lines form a loop structure connecting these external lines.

$$= SU_3 \cdot (2\pi)^{-3/2} \cdot i \int d^4k \operatorname{Tr} \left[\chi_{(k, P)}^{(1)} \left(k - \frac{P_1}{2} - m \right) \cdot \chi_{(k + \frac{P_2}{2}, P_2)}^{(2)} \left(k - \frac{P_2 - P_3}{2} - m \right) \cdot \chi_{(k - \frac{P_2}{2}, P_3)}^{(3)} \cdot \left(k + \frac{P_3}{2} - m \right) \right]. \quad (1)$$

We describe now the main features of this saturating model. The masses and the wave functions of the mesons are determined by solution of the BS equation with a smooth $-\gamma_5 \times \gamma_5$ type interaction kernel. Since after Wick rotation the kernel may be approximated by harmonic forces the bound states are characterized by the quantum numbers of the four-dimensional oscillator. Besides the orbital angular momentum l , a state is characterized by an $O(4)$ quantum number n_b , $n_b - l$ ($=$ even) counting the nodes of the oscillations in relative energy, and a hyperradial quantum number r counting the nodes of the oscillations in the relative four-momentum k^2 . The spin of the mesons results from a coupling of quark spins to the orbital angular momentum. The mass formula is for triplet states :

$$M^2 = 4(-\delta_V + 2N_b \sqrt{\beta}) = M_{o,t}^2 + 8\sqrt{\beta} N_b \quad (2.t)$$

$$N_b = n_b + 2r + 2 ; \quad n_b = 0, 1, 2, \dots ; \quad n_b - l = \text{even} ;$$

$$r = 0, 1, 2, \dots ; \quad j = l+1, l, l-1$$

and for singlet states :

$$M^2 = 4(-\delta_A + 2N \sqrt{\beta}) = M_{o,s}^2 + 8\sqrt{\beta} N \quad (2.s)$$

$$N = n + 2r + 2 ; \quad n = 0, 1, 2, \dots ; \quad n - j = \text{even}$$

$$r = 0, 1, 2, \dots ;$$

where δ and $\sqrt{\beta}$ are potential parameters, $\delta + m^2$ measuring the depth and β the curvature of the four-dimensional potential. The rotational excitations of the non-relativistic quark model lying on the leading Regge trajectories correspond to $n_b = l$, $r = 0$, the corresponding Regge slope being $8\sqrt{\beta}$. In addition to the radial excitations of the non-relativistic oscillator, our spectrum contains the states where the relativistic degree of freedom is excited.

The BS amplitudes for $-\gamma_5 \times \gamma_5$ interaction are of vector, axial-vector type in leading order in the quark mass. The Wick rotated, normalized amplitudes for the ground state mesons are of the form :

- pseudoscalar mesons

$$\chi_b(k, p) = \frac{\gamma_5 \not{p}}{M} \cdot \frac{4\pi}{\sqrt{\beta}} \cdot e^{-k^2/2\sqrt{\beta}} \cdot T, \quad (3.s)$$

- vector mesons

$$\chi_0(k, p) = \epsilon_{33} \cdot \frac{4\pi}{\sqrt{\beta}} \cdot e^{-k_E^2/2\sqrt{\beta}} \cdot T ; \quad k_E^2 = k_x^2 + k^2 \quad (3.t)$$

(e_{μ}^{s3} is the covariant polarization vector for spin one ; T denotes the familiar $3 \times \bar{3}$ quark wave functions of the mesons). The path of the k_0 integration in the decay matrix element, Eq. (1), may be deformed into the Euclidean region. Therefore we need only the Wick rotated BS amplitudes. The large components of the amplitudes of the excited states contain $O(4)$ spherical functions ⁹⁾ $y_{\ell\ell_3}^n(\Omega_4)$ and hyperradial functions :

$$R_{n,r}(k_E^2) = (2\pi)^2 \cdot \sqrt{\frac{2}{\beta}} \cdot \sqrt{\frac{\tau!}{(n+\tau+1)!}} \cdot \left(\frac{k_E}{\sqrt{\beta}}\right)^n \cdot L_{\tau}^{n+1}\left(\frac{k_E^2}{\sqrt{\beta}}\right) \cdot e^{-k_E^2/2\sqrt{\beta}}$$

and are given by ^{1), 5)}

$$\chi_0^{\Delta}(k, p) = \frac{\gamma_5 \gamma}{M} \cdot y_{jj_3}^n \cdot R_{n,r}(k_E^2) \\ (n = 0, 1, 2, \dots ; n-j \text{ even} ; r = 0, 1, 2, \dots) \quad (4.s)$$

$$\chi_0^t(k, p) = \epsilon_{33} \cdot y_{\ell\ell_3}^{n_\ell} \cdot (1s_3, \ell\ell_3 | jj_3) \cdot R_{n,r}(k_E^2) \\ (n_\ell = 0, 1, 2, \dots ; n_\ell - \ell = \text{even} ; r = 0, 1, 2, \dots \\ j = \ell+1, \ell, \ell-1). \quad (4.t)$$

The wave functions (4) are eigenfunctions of the orbital angular momentum $\vec{l}^2$, ℓ is equal to that of the non-relativistic classification of the states. We observe that the amplitudes do not contain the quark mass m . This is typical for strong binding in smooth potentials. The intuitive picture is that the quarks are oscillating around the equilibrium position ($\alpha = -m^2$), the bottom of the potential.

Now let us consider the calculation of the hadronic coupling constants according to Eq. (1). If we insert the BS amplitudes into this expression and look for the leading power in the quark mass, we see that the matrix element would be proportional to m^3 if the corresponding coefficient does not vanish. Of course, a saturating model is characterized by the vanishing of this coefficient together with the quadratic and linear

ones. Consequently, in order to perform a consistent calculation, the BS amplitudes have to be determined including all terms of order m^{-3} . For a fermion-antifermion BS equation with convolution type kernels approximated by an oscillator potential such a calculation can be carried through. It leads to an expression for $\mathcal{M}$ of the form

$$\begin{aligned} \mathcal{M} = m^3 \text{Tr} SU_3 \cdot (2\pi)^{-9/2} \cdot i \int d^4k \text{Tr} [& \Delta\chi^{(1)}(k, p_1) \chi_0^{(2)}(k + \frac{p_2}{2}, p_2) \chi_0^{(3)}(k - \frac{p_2}{2}, p_3) \\ & + \chi_0^{(1)}(k, p_1) \Delta\chi^{(2)}(k + \frac{p_2}{2}, p_2) \chi_0^{(3)}(k - \frac{p_2}{2}, p_3) \\ & + \chi_0^{(1)}(k, p_1) \chi_0^{(2)}(k + \frac{p_2}{2}, p_2) \Delta\chi^{(3)}(k - \frac{p_2}{2}, p_3)], \end{aligned} \quad (5.a)$$

where the χ_0 are given by the expressions (4.s), (4.t) and the $\Delta\chi$ derive from them by the operation

$$\Delta\chi = \frac{\varepsilon}{8m^3} [\not{p}, \chi] + \frac{\varepsilon'}{4m^3} \{ \not{k}, \chi_0 \} + \frac{\beta}{2m^3} \{ \not{p}, \chi \}, \quad (5.b)$$

$\varepsilon, \varepsilon'$ depend on the finer details of the interaction and are of order $\sqrt{\beta}$,

One might question whether the simple convolution kernels are sufficient to describe the physical situation to such a high accuracy. For instance, spin orbit splitting might require more complicated small terms in the interaction. Furthermore SU_3 breaking effects were not considered. All such modifications might show up in the parameters $\varepsilon, \varepsilon'$. Therefore, on this level, we consider the parameters $\varepsilon, \varepsilon'$ to be determined from experiment.

In Section 2 we perform an extensive comparison of the results following from Eqs. (5) with the experimental decay widths of the mesons. We postpone the formal derivation of the saturating model to Section 3. But before that we want to compare our model with the essential features of other approaches.

In the non-relativistic quark model decays into pions and vector mesons were calculated from matrix elements of one elementary meson field between two non-relativistic $q\bar{q}$ wave functions ^{6), 10)}. Relativistic wave functions were constructed with the kinematical methods of the $\tilde{U}(12)$ algebra ¹¹⁾. These wave functions have a Dirac spinor structure differing from ours. The decay into two particles is a collinear process. This

reduces $\tilde{U}(12)$ symmetry to $SU(6)_W$ ¹²⁾, which allows a simple group theoretical derivation of relations between coupling constants of mesons and baryons belonging to the same $\tilde{U}(12)$ multiplet. It was found that some of the predictions of $SU(6)_W$ symmetry are violated, in particular in the decay $B \rightarrow \omega + \pi$ ¹³⁾. This led to the discussion of broken $SU(6)_W$ ¹⁴⁾. Since our wave functions are not of $\tilde{U}(12)$ type, $SU(6)_W$ breaking terms appear naturally in our scheme. The oscillator degrees of freedom in a relativistic model were introduced by Feynman, Kislinger and Ravndal ¹⁵⁾. Their model is not considered as an approximation to field theory. Consequently, the couplings are calculated like in the non-relativistic quark model as transition matrix elements of one elementary field between two $\tilde{U}(12)$ bound state wave functions. This leads to different values according to which of the particles is treated as elementary. An attempt to find a more symmetric treatment is the quark pair creation model ¹⁶⁾ in which the hadronic decays are described by spontaneous pair creation matrix elements ¹⁷⁾ related to those appearing in the leptonic decays ¹⁸⁾. This intuitive treatment is not fully relativistic. Another attempt to embed the non-relativistic quark model into a relativistic scheme was recently proposed by Gell-Mann ¹⁹⁾: the "constituent" quarks of the non-relativistic model should be related to the "current" quarks of the relativistic model by a transformation ("Melosh transformation" ²⁰⁾). Modifications of the non-relativistic calculations of hadronic vertices due to the Melosh transformation were discussed by several authors ²¹⁾. A symmetrical and fully relativistic covariant treatment of all particles is a natural feature of our field theoretical scheme. Furthermore, the same dynamical patterns govern the determination of the spectrum, the matrix elements of the currents and the hadronic coupling constants.

2. - PHENOMENOLOGICAL APPLICATIONS

2.1. - The decays into two pseudoscalar mesons

Decays into two pseudoscalar mesons occur from meson resonances with natural parity $[\Pi = (-1)^j]$, only. In quark language these are the quark spin triplet states with $j = \ell \pm 1$. The BS amplitudes for these triplet states are given by Eq. (4.t), the small components $\Delta\chi$ by Eq. (49) and those of the pseudoscalar mesons by Eq. (46). In order to calculate the decay matrix elements $\mathcal{M}$ we have to insert these amplitudes in the expression (5.a) for $\mathcal{M}$

$$\mathcal{M} = SU_3 \cdot \left(\frac{4\pi}{\beta}\right)^3 \cdot \sqrt{4\pi} \cdot \sqrt{\frac{2^\ell}{(2\ell+1)!!}} \cdot (2\pi)^{-\frac{q_0}{2}} \cdot i \int dk \cdot e^{[k^2/2\beta + (k+\frac{p_2}{2})^2/2\beta + (k-\frac{p_2}{2})^2/2\beta]} \cdot \left(\frac{|\vec{k}|}{4\beta}\right)^\ell \cdot \text{Tr}^D \quad (6)$$

with

$$\begin{aligned} \text{Tr}^D = & \text{Tr} \left\{ [-\varepsilon_T^V \cdot \phi_1 \phi_2 \cdot Y_{\ell\ell_3} (1s_3, \ell_3/jj_3) + 2\hat{\varepsilon}_S^V \cdot |\vec{k}|(\ell-j) \sqrt{\frac{j+\ell+1}{2(2j+1)}} Y_{jj_3} - 4\beta \sqrt{\frac{\ell}{2\ell+1}} \delta_{j,\ell-1} \frac{2\ell+1}{|\vec{k}|} Y_{jj_3}] \cdot \gamma_5 \frac{p_2}{M_2} \cdot \gamma_5 \frac{p_3}{M_3} \right. \\ & + \phi_1 Y_{\ell\ell_3} (1s_3, \ell_3/jj_3) [-\varepsilon_P^A \cdot \gamma_5 M_2 + \hat{\varepsilon}_T^A \cdot \gamma_5 \frac{[p_2, k + \frac{p_2}{2}]}{M_2}] \gamma_5 \frac{p_3}{M_3} \\ & \left. + \phi_2 Y_{\ell\ell_3} (1s_3, \ell_3/jj_3) \gamma_5 \frac{p_2}{M_2} [-\varepsilon_P^A \cdot \gamma_5 M_3 + \hat{\varepsilon}_T^A \cdot \gamma_5 \frac{[p_3, k - \frac{p_3}{2}]}{M_3}] \right\} \\ = & \frac{1}{M_2 M_3} \cdot \left\{ \varepsilon_T^V \cdot e p_2 \cdot M_1^2 \cdot Y_{\ell\ell_3} (1s_3, \ell_3/jj_3) - 2\hat{\varepsilon}_S^V \cdot |\vec{k}|(\ell-j) \sqrt{\frac{j+\ell+1}{2(2j+1)}} \cdot Y_{jj_3} \cdot p_2 p_3 \right. \\ & + 4\beta \sqrt{\frac{\ell}{2\ell+1}} \cdot \delta_{j,\ell-1} \cdot \frac{2\ell+1}{|\vec{k}|} \cdot Y_{jj_3} \cdot p_2 p_3 \\ & + Y_{\ell\ell_3} (1s_3, \ell_3/jj_3) [\varepsilon_P^A \cdot e p_2 \cdot (M_2^2 + M_3^2) \\ & \left. + \hat{\varepsilon}_T^A \cdot (-4ek \cdot p_2 p_3 + e p_2 \cdot M_1^2 - 2ep_2 \cdot qk)] \right\}. \end{aligned}$$

For the calculation of this integral, we deform the path of the k_0 integral into the Euclidean region and obtain, for example :

$$i \int_{-\infty}^{+\infty} dk_0 \cdot e^{\frac{3}{2} k_0^2/\beta} \cdot e^{-k_0 q_0/2\beta} = \sqrt{\frac{2\pi}{3}} \cdot 4\beta \cdot e^{-q_0^2/24\beta} \quad (7.a)$$

The d^3k integrals are of the type

$$\begin{aligned} & \int d^3k \, e^{-3/2 \frac{\vec{k}^2}{\sqrt{\beta}}} \cdot e^{\vec{k}\vec{q}/2\sqrt{\beta}} \cdot |\vec{k}|^{L+s} Y_{LL_3}(\hat{k}) \\ &= \pi^{3/2} \cdot \left(\frac{2\sqrt{\beta}}{\sqrt{6}} \right)^{s+3} \cdot \left(\frac{|\vec{q}|}{6} \right)^L \cdot Y_{LL_3}(\hat{q}) \cdot \left(\frac{s}{2} \right)! \cdot e^{\vec{q}^2/24\sqrt{\beta}} \cdot L_{\frac{s}{2}}^{L+1/2} \left(-\frac{\vec{q}^2}{24\sqrt{\beta}} \right). \end{aligned} \quad (7.b)$$

After performing the Clebsch-Gordan gymnastics for the angular momenta and using the mass formula (2.t), we get the expression :

$$\begin{aligned} \mathcal{M} = & \mathcal{M}_3 \cdot \frac{16\pi\sqrt{2}}{9} \cdot e^{\ell/3} \cdot \sqrt{\frac{2^\ell}{(2\ell+1)!!}} \cdot \left(\frac{|\vec{q}|}{6\sqrt{\beta}} \right)^\ell \cdot Y_{jj_3} \cdot \\ & \cdot \left\{ (l-j) \sqrt{\frac{j+2\ell+1}{2(2j+1)}} \cdot |\vec{q}| \left[2\ell \frac{\sqrt{\beta}}{M_0^2} \cdot \varepsilon_0 + 1/3 \varepsilon_1 \right] \right. \\ & \left. + \delta_{j,\ell-1} \cdot (2\ell+1) \cdot \sqrt{\frac{j+1}{2j+1}} \cdot \frac{\sqrt{\beta}}{|\vec{q}|} \cdot p_2 p_3 \cdot \varepsilon_2 \right\} ; \quad q_\mu = (p_2 - p_3)_\mu \end{aligned} \quad (8)$$

with

$$\begin{aligned} \varepsilon_0 &= \left(\varepsilon_T^V - 1/3 \hat{\varepsilon}_S^V + 7/6 \hat{\varepsilon}_T^A \right) / \sqrt{\beta} \cdot e^{M_0^2/8\sqrt{\beta}} \\ \varepsilon_1 &= \left(3/2 \varepsilon_T^V + 1/2 \hat{\varepsilon}_S^V + \hat{\varepsilon}_T^A + 3\varepsilon_P^A \right) / \sqrt{\beta} \cdot e^{M_0^2/8\sqrt{\beta}} \\ \varepsilon_2 &= \left(-\hat{\varepsilon}_S^V - 2\hat{\varepsilon}_T^A + 6\sqrt{\beta} \right) / \sqrt{\beta} \cdot e^{M_0^2/8\sqrt{\beta}} \end{aligned} \quad (9)$$

This matrix element shows the following features :

- (i) as a consequence of our dynamical calculation, the range parameter $r_0 = 1/3 \beta^{1/4}$ is related to the Regge slope of the mass spectrum and has the value $r_0 \approx 0.8 \text{ (GeV)}^{-1}$;
- (ii) the factor $e^{\ell/3}$ represents a form factor effect resulting from the Gaussian radial wave functions ;
- (iii) together with the angular momentum factor $\sqrt{2^\ell/(2\ell+1)!!} (p_{\text{CM}} \cdot r_0)^\ell \cdot p_{\text{CM}}$ the factor $e^{\ell/3}$ yields a coupling of the orbital excited states (Regge recurrences) which decreases much slower than that resulting from models with square-well potentials and is comparable with the results of the dual resonance model ²²⁾.

Because of saturation, the decay matrix element $\mathcal{M}$ depends on the small components of the BS amplitudes and their parameters. These do not enter into the global structure of the meson spectrum [Eq. (2)]. As a consequence, we have to determine them from the decay widths. The parameter combination ε_0 measures the admixture of the term $2\sqrt{\beta}$, which would lead to an anomalous barrier factor. Therefore we put it equal to zero $\varepsilon_0 = 0$.

The matrix element for the decays of the angular momentum excited states with $j = l+1$ is now given by

$$M = S_{l3} \cdot \frac{16\pi\sqrt{2}}{9} \cdot e^{l/3} \cdot \sqrt{\frac{2^l}{(2l+1)!!}} \cdot \left(\frac{|q|}{4\sqrt{\beta}}\right)^l \cdot Y_{ji3}(q) \cdot \sqrt{\frac{j}{2j+1}} \cdot |q| \cdot \frac{\epsilon_1}{3} \quad (10)$$

This matrix element depends on ϵ_1 , we fix this parameter by comparison with the $\rho\pi\pi$ coupling constant. Using the value $g_{\rho\pi\pi}^2/4\pi = 2.53$ to which corresponds a ρ width $\Gamma_\rho = 146 \text{ MeV}$ ²³⁾ we get: $|\epsilon_1| = 2.26$. With our conventions, the decay matrix element is related to the width by the formula:

$$\Gamma = \frac{p_{cm}}{4M_i^2} \cdot \frac{1}{4\pi} \cdot \frac{1}{2j+1} \cdot \sum \int d\Omega_3 |M|^2; \quad p_{cm} = \frac{|q|}{2} \quad (11)$$

Inserting Eq. (10) results in:

$$\Gamma = S_{l3}^2 \cdot \frac{32\pi}{27} \cdot \frac{(4\sqrt{\beta})^3}{M_i^2} \cdot e^{2/3 l} \cdot \frac{2^l}{(2l+1)!!} \cdot \left(\frac{p_{cm}}{34\sqrt{\beta}}\right)^{2j+1} \cdot \frac{j}{2j+1} \cdot 4\epsilon_1^2 \quad (12)$$

Since all parameters are fixed we can calculate the widths of the $1^{--}, 2^{++}, 3^{--} \dots$ decays into two pseudoscalars. A comparison of these results with experiment is given in Table I and shown in Fig. 1. Good agreement is obtained for $\sqrt{\beta} = 0.2 \text{ GeV}^2$. This value has to be compared with the ones we got from the spectrum

$$\sqrt{\beta_s} = 0.175 \text{ GeV}^2,$$

$$\sqrt{\beta_t} = 0.150 \text{ GeV}^2.$$

In the R region, the ρ meson is established as a $3^{--}, I=1$ state. It should have K, ω, ϕ like partners to complete a nonet. Under the assumption of ideal mixing their masses are predicted from the relation ²⁴⁾:

$$M_{(I=1/2)}^2 - M_{(I=1)}^2 = M_{(\phi)}^2 - M_{(I=1/2)}^2 = 0.25 \text{ (GeV}^2\text{)};$$

$$M_{(\omega)}^2 = M_{(I=1)}^2,$$

which is fulfilled in the established nonets with ideal mixing. We get :

$$M_\omega = 1680 \text{ MeV}, \quad M_K = 1750 \text{ MeV}, \quad M_\phi = 1825 \text{ MeV}.$$

Experimental candidates for these resonances are :

$$\omega(1675), \quad K_N(1760), \quad \omega/\pi(1830).$$

With these assignments, we have calculated the partial widths of the 3^{--} nonet given in Table I.

The matrix element for the $j = l-1$ resonances

$$\begin{aligned} M = & S u_3 \cdot \frac{16\pi\sqrt{2}}{9} \cdot e^{q_3} \cdot \sqrt{\frac{2^l}{(2l+1)!!}} \cdot \left(\frac{19!}{64\beta}\right)^l \cdot Y_{l,j_3} \cdot \sqrt{\frac{j+1}{2j+1}} \cdot |q| \cdot \\ & \left\{ \frac{1}{3} \varepsilon_1 + (2l+1) \cdot \frac{4\sqrt{\beta}}{q^2} \cdot p_2 p_3 \cdot \varepsilon_2 \right\} \end{aligned} \quad (13)$$

consists of two terms. The one proportional to ε_2 leads to a barrier penetration factor p_{CM}^{2j+1} in contrast to the calculations with non-relativistic wave functions where a p_{CM}^{2j+5} factor appears. This leads to large decay widths which might be desirable for a resonance interpretation of the phenomena in the S wave phase shift of $\pi\pi$ and $K\pi$ scattering (25), (26). Since the phase shift analysis do not give reliable parameters for such a particle interpretation, we do not give a numerical analysis to these phenomena.

Our scheme provides a possible explanation for the relatively small decay width of the radially excited $S'(1600) \rightarrow 2\pi$ (25), (27). The matrix element for this decay is :

$$\begin{aligned} M_{S' \rightarrow 2\pi} = & S u_3 \cdot \frac{16\pi\sqrt{2}}{9} \cdot e^{2/3} \cdot Y_{1,j_3} \cdot \sqrt{\frac{1}{3}} \cdot |q| \cdot \frac{1}{3\sqrt{2}} \cdot \\ & \cdot \left\{ \frac{1}{3} \varepsilon_1 \cdot L_1^1(-q_{12}^2) - \frac{2p_2 p_3 \cdot \frac{1}{3} \varepsilon_2 + 8\hat{\varepsilon}_T^A \cdot e^{H_0^2/8\sqrt{\beta}}}{H_0^2} \right\} \\ = & S u_3 \cdot \frac{16\pi\sqrt{\beta}}{9} \cdot e^{2/3} \cdot Y_{1,j_3} \cdot \frac{|q|}{2\sqrt{2}} \cdot \left\{ \frac{1}{3} \varepsilon_1 \cdot L_1^1(-q_{12}^2) - \frac{16}{3} \frac{\sqrt{\beta}}{H_0^2} \cdot \varepsilon_2' \right\} \end{aligned} \quad (14)$$

with :

$$\varepsilon_2' = (-\hat{\varepsilon}_S^V - 1/2 \hat{\varepsilon}_T^A + 6\sqrt{\beta})/\sqrt{\beta} \cdot e^{H_0^2/8\sqrt{\beta}}.$$

If we put $\varepsilon'_2 = 0$ then the existence of a node in the radial function, i.e., the neighbourhood of the zero in the Laguerre polynomial leads to the relatively small $\Gamma_{\rho \rightarrow 2\pi} \approx 80 \text{ MeV}$ to be compared with the very rough experimental estimates of about 50 MeV.

2.2. - The decays into pseudoscalar and vector mesons

A) - Decays of the natural parity resonances ($j = l+1$)

Proceeding in the same way as in the last section, we obtain, after evaluation of the Dirac traces, integrals and angular momentum coefficients for the decay matrix element, the expression :

$$\begin{aligned} \mathcal{M} = & -i S_{13} \cdot \frac{16\pi\sqrt{2}}{9\sqrt{\beta}} \cdot \sqrt{\frac{0!}{(2l+1)!}} \cdot e^{i\phi_3} \cdot \left(\frac{1\bar{q}1}{6\sqrt{\beta}}\right)^l \cdot V_{j,M} \cdot e_{\varepsilon_3} (1\varepsilon_3, jM/jj_3) \cdot \\ & \cdot \sqrt{\frac{j+1}{2j+1}} \cdot \frac{1\bar{q}1}{2} \cdot M_1 \cdot \varepsilon_3, \quad (15) \\ \varepsilon_3 = & (\varepsilon_T^V + 1/3 \hat{\varepsilon}_T^A) / \sqrt{\beta} \cdot e^{i\phi_3/2} \end{aligned}$$

The structure of this matrix element with respect to barrier factors is similar to those for the decays into two pseudoscalar mesons. We take the parameter ε_3 from the $\omega\rho\pi$ coupling constant $g_{\omega\rho\pi} = 14 \text{ GeV}^{-1}$, which is determined via vector meson dominance from the $\omega \rightarrow \pi\gamma$ width. The last free parameter is in this way determined to $|\varepsilon_3| = 0.55$. The partial widths resulting from this matrix element are given in Table II. Again we find reasonable agreement between the values for $\sqrt{\beta}$ from the spectrum and those of the decay widths. Predictions for the 3^{--} states are also given in the Table.

B) - The A_1 and B meson decays

From the decays discussed above, the absolute values of the parameters ε_i have been determined. Therefore, we obtain the potential parameters ε_i^j up to ambiguities resulting from the unknown signs of the ε_1 and ε_3 . A possible solution is :

$$\hat{\varepsilon}_T^A = 3/2 \sqrt{\beta} ; \quad \varepsilon_T^V = 0 ; \quad \hat{\varepsilon}_S^V = 21/4 \cdot \sqrt{\beta} ; \quad \varepsilon_P^A = -2\sqrt{\beta} . \quad (16)$$

A discussion of the decays $A_1 \rightarrow \rho\pi$, $B \rightarrow \omega\pi$ and their helicity structure is now a test of our model and the choice of parameters. The BS amplitude of the A_1 meson is :

$$\begin{aligned} \chi_0^{A_1} &= i \frac{4\pi}{\sqrt{\beta}} \cdot e^{k^2/2\sqrt{\beta}} \cdot \varepsilon\left(\frac{p}{M}, \frac{k}{4\sqrt{\beta}}, \gamma, a\right) \\ \Delta\chi^{A_1} &= \frac{1}{4m^3} \cdot i \cdot \frac{4\pi}{\sqrt{\beta}} \cdot e^{k^2/2\sqrt{\beta}} \cdot p \cdot \varepsilon\left(\frac{p}{M}, \frac{k}{4\sqrt{\beta}}, \gamma, a\right) \quad *) \end{aligned} \quad (17)$$

Together with the BS amplitudes of the ρ and the π it gives the decay matrix element :

$$\begin{aligned} \mathcal{M} &= i S U_3 \cdot \left(\frac{4\pi}{\sqrt{\beta}}\right)^3 \cdot \frac{(2\pi)^{-1/2}}{4\sqrt{\beta}} \cdot \frac{4\beta}{5} \pi^2 \cdot e^{1/3} \cdot \frac{1}{4\sqrt{\beta}} \cdot \sqrt{\frac{3}{5}} \cdot \frac{1}{M_1} \cdot \\ &\quad \cdot \left\{ \frac{2}{3} \varepsilon_3 \sqrt{\beta} \cdot \varepsilon(\alpha p_1 p_2 a) \varepsilon(\alpha p_1 e_2 p_3) \right. \\ &\quad \left. + \frac{2}{3} \sqrt{\beta} \hat{\varepsilon}_T^A \cdot \varepsilon(\alpha\beta p_1 a) \varepsilon(\alpha\beta e_2 p_3) \right\}. \end{aligned}$$

For further discussion it is convenient to introduce the transverse and longitudinal couplings ²⁸⁾ :

$$\begin{aligned} (a, e)_T &= \frac{1}{M_1^2} \cdot \varepsilon(\alpha p_1 p_2 a) \varepsilon(\alpha p_1 p_2 e_2) \\ (a, e)_L &= \left(e_2 p_1 - \frac{p_1 p_2 \cdot e_2 p_2}{p_2^2} \right) \left(a p_2 - \frac{a p_1 \cdot p_1 p_2}{p_1^2} \right) \end{aligned} \quad (18)$$

With these, the matrix element $\mathcal{M}$ can be rewritten

$$\begin{aligned} \mathcal{M} &= i S U_3 \cdot \frac{16\sqrt{\pi}}{9\sqrt{\beta}} \cdot e^{1/3} \cdot \frac{M_1}{\sqrt{5}} \left\{ (a, e)_T \left(2\varepsilon_T^V - \frac{4}{3} \hat{\varepsilon}_T^A \right) - (a, e)_L \cdot 2\hat{\varepsilon}_T^A \right\} \\ &= i S U_3 \cdot (g_T + g_L). \end{aligned} \quad (19)$$

*) $a_\mu^{S_3}$ = covariant polarization vector of the A_1 meson ;

$$\varepsilon(p_1 p_2 p_3 p_4) = \varepsilon_{\alpha\beta\gamma\delta} p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta.$$

The relation of g_T, g_L to the helicity couplings g_0, g_1 is :

$$g_0 = p_{cm}^2 \cdot \frac{M_1}{M_2} \cdot g_L ; \quad g_1 = p_{cm}^2 \cdot g_T . \quad (20)$$

The parameters of our model yield for the longitudinal to transverse ratio :

$$\left| \frac{g_L}{g_T} \right|_{A_1}^2 = \frac{1}{2} \left| \frac{g_0}{g_1} \right|_{A_1}^2 = \left(\frac{\hat{\varepsilon}_T^A}{\varepsilon_T^V - \frac{2}{3} \varepsilon_T^A} \right)^2 = 2,25. \quad (21)$$

Therefore we expect that the A_1 decays dominantly in the longitudinal mode. The corresponding width is $\Gamma_{A_1} = 150 \text{ MeV}$.

As a last application, we treat the decay $B \rightarrow \omega \pi$. With help of the BS amplitude :

$$\begin{aligned} \chi_0^B &= \frac{4\pi}{V_B} \cdot e^{k^2/2V_B} \cdot \frac{\sqrt{2}}{4V_B} \cdot a k \cdot \gamma_5 \frac{\not{p}}{M} \\ \Delta \chi^B &= \frac{1}{4m^2} \cdot \frac{4\pi}{V_B} \cdot e^{k^2/2V_B} \cdot \frac{\sqrt{2}}{4V_B} \left[-a k \cdot \gamma_5 M \cdot \varepsilon_p^A + a k \cdot \gamma_5 \frac{[p, k]}{M} \right. \\ &\quad \left. + 4p \gamma_5 \frac{p \cdot k}{M} \right] , \end{aligned} \quad (22)$$

we obtain, proceeding in a similar way :

$$\left| \frac{g_L}{g_T} \right|_B^2 \approx 0 ; \quad \Gamma_{B \rightarrow \omega \pi} = 250 \text{ MeV}. \quad (23)$$

3. - THE DERIVATION OF THE MODEL

3.1. - Solution of the fermion-antifermion BS equation for a pseudo-scalar interaction

We want to solve the BS equation for a fermion-antifermion system ²⁹⁾

$$S_F^{-1}(p_1) \chi(k, p) \bar{S}_F^{-1}(p_2) = i \int K(k, k', p) \chi(k', p) dk' \quad (24)$$

$p = p_1 + p_2$ total momentum of constituents of mass m
 $k = \frac{1}{2}(p_1 - p_2)$ relative momentum

or more explicitly

$$\left(\frac{\not{p}}{2} + \not{k} - m\right) \chi(k, p) \left(\frac{\not{p}}{2} - \not{k} + m\right) = i \int K(k, k', p) \chi(k', p) dk' \quad (25)$$

for convolution-type kernels without derivative couplings. The general spin structure of the kernel then is :

$$K = K^{(1)} \cdot 1 \times 1 + K^{(2)} \cdot \gamma_\mu \times \gamma^\mu + K^{(3)} \cdot \gamma_{\mu\nu} \times \gamma^{\mu\nu} + K^{(4)} \cdot \gamma_5 \gamma_\mu \times \gamma_5 \gamma^\mu + K^{(5)} \cdot \gamma_5 \times \gamma_5. \quad (26)$$

It is rather convenient to rearrange this kernel by introducing projection operators π^i :

$$K = K^S \pi^S + K^V \pi^V + \dots + K^P \cdot \pi^P \quad (27)$$

with

$$\begin{pmatrix} \pi^S \\ \pi^V \\ \pi^T \\ \pi^A \\ \pi^P \end{pmatrix} = \frac{1}{32} \begin{pmatrix} 2 & 2 & 1 & -2 & 2 \\ 8 & -4 & 0 & -4 & -8 \\ 12 & 0 & -2 & 0 & 12 \\ 8 & 4 & 0 & 4 & -8 \\ 2 & -2 & 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \times 1 \\ \gamma_\mu \times \gamma^\mu \\ \gamma_{\mu\nu} \times \gamma^{\mu\nu} \\ \gamma_5 \gamma_\mu \times \gamma_5 \gamma^\mu \\ \gamma_5 \times \gamma_5 \end{pmatrix} \quad (27')$$

$$\pi^i \pi_j = \delta^{ij} \pi_j.$$

In order to simplify Eq. (24) we go into the centre-of-momentum frame $\vec{P} = 0$, $P_0 = M$. This does not imply any loss of generality because of the Lorentz covariance of Eq. (24). By applying the Lorentz boost

$$\begin{aligned} L^{-1}(P) \begin{pmatrix} k_0 \\ \vec{k} \end{pmatrix} &= \begin{pmatrix} kP/M \\ \vec{k} - \vec{P} \cdot \frac{kP + k_0 M}{M(M+P_0)} \end{pmatrix} \\ L^{-1}(P) \begin{pmatrix} \gamma_0 \\ \vec{\gamma} \end{pmatrix} &= \begin{pmatrix} P/M \\ \vec{\gamma} - \vec{P} \cdot \frac{P + \gamma_0 M}{M(M+P_0)} \end{pmatrix} \equiv \begin{pmatrix} P/M \\ \vec{\gamma}(P) \end{pmatrix}, \end{aligned} \quad (28)$$

we obtain $\chi(k, P)$ in an arbitrary Lorentz frame from the centre-of-mass system amplitude $\chi(k_0, \vec{k}, M)$ by simple substitution.

For the applications of this paper, it is sufficient to know $\chi(k, P)$ for Euclidean points k . Therefore we perform in the centre-of-mass system the rotation of the integration contour in the complex k_0' plane to the imaginary axis and substitute $k_0 \rightarrow ik_4$ ("Wick rotation"). By this the Lorentz four-vector $(k_0, \vec{k})$ gets replaced by an Euclidean four-vector $k_E(k_4, \vec{k}) = (ik_0, \vec{k})$; Eq. (24) then determines the analytic continuation of χ to Euclidean points.

The next step towards the solution of the BS equation consists in expanding χ into the "Dirac" covariants

$$\begin{aligned} \chi &= S + V + T + A + P \\ &= s \cdot 1 + v_\mu \gamma^\mu + t_{\mu\nu} \sigma^{\mu\nu} + a_\mu \gamma_5 \gamma^\mu + p \gamma_5. \end{aligned} \quad (29)$$

With this, the BS equation decomposes into ^{**) :}

$$\begin{aligned} i k S k - m^2 S - K^S S + M^2/4 \gamma_4 S \gamma_4 &= i m \{ k, V \} + \frac{iM}{2} (k T \gamma_4) \\ i k V k - m^2 V - K^V V + M^2/4 \gamma_4 V \gamma_4 &= i m \{ k, S \} - m \frac{M}{2} [\gamma_4, T] + \frac{iM}{2} (k A \gamma_4) \\ i k T k - m^2 T - K^T T + M^2/4 \gamma_4 T \gamma_4 &= i m \{ k, A \} - m \frac{M}{2} [\gamma_4, V] + \frac{iM}{2} (k S + P \gamma_4) \\ i k A k - m^2 A - K^A A + M^2/4 \gamma_4 A \gamma_4 &= i m \{ k, T \} - m \frac{M}{2} [\gamma_4, P] + \frac{iM}{2} (k V \gamma_4) \\ i k P k - m^2 P - K^P P + M^2/4 \gamma_4 P \gamma_4 &= -m \frac{M}{2} [\gamma_4, A] + \frac{iM}{2} (k T \gamma_4). \end{aligned} \quad (30)$$

*) Similarly the Dirac matrices $\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\vec{\gamma}^r = \begin{pmatrix} 0 & \vec{\sigma}_r \\ -\vec{\sigma}_r & 0 \end{pmatrix}$

get replaced by the Euclidean matrices $\gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^r = \begin{pmatrix} 0 & i\sigma_r \\ -i\sigma_r & 0 \end{pmatrix}$.

**) $(k T \gamma_4) = k T \gamma_4 - \gamma_4 T k$, etc.

Up to now we have performed the usual manipulations on BS equations. We proceed with the specifications of our model. In a previous paper ²⁾ we have shown that strong binding of heavy quarks ($M^2/4m^2 \ll 1$) in smooth potentials leads to a linear meson spectrum with a level spacing small compared to the quark mass. The expectation value of the "kinetic energy" k^2 for these strongly bound states is also small : $\langle k^2 \rangle \ll m^2$. This means that the quarks are moving in the neighbourhood of the bottom of the smooth potential, where it may be approximated by a parabola, i.e., harmonic potentials

$$K^i(R) = \alpha^i + \beta^i R^2; \quad (\text{sgn } \beta^i = -\text{sgn } \alpha^i); \quad R = \sqrt{x_4^2 + \vec{x}^2},$$

$$K^i(k-k') = \delta^{(4)}(k-k') (\alpha^i + \beta^i \square_k); \quad i = \{S, V, T, A, P\}. \quad (31)$$

The depth α of the attractive parts of the interaction potentials has to be approximately m^2 in order to yield strong binding.

As already pointed out in Section 1, we want to solve the BS equation for a pseudoscalar interaction. This means for the α^i, β^i

$$\alpha^{S,T,P} = m^2 + \delta^{S,T,P}; \quad \alpha^{V,A} = -m^2 - \delta^{V,A}; \quad \delta^i = O(m^0);$$

$$\beta^S = \beta^T = \beta^P = -\beta^V = -\beta^A = \beta \Rightarrow 1. \quad (32)$$

With these specifications, Eq. (30) becomes :

$$\begin{aligned} \text{for } S: & -2m^2 S - \square S - \delta_S S + M^2/4 \gamma_4 S \gamma_4 = i m \{ \not{A}, V \} + \frac{iM}{2} (\not{A} T \gamma_4) \\ \text{for } V: & -\square V - \delta_V V + M^2/4 \gamma_4 V \gamma_4 = i m \{ \not{A}, S \} - \frac{mM}{2} [\gamma_4, T] + \frac{iM}{2} (\not{A} A \gamma_4) \\ \text{for } T: & -2m^2 T - \square T - \delta_T T + M^2/4 \gamma_4 T \gamma_4 = i m \{ \not{A}, A \} - \frac{mM}{2} [\gamma_4, V] + \frac{iM}{2} (\not{A} S P \gamma_4) \\ \text{for } A: & -\square A - \delta_A A + M^2/4 \gamma_4 A \gamma_4 = i m \{ \not{A}, T \} - \frac{mM}{2} [\gamma_4, P] + \frac{iM}{2} (\not{A} V \gamma_4) \\ \text{for } P: & -2m^2 P - \square P - \delta_P P + M^2/4 \gamma_4 P \gamma_4 = -\frac{mM}{2} [\gamma_4, A] + \frac{iM}{2} (\not{A} T \gamma_4). \end{aligned} \quad (33)$$

Equation (33) shows the following properties. In the equations for V and A the quark mass squared gets compensated by the depth of the attractive interaction whereas in the equations for S, T, P this does not occur. Therefore for the strongly bound states V and A are the large amplitudes, they describe the oscillations of the $q\bar{q}$ system around its equilibrium position. S, T, P are not in resonance, they perform forced oscillations with small amplitudes. This picture shows us the method of solving Eq. (33); elimination of the small components. We first solve Eq. (33.1,3,5) for S, T, P :

$$\begin{aligned} S &= \frac{1}{k^2 - 2m^2 - \delta_s - \square + M^2/4} \left[i m \{ \not{k}, V \} + \frac{iM}{2} (\not{k} T \gamma_4) \right] \\ T &= \frac{1}{\not{k} \times \not{k} - 2m^2 - \delta_T - \square + \gamma_4 \times \gamma_4 \cdot M^2/4} \left[i m \{ \not{k}, A \} - \frac{mM}{2} [\gamma_4, V] + \frac{iM}{2} (\not{k} S P \gamma_4) \right] \\ P &= \frac{1}{-k^2 - 2m^2 - \delta_P - \square - M^2/4} \left[- \frac{mM}{2} [\gamma_4, A] + \frac{iM}{2} (\not{k} T \gamma_4) \right]. \end{aligned} \quad (34)$$

Since we want to solve the BS equation (24) with an accuracy of the third order in m^{-1} we expand the right-hand side of Eq. (34) into powers of $1/2m^2$:

$$\begin{aligned} S &= \left(-\frac{1}{2m^2} - \frac{1}{4m^4} (k^2 - \delta_s - \square + M^2/4) \right) \left[i m \{ \not{k}, V \} + \frac{iM}{2} (\not{k} T \gamma_4) \right] \\ T &= \left(-\frac{1}{2m^2} - \frac{1}{4m^4} (\not{k} \times \not{k} - \delta_T - \square + M^2/4 \gamma_4 \times \gamma_4) \right) \left[i m \{ \not{k}, A \} - \frac{mM}{2} [\gamma_4, V] + \frac{iM}{2} (\not{k} S P \gamma_4) \right] \\ P &= \left(-\frac{1}{2m^2} - \frac{1}{4m^4} (-k^2 - \delta_P - \square - M^2/4) \right) \left[- \frac{mM}{2} [\gamma_4, A] + \frac{iM}{2} (\not{k} T \gamma_4) \right]. \end{aligned}$$

By reinsertion of these expressions in the right-hand side of this equation, the small components get expressed completely in terms of V and A. Putting them into Eq. (29) we get for the full BS amplitude :

$$\begin{aligned} \chi &= \frac{1}{2m} \left(1 + \frac{k^2 - \square - M^2/4}{4m^2} \right) \left[(\not{k}_2 - i\not{k} + m) \chi_0 - \chi_0 (\not{k}_2 + i\not{k} - m) \right] \\ &\quad - i \frac{P \not{k}}{4m^3} \left[(\not{k}_2 - i\not{k} + m) \chi_0 + \chi_0 (\not{k}_2 + i\not{k} - m) \right] \\ &\quad - \frac{1}{8m^3} \delta_T [\not{k}, V] - \frac{1}{8m^3} \delta_P [\not{k}, A] + \frac{i}{m^3} \delta_s \{ \not{k}, V \} + \frac{i}{m^3} \delta_T \{ \not{k}, A \} \end{aligned} \quad (35)$$

(χ_0 denotes any linear combination of V and A).

The equations for the large components V, A are obtained by insertion of these expressions (34) into Eqs. (33.2) and (33.4) :

$$\begin{aligned}
 (-k^2 + \square + \delta_V + M^2/4) V &= \frac{1}{4m^2} \left\{ (k^L + k^R) (k^2 - \square - \delta_S - M^2/4) \{ k, V \} + 2k^2 M^2 V \right. \\
 &\quad - M^2/4 (k^2 - \square - \delta_T - M^2/4) [\gamma_4, [\gamma_4, V]] \\
 &\quad \left. + i M/2 (k^2 - \square - \delta_T - M^2/4) [\gamma_4, \{ \not{q}, A \}] \right\} \\
 (-k^2 + \square + \delta_A + M^2/4) A &= \frac{1}{4m^2} \left\{ (k^L + k^R) (k^2 - \square - \delta_T - M^2/4) \{ k, A \} + 2k^2 M^2 A \right. \\
 &\quad - \frac{M^2}{4} (k^2 - \square - \delta_P - M^2/4) [\gamma_4, [\gamma_4, A]] \\
 &\quad \left. + i M/2 (k^2 - \square - \delta_T - M^2/4) \{ k, [\gamma_4 V] \} \right\}. \tag{36}
 \end{aligned}$$

These are the dynamical equations of the pseudoscalar model. They determine the spectrum and the large components of the BS amplitudes and include all terms of order $1/m^2$.

Neglecting the (small) coupling terms on the right-hand side of Eq. (36) gives the spectrum and wave functions which were presented in Section 1, Eqs. (2) to (4).

What therefore remains to be discussed is the influence of the terms proportional to m^{-2} in Eq. (36). We may divide them into two classes :

- (i) those which couple V to A , i.e., $[\not{P}, \{ \not{A}, A \}]$, $\{ \not{A}, [\not{P}, V] \}$; their effect in leading order m^{-2} is proportional to $\text{Tr } \gamma^\mu V_\mu$. $[\not{P}, \{ \not{A}, \gamma_5 \gamma_\nu A^\nu \}] = M \cdot (\vec{q} \times \vec{A} \cdot \vec{V})$; the matrix elements of this operator between degenerate states vanish ; therefore it gives a contribution to the spectrum of order m^{-4} and a coupling of V and A of order m^{-2} ; consequently the amplitudes χ_0 of the triplet states are still of vector type, those of the singlet states of axial vector type with only small mixing of these structures ; we shall see that for the applications to the mesonic vertices this admixture is unimportant ;

(ii) the remaining terms of the right-hand side of Eq. (36) do possess diagonal matrix elements and therefore contribute to the spectrum in order m^{-2} ; they cause a fine structure splitting of LS type the magnitude of which is too small for heavy quarks to agree with experiment ; a realistic description of fine structure therefore needs the inclusion of additional couplings in the kernel K and the consideration of the effects of mesonic forces ; these terms yield also corrections to V_0, A_0 but, of course, they have the same spin structure as the leading amplitudes and therefore we expect and shall see in the next section that they do not contribute to the leading orders of the mesonic decays.

The result of this discussion is the following. The triplet-singlet amplitudes have the form :

$$\begin{aligned}\chi_0^t &= V_0 + 1/m^2 V_2 + 1/m^2 A_2' \\ \chi_0^s &= A_0 + 1/m^2 A_2 + 1/m^2 V_2',\end{aligned}\tag{37}$$

with V_0, A_0 given by Eq. (4) and V_2, A_2, V_2', A_2' being determined by Eq. (36).

Our discussion of the BS equation did not include the treatment of $SU(3)$ breaking and realistic spin orbit splitting. Therefore, in principle, our calculations are valid only in a $SU(6)$ symmetric world. In spite of these short-comings, we want to compare our results with experiment, applying the old recipe : take the experimental masses to calculate the phase space and centrifugal barrier factors and calculate the dynamical parts of the meson couplings with degenerate masses.

3.2. - The proof of saturation

In the last section we have calculated the BS amplitudes of the pseudoscalar model. The result is given in Eq. (36). Using the mass formula (2) we can write it (in Lorentz metric) :

$$\begin{aligned} \chi = & \frac{1}{2m} \left[\bar{S}^{-1}(\frac{p}{2} + k) \chi_0 - \chi_0 S^{-1}(\frac{p}{2} - k) \right] \\ & + \frac{pk}{4m^3} \left[\bar{S}^{-1}(\frac{p}{2} + k) \chi_0 + \chi_0 S^{-1}(\frac{p}{2} - k) \right] \\ & + \frac{1}{8m^3} (k^2 - \square - M^2/4) \left[\left[\frac{p}{2}, \chi_0 \right] + \left[k, \chi_0 \right] \right] \\ & - \frac{1}{8m^3} \left[\delta_T V_0 + \delta_P A_0 \right] + \frac{1}{4m^3} \left\{ k, \delta_S V_0 + \delta_P A_0 \right\}. \end{aligned} \quad (38)$$

We shall use now these amplitudes to determine the mesonic vertices $\mathcal{M}$. We proceed in two steps. First we show that the amplitudes (36) of the pseudoscalar model yield saturation, i.e., $\mathcal{M}$ is of order m^0 . Second, we examine the meson vertex explicitly.

The terms in the second to the fourth line of Eq. (38) are of order m^{-3} therefore they contribute to $\mathcal{M}$ at most in order m^0 . The first term of Eq. (38)

$$\frac{1}{2m} \left[\bar{S}^{-1}(\frac{p}{2} + k) \chi_0 - \chi_0 S^{-1}(\frac{p}{2} - k) \right] \quad (39)$$

is of leading order m^0 and therefore might give an $\mathcal{M}$ of order m^3 . To show that we have saturation is just to prove that this term contributes only in order m^0 to $\mathcal{M}$. In order to demonstrate this we calculate with this term the trace over the Dirac matrices in Eq. (1) :

$$\begin{aligned} Tr^{(1)} = & \frac{1}{8m^3} \cdot Tr \left[\bar{S}^{-1}(\frac{p}{2} + k) \chi_{01} - \chi_{01} S^{-1}(\frac{p}{2} - k) \right] S^{-1}(k - \frac{p}{2}) \\ & \cdot \left[\bar{S}^{-1}(\frac{p}{2} + k) \chi_{02} - \chi_{02} S^{-1}(\frac{p}{2} - k) \right] S^{-1}(k - \frac{q}{2}) \\ & \cdot \left[\bar{S}^{-1}(-\frac{q}{2} + k) \chi_{03} - \chi_{03} S^{-1}(-\frac{p}{2} - k) \right] S^{-1}(\frac{p}{2} + k), \\ & (q = p_2 - p_3). \end{aligned}$$

We use the properties of the free inverse propagators

$$S^{-1}(k) = k - m ; \quad \bar{S}^{-1}(k) = k + m = - S^{-1}(-k)$$

$$S^{-1}(k) \bar{S}^{-1}(k) = \bar{S}^{-1}(k) S^{-1}(k) = D^{-1}(k) = k^2 - m^2$$

to evaluate the above expression :

$$\begin{aligned} \text{Tr}^{(n)} = \frac{1}{8m^3} \text{Tr} \bigg[& 2 \chi_{01} \chi_{02} \chi_{03} \cdot D^{-1}(\frac{p_1}{2} + k) D^{-1}(\frac{p_1}{2} - k) D^{-1}(q_2 - k) \\ & + \bar{S}^{-1}(\frac{p_1}{2} + k) \chi_{01} \chi_{02} S^{-1}(\frac{q_2}{2} - k) \chi_{03} \cdot D^{-1}(\frac{p_1}{2} + k) D^{-1}(\frac{p_1}{2} - k) \\ & + \chi_{01} S^{-1}(-\frac{p_1}{2} + k) \chi_{02} \bar{S}^{-1}(-\frac{q_2}{2} + k) \chi_{03} \cdot D^{-1}(\frac{q_2}{2} - k) D^{-1}(-\frac{p_1}{2} - k) \\ & + \bar{S}^{-1}(\frac{p_1}{2} + k) \chi_{01} S^{-1}(-\frac{p_1}{2} + k) \chi_{02} \chi_{03} \cdot D^{-1}(\frac{q_2}{2} - k) D^{-1}(-\frac{p_1}{2} - k) \\ & + \chi_{01} \bar{S}^{-1}(-\frac{p_1}{2} + k) \chi_{02} \chi_{03} S^{-1}(\frac{p_1}{2} + k) \cdot D^{-1}(-\frac{p_1}{2} + k) D^{-1}(k - \frac{q_2}{2}) \\ & + \chi_{01} \bar{S}^{-1}(-\frac{p_1}{2} + k) \chi_{02} S^{-1}(k - q_2) \chi_{03} \cdot D^{-1}(\frac{p_1}{2} + k) D^{-1}(\frac{p_1}{2} - k) \\ & + \chi_{01} \chi_{02} \bar{S}^{-1}(-\frac{q_2}{2} + k) \chi_{03} S^{-1}(k + \frac{p_1}{2}) \cdot D^{-1}(-\frac{p_1}{2} + k) D^{-1}(q_2 - k) \bigg]. \end{aligned}$$

Since in the pseudoscalar model the amplitudes of the singlet - as well as of the triplet - states are composed of an odd number of Dirac matrices ($\gamma_5 \gamma_\mu, \gamma_\mu$ type resp.), we have the relations :

$$\text{Tr} \chi_{01} \chi_{02} \chi_{03} = 0$$

and

$$\text{Tr} \chi_{01} \chi_{02} S^{-1}(k - q_2) \chi_{03} \bar{S}^{-1}(k + \frac{p_1}{2}) = m \text{Tr} \chi_{01} [\chi_{02} (k - q_2) \chi_{03} - \chi_{02} \chi_{03} (k + \frac{p_1}{2})]$$

⋮

Therefore

$$\begin{aligned} \text{Tr}^{(n)} = \frac{1}{8m^2} \text{Tr} \chi_{01} \bigg\{ & \chi_{02} (k - q_2) \chi_{03} \cdot [2 D^{-1}(\frac{p_1}{2} + k) D^{-1}(\frac{p_1}{2} - k) - D^{-1}(q_2 - k) D^{-1}(-\frac{p_1}{2} - k) - D^{-1}(-\frac{p_1}{2} + k) D^{-1}(q_2 - k)] \\ & + \chi_{02} \chi_{03} (k + \frac{p_1}{2}) [2 D^{-1}(-\frac{p_1}{2} + k) D^{-1}(q_2 - k) - D^{-1}(\frac{p_1}{2} + k) D^{-1}(\frac{p_1}{2} - k) - D^{-1}(q_2 - k) D^{-1}(\frac{p_1}{2} + k)] \\ & + (-\frac{p_1}{2} + k) \chi_{02} \chi_{03} [2 D^{-1}(\frac{q_2}{2} - k) D^{-1}(-\frac{p_1}{2} - k) - D^{-1}(\frac{p_1}{2} - k) D^{-1}(\frac{q_2}{2} - k) - D^{-1}(\frac{p_1}{2} + k) D^{-1}(\frac{p_1}{2} - k)] \bigg\}. \end{aligned}$$

With

$$\begin{aligned} & [2 \mathcal{D}'(\frac{p}{2} + k) \mathcal{D}'(\frac{p}{2} - k) - \mathcal{D}'(q/2 - k) \mathcal{D}'(-\frac{p}{2} - k) - \mathcal{D}'(-\frac{p}{2} + k) \mathcal{D}'(\frac{q}{2} - k)] \\ & = -2m^2 \left(\frac{M_1^2}{4} - \frac{q^2}{4} + qk \right) \end{aligned}$$

finally, we obtain :

$$\begin{aligned} \text{Tr}^{(4)} = & -\frac{1}{8} \text{Tr} \left\{ \chi_{o1} \left[2 \chi_{o2} \left(k - \frac{q}{2} \right) \chi_{o3} - \chi_{o2} \chi_{o3} \left(k + \frac{q}{2} \right) + \left(\frac{q}{2} - k \right) \chi_{o2} \chi_{o3} \right] \cdot \right. \\ & \cdot \left(\frac{M_1^2}{4} - \frac{q^2}{4} + qk \right) \\ & \left. - \chi_{o1} \left[\chi_{o2} \chi_{o3} \left(k + \frac{q}{2} \right) + \left(\frac{q}{2} - k \right) \chi_{o2} \chi_{o3} \right] \cdot 3 p_1 k \right\}. \quad (40) \end{aligned}$$

The fact that in $\text{Tr}^{(1)}$ only terms of order m^0 survive proves the saturation in the pseudoscalar model. It is the only model with this property ¹⁾. We want to point out that this saturation occurs in all mesonic vertices : triplet-triplet-triplet, triplet-triplet-singlet, triplet-singlet-singlet, singlet-singlet-singlet.

Saturation is independent of the orbital structure of the bound states. This shows that it is mainly connected with the relativistic spinor structure of the amplitudes.

3.3. - Evaluation of the decay matrix element

In Section 3.2, we have calculated the contribution of the large components of the BS amplitudes to the mesonic decay matrix element. Now we evaluate the contribution of the small components. These - being of order m^{-3} - are easy to treat because we are interested only in the leading order m^0 of $\mathcal{M}$. We consider first the second line of Eq. (15). It yields the Dirac trace :

$$\begin{aligned} \text{Tr}^{(4)} = & \frac{1}{4m^3} \text{Tr} \left\{ p_1 k \left[\bar{S}'(\frac{p}{2} + k) \chi_{o1} + \chi_{o1} \bar{S}'(\frac{p}{2} - k) \right] \chi_{o2} \chi_{o3} \right. \\ & + p_2 \left(k + \frac{p}{2} \right) \left[\bar{S}'(\frac{p}{2} + k) \chi_{o2} - \chi_{o2} \bar{S}'(q/2 - k) \right] \chi_{o3} \chi_{o1} \\ & \left. + p_3 \left(k - \frac{p}{2} \right) \left[\bar{S}'(-\frac{q}{2} + k) \chi_{o3} - \chi_{o3} \bar{S}'(-\frac{p}{2} - k) \right] \chi_{o1} \chi_{o2} \right\}. \quad (41) \end{aligned}$$

Therefore :

$$T_r^{(1)} + T_r^{(2)} = 0 \quad (42)$$

In order to evaluate the contribution of the third and fourth lines of Eq. (15) we rewrite these terms for triplets and singlets separately :

$$\begin{aligned} \Delta \chi^t &= \frac{1}{8m^2} (k^2 - \square - M^2/4) ([\varphi_{1/2}, V_0] + \{k, V_0\}) - \frac{1}{8m^2} \delta_T [\varphi, V_0] + \frac{\delta_S}{4m^2} \{k, V_0\} \\ &= \frac{\delta_V - \delta_T}{8m^2} [\varphi, V_0] + \frac{\delta_V - \delta_S}{4m^2} \{k, V_0\} + \frac{1}{2m^2} \{\varphi, V_0\} \\ \Delta \chi^s &= \frac{\delta_A - \delta_T}{8m^2} [\varphi, A_0] + \frac{\delta_A - \delta_T}{4m^2} \{k, A_0\} + \frac{1}{2m^2} \{\varphi, A_0\} \end{aligned} \quad (43)$$

The corresponding trace is :

$$\begin{aligned} T_r^{(3)} &= m^3 \text{Tr} \left\{ \Delta \chi_{01}(k, p_1) \chi_{02}(k + \frac{p_2}{2}, p_2) \chi_{03}(k - \frac{p_2}{2}, p_3) \right. \\ &\quad + \chi_{01}(k, p_1) \Delta \chi_2(k + \frac{p_2}{2}, p_2) \chi_{03}(k - \frac{p_2}{2}, p_3) \\ &\quad \left. + \chi_{01}(k, p_1) \chi_{02}(k + \frac{p_2}{2}, p_2) \Delta \chi_3(k - \frac{p_2}{2}, p_3) \right\}. \end{aligned} \quad (44)$$

The complete decay matrix element is given by :

$$\begin{aligned} \mathcal{M} &= m^3 \cdot S U_3 \cdot (2\pi)^{-9/2} i \int dk \cdot T_r^{(3)} \\ &= m^3 \cdot S U_3 \cdot (2\pi)^{-9/2} i \int dk \cdot \text{Tr} \left\{ \Delta \chi_{01}(k, p_1) \chi_{02}(k + \frac{p_2}{2}, p_2) \chi_{03}(k - \frac{p_2}{2}, p_3) \right. \\ &\quad + \chi_{01}(k, p_1) \Delta \chi_2(k + \frac{p_2}{2}, p_2) \chi_{03}(k - \frac{p_2}{2}, p_3) \\ &\quad \left. + \chi_{01}(k, p_1) \chi_{02}(k + \frac{p_2}{2}, p_2) \Delta \chi_3(k - \frac{p_2}{2}, p_3) \right\}. \end{aligned} \quad (45)$$

As a consequence of saturation, the decay matrix element is not given by a convolution type integral of three BS amplitudes. On the contrary, it is proportional to the small components $\Delta \chi$ of the wave functions. As can be seen from Eq. (43), the $\Delta \chi$'s have orbital angular momenta differing from those of the non-relativistic coupling. Therefore we expect different structures of our decay matrix elements in comparison to those of the non-relativistic models. As an example we give explicitly the χ_0 and $\Delta \chi$ of the ground state mesons :

- Pseudoscalars

$$\begin{aligned}\chi_0^{\text{PS}}(k, p) &= \frac{4\pi}{\sqrt{\beta}} e^{k^2/2\sqrt{\beta}} \cdot \gamma_5 \frac{p}{M}, \\ \Delta\chi_0^{\text{PS}}(k, p) &= \frac{1}{4m^3} \cdot \frac{4\pi}{\sqrt{\beta}} \cdot e^{k^2/2\sqrt{\beta}} (-\varepsilon_p^A \cdot \gamma_5 M + \hat{\varepsilon}_T^A \cdot \gamma_5 \frac{[p, k]}{M}),\end{aligned}\quad (46)$$

- Vector mesons :

$$\begin{aligned}\chi_0^{\text{V}}(k, p) &= \frac{4\pi}{\sqrt{\beta}} e^{k^2/2\sqrt{\beta}} \cdot \phi_{s_3}, \\ \Delta\chi_0^{\text{V}}(k, p) &= \frac{1}{4m^3} \cdot \frac{4\pi}{\sqrt{\beta}} e^{k^2/2\sqrt{\beta}} (-\varepsilon_T^V \phi p + \hat{\varepsilon}_S^V \cdot 2ek),\end{aligned}\quad (47)$$

and the orbitally excited meson resonances :

- Singlets :

$$\begin{aligned}\chi_0^{\text{S}}(k, p) &= \frac{4\pi}{\sqrt{\beta}} e^{k^2/2\sqrt{\beta}} \cdot \sqrt{4\pi} \cdot \left(\frac{|\vec{k}|}{\sqrt{\beta}}\right)^j \cdot \sqrt{\frac{2^j}{(2j+1)!!}} \cdot Y_{jj_3} \cdot \gamma_5 \frac{p}{M}, \\ \Delta\chi_0^{\text{S}}(k, p) &= \frac{1}{4m^3} \cdot \frac{4\pi}{\sqrt{\beta}} \cdot e^{k^2/2\sqrt{\beta}} \cdot \sqrt{4\pi} \cdot \left(\frac{|\vec{k}|}{\sqrt{\beta}}\right)^j \cdot \sqrt{\frac{2^j}{(2j+1)!!}} \cdot \\ &\quad [-\varepsilon_p^A \cdot \gamma_5 M \cdot Y_{jj_3} + \hat{\varepsilon}_T^A \cdot \gamma_5 \frac{[p, k]}{M} Y_{jj_3} \\ &\quad - 4\beta \sqrt{\frac{j}{2j+1}} \cdot \frac{2j+1}{|\vec{k}|} \cdot Y_{j-1, L_3} (1s_3, jj_3 | j-1 L_3) \gamma_5 \frac{[p, \gamma_{s_3}]}{M}],\end{aligned}\quad (48)$$

- Triplets :

$$\begin{aligned}\chi_0^{\text{T}}(k, p) &= \frac{4\pi}{\sqrt{\beta}} e^{k^2/2\sqrt{\beta}} \cdot \sqrt{4\pi} \cdot \left(\frac{|\vec{k}|}{\sqrt{\beta}}\right)^l \cdot \sqrt{\frac{2^l}{(2l+1)!!}} Y_{ll_3} \cdot \phi_{s_3} (1s_3, ll_3 | jj_3), \\ \Delta\chi_0^{\text{T}}(k, p) &= \frac{1}{4m^3} \cdot \frac{4\pi}{\sqrt{\beta}} \cdot e^{k^2/2\sqrt{\beta}} \cdot \sqrt{4\pi} \cdot \left(\frac{|\vec{k}|}{\sqrt{\beta}}\right)^l \cdot \sqrt{\frac{2^l}{(2l+1)!!}} \cdot \\ &\quad [-\varepsilon_T^V \cdot \phi p \cdot Y_{ll_3} (1s_3, ll_3 | jj_3) + 2\hat{\varepsilon}_S^V \cdot |\vec{k}| (l \cdot j) \sqrt{\frac{j+l+1}{2(2j+1)}} \cdot Y_{jj_3} \\ &\quad - 4\beta \sqrt{\frac{l}{2l+1}} \cdot \delta_{jj_3} \cdot \frac{2l+1}{|\vec{k}|} \cdot Y_{jj_3}],\end{aligned}\quad (49)$$

with :

$$\begin{aligned}\varepsilon_j^i &= \delta_i - \delta_j \\ \hat{\varepsilon}_j^i &= \varepsilon_j^i + 2\sqrt{\beta}.\end{aligned}$$

4. - CONCLUSION

We have applied our dynamical model of heavy quarks in the framework of relativistic field theory to the problem of mesonic couplings. Our achievements are twofold.

(i) There is the general problem to derive mesonic forces of conventional strength in a model with strongly bound heavy quarks. We gave a solution of this problem by showing that in a BS model with a $\gamma_5 \times \gamma_5$ interaction the spin dependence of the BS amplitudes leads to saturation of the superstrong quark binding forces. This means that the mesonic vertices are independent of the quark mass, a fact which we found also for the spectrum, meson wave functions and matrix elements of the electromagnetic current. As a matter of fact, the main features of hadron physics are in our model independent of the value of the mass of the heavy quarks. This property could open the possibility of a "quark theory without free quarks".

(ii) In our phenomenological discussion we found the qualitative features of our model quite satisfactory. It describes consistently the relation between the range parameter in the barrier penetration factor and the level spacing (Regge slope) of the meson spectrum. This is a non-trivial statement because we consider also the decays of the mesons in the R region where the range parameter enters in a high power. Furthermore the relativistic spin structure of the wave functions allows for an explanation of some observed deviations from the non-relativistic quark model : so we get an almost pure transverse $B \rightarrow \omega \pi$ decay. We can fix the free parameters in such a way that we get reasonable numerical agreement for nearly all meson decays.

We do not want to emphasize too much the numerical success of this model. There are still many improvements necessary. A more refined model should describe fine structure and $SU(3)$ breaking effects correctly. Particle mixing should be considered particularly in the ξ' region with several degenerate states. Finally it is desirable to determine the free parameters by more general principles.

Decay	$M^2(\text{GeV}^2)$	$p_{\text{CM}}(\text{GeV})$	$\Gamma_{\text{exp}}(\text{GeV})$	$\Gamma_{\text{th}}(\text{GeV})$	
				$\sqrt{s}=0.15\text{GeV}^2$	$\sqrt{s}=0.20\text{GeV}^2$
$\varrho \rightarrow 2\pi$	0.59	0.359	$0.146 \pm 7\%$	input	input
$K^* \rightarrow K\pi$	0.80	0.288	$0.050 \pm 2\%$	0.043	0.043
$\phi \rightarrow K^+K^-$	1.04	0.127	$0.00196 \pm 7\%$	0.00185	0.00185
$K_L K_S$		0.110	$0.00147 \pm 10\%$	0.0012	0.0012
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$A_2 \rightarrow \eta\pi$	1.72	0.529	$0.0153 \pm 14\%$	0.018	0.014
$K\bar{K}$		0.428	$0.0047 \pm 15\%$	0.009	0.0067
$K^{**} \rightarrow K\pi$	2.02	0.616	$0.055 \pm 6\%$	0.074	0.055
$K\eta$		0.482	$0.002 \pm 100\%$	0.002	0.002
$f \rightarrow 2\pi$	1.61	0.619	$0.130 \pm 10\%$	0.18	0.14
$K\bar{K}$		0.394	$0.008 \pm 60\%$	0.010	0.008
$f' \rightarrow K\bar{K}$	2.29	0.572	$0.040 \pm 25\%$	0.060	0.045
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$g \rightarrow 2\pi$	2.82	0.828	$0.064 \pm 20\%$	0.135	0.076
$K\bar{K}$		0.677	$0.005 \pm >20\%$	0.016	0.009
$K_N(1763) \rightarrow K\pi$	3.10	0.80		0.070	0.040
$K\eta$		0.706		0.030	0.020
$\omega(1675) \rightarrow K^+K^-$	2.77	0.672		0.007	0.004
$K_L K_S$		0.677		0.007	0.004
$\phi(1830) \rightarrow K^+K^-, K_L K_S$	3.34	0.77		0.030	0.020

TABLE I : Decays into two pseudoscalar mesons

Decay	$M^2(\text{GeV})^2$	$p_{\text{CM}}(\text{GeV})$	$\Gamma_{\text{exp}}(\text{GeV})$	$\Gamma_{\text{th}}(\text{GeV})$ ($\sqrt{s}=0.15\text{GeV}^2$)	$\Gamma_{\text{th}}(\text{GeV})$ ($\sqrt{s}=0.20\text{GeV}^2$)
$A_2 \rightarrow \rho\pi$	1.72	0.413	$0.072 \pm 10\%$	0.11	0.082
$K^{*+} \rightarrow K^+\pi$	2.02	0.415	$0.0295 \pm 10\%$	0.038	0.0029
ρK		0.319	$0.0092 \pm 30\%$	0.011	0.008
ωK		0.304	$0.0044 \pm 40\%$	0.0029	0.002
$f' \rightarrow K^*K$	2.29	0.294	< 0.04	0.019	0.014
$g \rightarrow \omega\pi$	2.82	0.66		0.14	0.062
$K^*\bar{K} + \bar{K}^*K$		0.46		0.011	0.005
$K(1760) \rightarrow K^*\pi$	3.10	0.655		0.11	0.049
ρK		0.60		0.055	0.024
ωK		0.58		0.012	0.005
ϕK		0.43		0.004	0.002
$\omega(1675) \rightarrow \rho\pi$	2.77	0.645	$< 0.141 \pm 10\%$	0.400	0.18
K^*K		0.46		0.011	0.005
$\phi(1830) \rightarrow K^*K$	3.34	0.58		0.09	0.043

TABLE II : Decays into a vector meson and a pseudoscalar meson

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