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CANONICAL DIMENSIONS IN
ASYMPTOTICALLY CONFORMAL INVARIANT THEORIES

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A B S T R A C T

It is shown that canonical dimensions in the product of currents near the light-cone imply a free field structure for the Wightman functions of currents in the underlying skeleton theory.

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Present experimental evidence seems increasingly to support the idea that Nature is described by local Quantum Field Theory, not necessarily a Lagrangian one. A possible way to probe such a field theory is, as pointed out by Gell-Mann ¹⁾, to measure matrix elements of available observable local quantum operators, that is weak and electromagnetic currents, energy momentum tensor and so on. In such a framework, following Wilson ²⁾, we assume that hadron field theory depends on a suitable set of parameters (coupling constants, slope of Regge trajectories, etc...) and that the choice of parameters which makes the theory scale and conformal invariant is relevant for the short distance behaviour of Wightman functions (skeleton theory). Deep inelastic electron and neutrino scattering ³⁾ seems to share properties expected from a canonical light-cone expansion ⁴⁾ of product of currents and is not in contradiction with the algebraic structure of light-cone algebra abstracted from a free-quark model ⁵⁾. It is therefore of interest to investigate, as deep as possible, the general structure of the skeleton implied by canonical operator products.

In this note we show that in the skeleton theory all Wightman functions of currents can be exactly computed and their structure turns out to be the same as in free field theory apart from a choice of some normalization coefficients which depend on the structure of the light-cone algebra. In particular if such coefficients are abstracted from a Lagrangian model, fundamental free massless fields can be uniquely reconstructed, apart from their statistics, in such a way that currents can be regarded as suitable bilinear products. This would mean for example that if we assume a quark-like structure for the algebra of currents on the light-cone, in the skeleton theory quarks exist and are free. However, existence of quarks in the real world is not implied by this result as it strongly depends on the mass-shell properties of the real theory which we completely ignore. If the theory underlying the currents is a renormalizable one, the Wilson skeleton theory is identified with the massless Gell-Mann - Low limit ⁶⁾ supposed to be ultraviolet stable. Our result implies that Bjorken scaling requires an asymptotically free theory. Indications that this phenomenon in fact occurs in renormalizable field theories have already been given ^{7),8),9)}.

Our conclusion seems to be particularly strong as it has been recently realized that the only renormalizable asymptotically free theories (with a stable vacuum) are non-Abelian gauge theories ^{10),11)}, so they are the only possible theories which can exhibit canonical scaling.

Let us now specify the assumptions we will need in the sequel. For simplicity we confine ourselves to a world with no internal quantum numbers. Moreover, we also consider scalar currents and abstract the canonical expansion as if the current were a bilinear product of scalar fields. The more general case where spin and internal quantum numbers are introduced will be discussed elsewhere ¹²⁾.

The canonical light-cone expansion then reads

$$J(x_1)J(x_2) \approx \frac{R}{(x_1-x_2)^2 \approx 0 \left[-(x_1-x_2)^2 + i\varepsilon(x_{01}-x_{02}) \right]} + \frac{1}{-(x_1-x_2)^2 + i\varepsilon(x_{01}-x_{02})} J(x_1, x_2) + \text{regular terms} \quad (1)$$

with the normalization condition of the twist two bilocal operator

$$J(x, x) = K J(x) \quad (2)$$

Since we want to abstract only the structure of the singularities we do not commit ourselves with a particular choice of coefficients R and K and we shall treat them as dynamical parameters subject only to the constraints that they must give a consistent theory. For example, from positivity $R > 0$. Property (2) is a normalization condition of the bilocal operator, which in more realistic examples with vector currents equipped by internal quantum numbers is provided by e.t. current algebra.

In the skeleton theory expansion (1) is required to be covariant under scale and special conformal transformations ¹³⁾. Near the light-cone this implies

$$J(x_1, x_2) \approx \sum_{N \text{ even}} \sum_{n \text{ even}} \frac{1}{n!} \left(-\frac{1}{2}\right)^{n/2} \frac{(n-1)!!}{(N+3/2)^{n/2}} z^{\alpha_1} \dots z^{\alpha_n} (z \cdot \partial_R)^n J_{\alpha_1 \dots \alpha_N}(R) \quad (3)$$

where $z = x_1 - x_2/2$, $R = x_1 + x_2/2$ are the relative and baricentric co-ordinates respectively, and $J_{\alpha_1 \dots \alpha_N}(R)$ are twist two local $(N/2, N/2)$ operators irreducible under conformal transformations. The crucial point is that, by conformal invariance, we can uniquely extend the bilocal (3) to arbitrary relative distances z^2 , as follows ¹⁴⁾

$$J(x_1, x_2) = \sum_{N \text{ even}} \sum_{m \text{ even}} \sum_n \left(-\frac{1}{2}\right)^{m+\frac{n}{2}} \frac{(n-1)!!}{[2(N+m)+1]!!} \frac{(2N+1)!!}{(N+m+\frac{3}{2})^{\frac{n}{2}}} \frac{1}{m!} \frac{1}{m!} x$$

$$\times Z^{\alpha_1} \dots Z^{\alpha_N} (Z \cdot \partial_R)^n (Z^2 \square_R)^m J_{\alpha_1, \dots, \alpha_N}^{(R)} \quad (4)$$

We assume that (4) defines a bilocal operator in the true sense of the word, i.e., it commutes with every local (or bilocal) operator when all possible relative distances are space-like. This is a consequence of the fact that (4) is bilocal for Z^2 near 0 and that conformal covariance must be consistent with locality. A manifest property of (4) is

$$J(x_1, x_2) = J(x_2, x_1) \quad (5)$$

It has been shown in Ref. 7) that twist two tensors are divergenceless in a conformal invariant theory. As a consequence of this property we derive

$$\square_{x_1} J(x_1, x_2) = \square_{x_2} J(x_1, x_2) = 0 \quad (6)$$

This follows from the fact that Eq. (6) is actually true for each term of the N sum appearing in Eq. (4). The calculation showing this is lengthy but completely straightforward and will be reported elsewhere ¹²⁾.

Let us now consider the matrix element

$$\langle 0 | J(x_1, x_2) J(x_3, x_4) \dots J(x_{n-1}, x_n) | 0 \rangle \quad (7)$$

The most general form allowed by conformal symmetry is

$$\langle 0 | J(x_1, x_2) J(x_3, x_4) \dots J(x_{n-1}, x_n) | 0 \rangle =$$

$$= \frac{1}{(x_1 - x_2)^2} \frac{1}{(x_3 - x_4)^2} \dots \frac{1}{(x_{n-1} - x_n)^2} f(h_{2i}, h_{3i}, h_{ij}) \quad (8)$$

$i, j = 4, \dots, n$

where f is a function of $n(n-3)/2$ harmonic ratios defined as follows:

$$h_{2i} = \frac{(x_1 - x_2)^2 (x_3 - x_i)^2}{(x_2 - x_3)^2 (x_1 - x_i)^2}; \quad h_{3j} = \frac{(x_1 - x_2)^2 (x_3 - x_j)^2}{(x_1 - x_3)^2 (x_2 - x_j)^2}; \quad h_{ij} = \frac{(x_1 - x_i)^2 (x_i - x_j)^2}{(x_1 - x_i)^2 (x_2 - x_j)^2} \quad i < j$$

Of course, due to the fact that the singularities at vanishing $(x_1 - x_2)^2 \dots (x_{n-1} - x_n)^2$ are only apparent because of the regularity of the bilocal at these points, the function f must satisfy some threshold conditions in the harmonic ratios. Equation (6) implies the following set of second order differential equations:

$$\left[\sum_i \left(\frac{h_{2i}}{h_{3i}} \frac{\partial^2}{\partial h_{2i}^2} + \frac{h_{3i}^2}{\partial h_{3i}^2} \right) + \sum_{i,j} h_{2i} h_{3j} \left(\frac{h_{2j}}{h_{3j}} + \frac{h_{2i}}{h_{3i}} - \frac{h_{2i} h_{3j} h_{ij}}{h_{3j}} \right) \frac{\partial^2}{\partial h_{2i} \partial h_{3j}} \right. \\ \left. + 2 \sum_{i,j} h_{3i} h_{3j} \frac{\partial^2}{\partial h_{3i} \partial h_{3j}} + \sum_{i,j} h_{2i} h_{3j} \left(1 + \frac{h_{2i}}{h_{3i}} - h_{2i} \right) \frac{\partial^2}{\partial h_{2i} \partial h_{3j}} \right] f^{(9)} \\ \times f(h_{2i}, h_{3j}, h_{ij}) = 0$$

Moreover, Eq. (5) requires a certain permutation symmetry on f and causality adds very strong constraints which will be examined below. These conditions imply that all n point Wightman functions of bilocal operators are completely disconnected. In order to show that, let us begin the study of the four-point function:

$$\langle 0 | f(x_1, x_2) f(x_3, x_4) | 0 \rangle = \frac{1}{(x_1 - x_2)^2} \frac{1}{(x_3 - x_4)^2} f(h_{24}, h_{34}) \quad (10)$$

Equation (5) implies $f(h_{24}, h_{34}) = f(h_{34}, h_{24})$. Equation (9) reads in this case

$$\left[\frac{h_{24}^2}{h_{34}} \frac{\partial^2}{\partial h_{24}^2} + \frac{h_{34}^2}{h_{24}} \frac{\partial^2}{\partial h_{34}^2} + (h_{24} + h_{34} - h_{24} h_{34}) \frac{\partial^2}{\partial h_{24} \partial h_{34}} \right] f = 0 \quad (11)$$

The most general solution of Eq. (11) compatible with the light-cone expansion is ¹²⁾

$$f(h_{24}, h_{34}) = \int_0^1 d\sigma \frac{h(\sigma)}{\sigma^2 + \sigma(h_{24}^{-1} - h_{34}^{-1} - 1) + h_{34}^{-1}} \quad (12)$$

where symmetry of f implies $h(\sigma) = h(1 - \sigma)$.

End point singularities of (12) correspond to light-cone singularities of Wightman functions. However, there are unwanted branch cut singularities at Euclidean points in the region

$$(h_{24} - h_{34})^2 + h_{24} h_{34} (h_{24} h_{34} - 2 h_{24} - 2 h_{34}) = 0$$

$$|h_{24} - h_{34}| < h_{24} h_{34} \quad h_{24}, h_{34} > 0 \quad (13)$$

which cannot be avoided by shifting the contour integration in (12).

The only possible choice of $h(\sigma)$ compatible with locality is therefore a combination of derivatives of δ functions concentrated at the end points. However, a careful analysis shows that derivatives of δ functions give rise to light-cone singularities not compatible with light-cone expansion (1) so we get that the only possible choice is

$$f(h_{24}, h_{34}) = \gamma (h_{34} + h_{24}) \quad (14)$$

This gives for Eq. (10)

$$\langle 0 | f(x_1, x_2) f(x_3, x_4) | 0 \rangle = \gamma \left[\frac{1}{(x_1 - x_3)^2} \frac{1}{(x_2 - x_4)^2} + \frac{1}{(x_1 - x_4)^2} \frac{1}{(x_2 - x_3)^2} \right] \quad (15)$$

with $\gamma = \frac{1}{2} K^2 R$ as a consequence of Eq. (2).

Using similar techniques one can extend the proof to arbitrary n point functions, thanks to the following selection rule: a conformal covariant vertex $[O(4,2) \text{ Clebsch-Gordon coefficient}] V_{N_1 N_2}(x_1, x_2, x_3)$ such that $\square_{x_1} V_{N_1 N_2}(x_1, x_2, x_3) = 0$, vanishes unless $|\tau_{N_1} - \tau_{N_2}| = 1$ being τ_{N_1}, τ_{N_2} the twists associated to points x_2, x_3 respectively ¹²⁾.

The n point Wightman functions thus turn out to be linear combination of products of elementary free-massless two-point functions, whose coefficients are fixed, in term of R and K by bilocality, clustering property and Eqs. (2) and (5).

Since the Wightman functions of bilocal quantities have the same functional form as in free-field theory one may wonder if there is a set of underlying massless free fields such that

$$J(x_1, x_2) = \sum_{i=1}^N c_i : \varphi_i(x_1) \varphi_i(x_2) : \quad (16)$$

from condition (2) we get

$$J(x) = \frac{1}{K} \sum_{i=1}^N c_i : \varphi_i^2(x) : \quad (17)$$

From (1) we have the consistency condition: $c_i = \pi^2 K^2$ and the number of fields is $N = 8R/K^2$.

So, if the ratio R/K^2 is an integer as it would happen if R and K are abstracted from a Lagrangian field theory, then we can actually introduce a set of fields such that the currents are bilinear products of them. The only ambiguity which remains is that nowhere we must assume commutation or anticommutation relations among different fields so we cannot distinguish between multiplets of ordinary fields or a parafield.

If the ratio R/K^2 is not an integer there is no way to reconstruct fundamental fields; however, Wightman functions of currents can be computed by equal point limits of bilocal Wightman functions and they correspond to an essentially free theory¹⁵⁾. We think that this second situation is not in contradiction with general principles of field theory and indeed we suspect that this should correspond to a scale invariant theory with continuous mass spectrum.

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67, 1 (1973).
- 14) The infinite sequence of derivative on the right-hand side of Eq. (4)
is defined by the generating function:

$$\frac{(2N+1)!!}{2^{N+1} N!} \int_0^1 d\lambda (1-\lambda)^N {}_0F_1(N+1; -\frac{z^2}{4} \square_R (1-\lambda)) e^{\lambda z \cdot \partial_R}.$$

Conformal covariance of this expression is evident in six-dimensional
formalism of DIRAC [P.A.M. Dirac, Ann.Math. 37, 429 (1936) and Ref. 13)]
where it becomes $[\gamma \cdot \gamma' \square'_6 - 2 \gamma \cdot \partial' (1 + \gamma \cdot \partial')]^{-1-N}$.

- 15) The coefficients R and K determine the normalizations of two and three-point functions of currents $J(x)$ (whose space-time structure is uniquely fixed by conformal symmetry) which in realistic theories have a direct physical meaning.