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HIGH ENERGY PROPERTIES OF A CLASS OF UNITARY EIKONAL MODELS
FOR MULTIPRODUCTION

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A B S T R A C T

The high energy properties of a simple class of unitary, crossing symmetric eikonal models of multiproduction are discussed on the basis of the general closed expression given for the S matrix elements in a previous publication. In particular, the high energy behaviour of the multiplicity moments is discussed and it is shown that the K.N.O. scaling relation emerges in a very natural fashion in this class of models.

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1. INTRODUCTION

In high energy nucleon-nucleon scattering it is natural to distinguish between three types of processes; elastic scattering, quasi-elastic nucleon resonance production (with subsequent decay) and the truly inelastic events where one or several low c.m. energy mesons are produced. In order to facilitate the discussion of the general properties of multi-particle events, we make the assumption that, for our purposes, one need not differentiate between elastic and quasi-elastic events. This leads to the simplified picture where there are only two types of processes viz. elastic scattering and nucleon-nucleon collision accompanied by the production of one or several low energy mesons ¹⁾. We shall refer to this as pionization. As a consequence of this approximation, the unitarity equations for the scattering amplitude simplify tremendously. With the additional assumption of transverse momentum cut-off the S matrix becomes diagonal in the nucleon impact parameter. In fact, a number of models of this kind, satisfying many-particle unitarity, have been known for some time ^{2),3)}. In a recent paper ⁴⁾, which we shall refer to as I, we have derived the general closed formula for the scattering amplitude ⁵⁾ for all models of this kind, satisfying crossing symmetry in the meson variables and many-particle unitarity in the approximations described above. The general solution was given in terms of the following input quantities; N five-point functions describing for fixed nucleon rapidity $Y = \ln(s/m^2)$ and fixed nucleon impact parameter $\vec{B}$, the wave functions in which the emitted mesons are produced and a generalized phase shift

$\delta(Y, B, x_1, \dots, x_N) = \delta^*(Y, B, x_1, \dots, x_N)$ where the variables x_j , $j = 1, 2, \dots, N$, label, for given Y and B, the many-particle eigenstates of the scattering operator S. [The eigenvalues being given by $S(Y, \vec{B}, x_1, \dots, x_N) = e^{2i\delta(Y, B, x_1, \dots, x_N)}$.]

The purpose of the present paper is to illustrate the power of the formula for the general solution given in paper I, by considering the high energy properties of a large class models. The class of models we are going to discuss here are characterized by the property that the generalized phase shift is independent of Y and B, i.e., $\delta = \delta(x_1, \dots, x_N)$. We shall refer to this class of models as geometrical since in these models there is no explicit dependence on the structure of the nucleons. The Y and B dependence of the scattering amplitude comes about entirely via the Y and B dependence of the meson wave functions.

We shall start with the geometrical models of a single mode ($N = 1$). We shall show that one can define in a natural fashion two regions in the nucleon impact parameter space. A "grey" region, which gives an energy independent asymptotic contribution to the elastic and inelastic cross-sections, to the partial cross-sections σ_n of producing n mesons, and to the meson multiplicity moments (including the multiplicity itself). A "black" region with the following properties: it gives for large Y an equal contribution to the elastic and to the inelastic cross-sections (shadow scattering in the elastic channel) while its contribution to any one of the individual σ_n ($n \geq 1$) goes to zero. Its contribution makes, however, the meson multiplicity and the multiplicity moments to increase with increasing energy.

Afterwards, we shall turn to the more general case of $N > 1$ modes and show that again one can make the same kind of analyses and reach the same conclusions.

We shall also discuss the K.N.O. scaling relation ⁶⁾ for the multiplicity moments

$$\langle n^m \rangle = G_m \langle n \rangle^m$$

and show that it holds for the geometrical models provided that the generalized phase shift $\delta(x)$ behaves like a positive power ≥ 1 as $|x| \rightarrow \infty$.

Assuming now that the grey region is absent we arrive at the conclusion that the geometrical models are in agreement with some of the most prominent qualitative features in high energy meson production; non-vanishing inelastic cross-section, asymptotically vanishing σ_n , increasing $\langle n^m \rangle$ and the K.N.O. scaling relation.

The solvable models of references (2) and (3) are geometrical models of a single mode i.e., they are included in our discussion. In the multiperipheral like solvable models ⁷⁾ the phase shift δ contains explicitly Y and B and they are therefore outside the scope of the present paper. It seems, however, that our method could even be used to analyze in general terms the high energy behaviour of these more complicated models. In this exploitory investigation we have purposely restricted ourselves to the simplest class of models for which it is possible to obtain particularly transparent results.

2. PRELIMINARIES

We consider the scattering of two spinless isoscalar nucleons of mass m and four-momenta p_a and p_b . The outgoing nucleon momenta we denote by p'_a and p'_b and the momenta of the produced isoscalar mesons by q_j . We work in the c.m. system and choose the z axis along the beam direction. For any four-vector q the two-dimensional transverse component will be denoted by $\vec{q}$. We use the variables

$$Y = \ln s/m^2$$

where $\sqrt{s}$ is the c.m. total energy, the transverse nucleon momentum transfer

$$\vec{P}' = \frac{1}{2} (\vec{P}'_a - \vec{P}_a) - \frac{1}{2} (\vec{P}'_b - \vec{P}_b) \quad (1)$$

the meson rapidities y_j and the transverse meson momenta $\vec{q}_j$. We introduce the n meson phase-space element $d\psi_n$ by

$$d\psi_n = \frac{1}{n!} (2(2\pi)^3)^{-n} \prod_{j=1}^n dy_j d^2 q_j \quad (2)$$

and write the partial differential cross-section for producing $n=0,1,\dots$ mesons as

$$\frac{d\sigma_n}{d^2 P' d\psi_n} = \frac{1}{4s^2} (2\pi)^{-2} \left| \hat{M}_n(Y, \vec{P}'; y_1, \vec{q}_1, \dots, y_n, \vec{q}_n) \right|^2. \quad (3)$$

The scattering amplitude in the nucleon impact parameter space, T_n , is defined by

$$\hat{M}_n(Y, \vec{P}'; y_1, \vec{q}_1, \dots, y_n, \vec{q}_n) = 2is \int d^2 B e^{i\vec{P}' \cdot \vec{B}} T_n(Y, \vec{B}; y_1, \vec{q}_1, \dots, y_n, \vec{q}_n).$$

In the present paper we wish to analyze the consequences of unitarity and pionization. Therefore, we assume that the production of the fragmentation region mesons proceeds via quasi-elastic nucleon resonance production which we then lump together with the truly elastic events. Thus, we require the meson rapidities to satisfy

$$|y_j| \leq \frac{1}{2} (1 - \varepsilon) Y$$

In addition we require the average meson multiplicity to grow less fast than $s^{\varepsilon/2}$ and we assume all transverse momenta to be small. By a model of N modes, $W_j(Y, \vec{B}, y, \vec{q})$, $j=1, 2, \dots, N$ we mean a model in which a meson, for a given value of Y and $\vec{B}$, can only be emitted in a state described by one of the N wave functions W_j . As shown in paper I, one may assume without loss of generality, the W_j to satisfy the orthogonality relation

$$\int dy \int d^2 q W_j^*(Y, \vec{B}, y, \vec{q}) W_i(Y, \vec{B}, y, \vec{q}) = 8\pi^2 \delta_{ij} \Delta_j(Y, \vec{B}) \quad (4)$$

and the reality condition

$$W_j^*(Y, \vec{B}, y, -\vec{q}) = W_j(Y, \vec{B}, y, \vec{q}) \quad (5)$$

If the number of modes is finite (this is perhaps not a serious restriction since even models with an infinite number modes might be thought of in the present scheme by taking a finite but sufficiently large N) we can use the results of paper I, where it was shown that the general expression for the scattering amplitude of a unitary, crossing symmetric model with N modes W_j , is given for $n=1, 2, \dots$, by

$$\begin{aligned} T_n(Y, \vec{B}; y_1, \vec{q}_1, \dots, y_n, \vec{q}_n) &= \sum_Q Q \sum_{\lambda_1=0}^{\infty} \dots \sum_{\lambda_N=0}^{\infty} \delta_{n, \sum \lambda_j} \\ & W_1^*(Y, \vec{B}, y_1, \vec{q}_1) \dots W_{\lambda_1}^*(Y, \vec{B}, y_{\lambda_1}, \vec{q}_{\lambda_1}) \dots W_N^*(Y, \vec{B}, y_{n-\lambda_N}, \vec{q}_{n-\lambda_N}) \\ & \dots W_N^*(Y, \vec{B}, y_n, \vec{q}_n) \left(\frac{1}{\sqrt{\pi}} \right)^N \int_{-\infty}^{+\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_N e^{-(x_1^2 + \dots + x_N^2)} \\ & \prod_{j=1}^N \left(\frac{1}{\lambda_j!} \left(\frac{1}{2\sqrt{2\Delta_j}} \right)^{\lambda_j} \right) S(Y, \vec{B}, x_1 \sqrt{2\Delta_1}, \dots, x_N \sqrt{2\Delta_N}) , \end{aligned} \quad (6)$$

where the Q are the permutations of the meson momenta and where $\Delta_j(Y, \vec{B})$ is the quantity as defined before by Eq. (4). The function $S(Y, \vec{B}, x_1, \dots, x_N)$ is arbitrary apart from the requirement of its absolute value being equal to one. If one puts

$$S(Y, \vec{B}, x_1, \dots, x_N) = e^{2i\delta(Y, \vec{B}, x_1, \dots, x_N)} \quad (7)$$

the function δ being real, then, as shown in I, the $\delta(Y, \vec{B}, x_1, \dots, x_N)$ has the meaning of a generalized phase shift; the variables x_j label, for given Y and $\vec{B}$, the multi-particle eigenstates of the unitary scattering operator, the eigenvalues of which being simply given by $e^{2i\delta}$.

For the elastic amplitude $T_0(Y, \vec{B}) = T(Y, \vec{B})$ we obtained that

$$T(Y, \vec{B}) = \left(\frac{1}{\sqrt{\pi}} \right)^{N+\infty} \int_{-\infty}^{+\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_N e^{-(x_1^2 + \dots + x_N^2)} \quad (8)$$

$$(1 - S(Y, \vec{B}, x_1 \sqrt{2\Delta_1}, \dots, x_N \sqrt{2\Delta_N})).$$

For the derivation of these expressions and for further details we refer to paper I.

The class of models we are going to discuss in this article is characterized by the independence of the phase shift δ on Y and $\vec{B}$, i.e.,

$$\delta = \delta(x_1, \dots, x_N) \quad (9)$$

We shall refer to the models of this type as geometrical since, according to Eqs. (6) and (8), for these models the scattering amplitude depends on Y and $\vec{B}$ only through W_j and Δ_j , i.e., only through the wave-functions of the emitted mesons.

3. HIGH ENERGY BEHAVIOUR OF THE SINGLE MODE GEOMETRICAL MODELS

For this type of models the scattering amplitude depends only on two quantities: a single meson wave-function $W(Y, B, y, \vec{q})$ and the generalized phase shift $\delta(x)$ which is independent of Y and B .

According to paper I, the total cross-section, the elastic cross-section and the partial cross-section of producing n mesons are given by the expressions

$$\begin{aligned} \sigma_{tot} &= 2 \int d^2 B \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx e^{-x^2} (1 - \operatorname{Re} e^{2i\delta(x\sqrt{2\Delta})}) , \\ \sigma_{el} &= \int d^2 B \left| \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx e^{-x^2} (1 - e^{2i\delta(x\sqrt{2\Delta})}) \right|^2 , \\ \sigma_{tot} - \sigma_{el} &= \sigma_{in} = \int d^2 B (1 - \left| \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx e^{-x^2} e^{2i\delta(x\sqrt{2\Delta})} \right|^2) \end{aligned} \quad (10)$$

and

$$\sigma_n = \frac{1}{2^n} \frac{1}{n!} \int d^2 B \left| \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx e^{-x^2} \frac{\partial^n}{\partial x^n} e^{2i\delta(x\sqrt{2\Delta})} \right|^2 . \quad (11)$$

The meson multiplicity moments

$$F_n = \frac{1}{\sigma_{in}} \sum_{n'=n}^{\infty} \frac{n'!}{(n'-n)!} \sigma_{n'}$$

and the meson distribution function

$$g_n(Y, y_1, \vec{q}_1, \dots, y_n, \vec{q}_n) = \sum_{n'=n}^{\infty} \int d^2 p' \int d\psi_{n'-n} \frac{d\epsilon_{n'}}{d^2 p' d\psi_{n'}}$$

normalized as $\int d\psi_n g_n = (\epsilon_{in}/n!) F_n$, are given, again according to paper I, by

$$F_n = \frac{1}{\epsilon_{in}} \frac{1}{2^n} \frac{1}{\sqrt{\pi}} \int d^2 B \int_{-\infty}^{+\infty} dx e^{-x^2} \left| \frac{\partial^n}{\partial x^n} e^{2i\delta(x\sqrt{2\Delta})} \right|^2 \quad (12)$$

and

$$g_n(Y, y_1, \vec{q}_1, \dots, y_n, \vec{q}_n) = \int d^2 B \prod_{j=1}^n \left| \frac{w(Y, \vec{B}, y_j, \vec{q}_j)}{2\sqrt{2\Delta}} \right|^2 \quad (13)$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx e^{-x^2} \left| \frac{\partial^n}{\partial x^n} e^{2i\delta(x\sqrt{2\Delta})} \right|^2.$$

The generalized phase shift is of course real.

The asymptotic properties of the model for large Y depend on two things: the behaviour of $\Delta(Y, \vec{B})$ for large Y and fixed $\vec{B}$ and the behaviour of the phase shift $\delta(x)$ as $x \rightarrow \infty$.

Let us first discuss $\Delta(Y, \vec{B})$. We assume that for fixed $\vec{B}$ and $Y \rightarrow \infty$ the function $\Delta(Y, \vec{B})$ either converges or diverges (we exclude oscillating behaviour). We denote by $\mathcal{L}^b$ the set of $\vec{B}$ values for which $\Delta(Y, \vec{B})$ diverges, by $\mathcal{L}^g$ the set for which $\Delta(Y, \vec{B})$ goes to a constant different from zero and by $\mathcal{L}^t$ the set for which $\Delta(Y, \vec{B})$ converges to zero as $Y \rightarrow \infty$ i.e.,

$$\lim_{Y \rightarrow \infty} \Delta(Y, \vec{B}) = \infty \quad \text{if } \vec{B} \in \mathcal{L}^b,$$

$$\lim_{Y \rightarrow \infty} \Delta(Y, \vec{B}) = \text{constant} \neq 0 \quad \text{if } \vec{B} \in \mathcal{L}^g \quad (14)$$

$$\lim_{Y \rightarrow \infty} \Delta(Y, \vec{B}) = 0 \quad \text{if } \vec{B} \in \mathcal{L}^t$$

We denote by $\mathcal{A}^b(\vec{B})$, $\mathcal{A}^g(\vec{B})$ and $\mathcal{A}^t(\vec{B})$ the characteristic functions of these sets. Since we have excluded oscillating behaviour one has that

$$\mathcal{A}^b(\vec{B}) + \mathcal{A}^g(\vec{B}) + \mathcal{A}^t(\vec{B}) = 1. \quad (15)$$

For simplicity we assume that $\mathcal{A}^b$ and $\mathcal{A}^g$ have compact supports.

We shall refer to the regions in $\vec{B}$ space given by $\mathcal{A}^b(\vec{B}) = 1$, $\mathcal{A}^g(\vec{B}) = 1$ and $\mathcal{A}^t(\vec{B}) = 1$ as "black", "grey" and "transparent" regions. We discuss separately the contributions from these three regions.

The "transparent" region

Remembering Eq. (14) it follows immediately from Eq. (10) that the contribution of this region to the inelastic cross-section goes to zero as $Y \rightarrow \infty$. Likewise, it follows from Eq. (12) that the contribution to the multiplicity moments also goes to zero at infinite energy. Its contribution to σ_{tot} will also vanish provided $\delta(0) = n\pi$ which we shall assume from now on.

The "grey" region

For this region $\vec{B} \in \mathcal{A}^g$. Recalling equations (10) and (12) we see that this region gives a non-vanishing energy independent asymptotic contribution to all the quantities σ_{tot} , σ_{el} , σ_{in} , σ_n and F_n . This is a very undramatic behaviour. Without a detailed knowledge of $\delta(x)$ we can say nothing more about it.

The "black" region

This is the region of $\vec{B}$ values for which $\Delta(Y, \vec{B})$ diverges as $Y \rightarrow \infty$. For any quantity of the form

$$r = \int d^2 B R(\vec{B}) \quad (16)$$

we denote the contribution of the "black" region by r^b , i.e.,

$$r^b = \int d^2 B \mathcal{L}^b(\vec{B}) R(\vec{B}) . \quad (17)$$

The behaviour of the "black" region depends critically on the behaviour of $\delta(x)$ as $x \rightarrow \pm \infty$. For notational convenience ⁸⁾ we assume that $\delta(x)$ behaves in the same way at $x = +\infty$ and $x = -\infty$.

We distinguish between two cases.

Case I: The phase shift $\delta(x)$ increases faster than some positive power as $x \rightarrow \infty$, i.e., there is some $\gamma > 0$ such that

$$\lim_{x \rightarrow \pm \infty} \left| \frac{\delta(x)}{|x|^\gamma} \right| \rightarrow \infty \quad (18)$$

Case II: $\lim_{x \rightarrow \pm \infty} \delta(x) = \delta(\infty)$ exists.

Between cases I and II there is the intermediate situation where $\delta(x)$ increases at infinity slower than an arbitrary power. For such a behaviour it is difficult to give a unified treatment. Fortunately it does not seem to be very interesting from the physical point of view since a logarithmic increase for example can be shown to lead to an oscillating total cross-section at high energy. We shall therefore discard this possibility.

First we discuss case I. We note that

$$\begin{aligned} \lim_{\Delta \rightarrow \infty} \int_{-\infty}^{+\infty} dx e^{-x^2} x^n e^{2i\delta(x\sqrt{2\Delta})} &= \\ &= \lim_{\Delta \rightarrow \infty} \left(\frac{1}{\sqrt{2\Delta}} \right)^{n+1} \int_{-\infty}^{+\infty} dx e^{-x^2/2\Delta} x^n e^{2i\delta(x)} = 0 \end{aligned} \quad (19)$$

for any $n > -1$, since due to the relatively fast oscillation of $e^{2i\delta(x)}$ at infinity the limit

$$\lim_{\Delta \rightarrow \infty} \int_{-\infty}^{+\infty} dx e^{-x^2/2\Delta} x^n e^{2i\delta(x)}$$

can be shown to exist for case I.

From this it follows that

$$\frac{1}{2} G_{\text{tot}}^b = G_{\text{el}}^b = G_{\text{in}}^b = \int d^2 B \mathcal{L}^b(\vec{B}). \quad (20)$$

Thus, the "black" region for case I corresponds to shadow scattering (maximal inelastic cross-section allowed by unitarity, the elastic amplitude is pure imaginary).

Each individual G_n goes to zero as $Y \rightarrow \infty$. This can be seen by n fold partial integration in the right-hand side of Eq. (11) in conjunction with Eq. (19).

The average meson multiplicity and its moments diverge as $Y \rightarrow \infty$ since according to Eq. (12)

$$F_n^b \rightarrow \text{constant} \int d^2 B \mathcal{L}^b(\vec{B}) (\sqrt{2\Delta})^{2n-1} \int_{-\infty}^{+\infty} dx e^{-x^2/2\Delta} \quad (21)$$

$$\left| \frac{\partial^n}{\partial x^n} e^{2i\delta(x)} \right|^2 = \text{constant} \int d^2 B \mathcal{L}^b(\vec{B}) (\sqrt{2\Delta})^{2n-1} \int_{-\infty}^{+\infty} dx \left| \frac{\partial^n}{\partial x^n} e^{2i\delta(x)} \right|^2 \rightarrow \infty$$

A more detailed account of the behaviour of F_n will be given in the next section where we are going to discuss the K.N.O. scaling relation in the context of the present model.

Let us now turn to case II. Since now $\lim_{x \rightarrow \pm \infty} \delta(x) = \delta(\infty)$ exists we obtain first of all that

$$\lim_{Y \rightarrow \infty} G_{\text{tot}}^b = \lim_{Y \rightarrow \infty} G_{\text{el}}^b = 4(\sin \delta(\infty))^2 \int d^2 B \mathcal{L}^b(\vec{B}) \quad (22)$$

implying

$$G_{\text{in}}^b \rightarrow 0$$

and

$$G_n^b \rightarrow 0. \quad (23)$$

For the multiplicity moments one has that

$$F_n^b \rightarrow \text{constant} \int d^2 B \mathcal{L}^b(\vec{B}) (\sqrt{2\Delta})^{2n-1} \int_{-\infty}^{+\infty} dx \left| \frac{\partial^n}{\partial x^n} e^{2i\delta(x)} \right|^2 \quad (24)$$

which gives, upon assuming that $\Delta(Y, \vec{B})$ is a slowly varying function of $\vec{B}$ for $\vec{B} \in \mathcal{L}^b$ and Y large, that

$$\langle n^m \rangle = \text{constant} \langle n \rangle^{2m-1}, \quad (25)$$

since for case II the integral $\int_{-\infty}^{+\infty} dx \left| \frac{\partial^n}{\partial x^n} e^{2i\delta(x)} \right|^2$

is finite. Such a rapid increase of $\langle n^m \rangle$ with the mean multiplicity $\langle n \rangle$ seems to be ruled out experimentally.

The main results of our considerations are summarized in the Table.

4. MULTIPLICITY SCALING IN THE SINGLE MODE GEOMETRICAL MODELS

The K.N.O. scaling relation ⁶⁾ for the multiplicity moments reads as

$$\lim_{Y \rightarrow \infty} \frac{\langle n^m \rangle}{\langle n \rangle^m} = C_m = \text{constant} . \quad (26)$$

It was originally derived from the assumption that all the meson distribution functions approach a limit at high energy. It seems to be well satisfied experimentally ⁹⁾. We shall now show that a relation of this type can very easily be obtained in the present scheme. We have to make two assumptions

a) $\lim_{x \rightarrow \pm \infty} \delta(x) = \frac{\beta}{2} |x|^\gamma, \quad \beta, \gamma \text{ constant}, \quad \gamma \geq 1$

b) The function $\Delta(Y, \vec{B})$ is a slowly varying function of $\vec{B}$ for Y large and $\vec{B} \in \mathcal{L}^b$.

In fact Eq. (12) gives that

$$F_m \rightarrow F_m^b = \frac{1}{6_{in}} \frac{1}{2^m} \frac{1}{\sqrt{\pi}} 2 \int d^2 B \mathcal{L}^b(\vec{B}) \int_0^\infty dx e^{-x^2} (\beta \gamma (\sqrt{2\Delta})^\gamma)^{2m} x^{2(\gamma-1)m}$$

which, because of assumption b) can be written as

$$F_m \rightarrow \frac{6_{in}^b}{6_{in}} \left(\frac{\gamma \beta (\sqrt{2\Delta})^\gamma}{\sqrt{2}} \right)^{2m} \frac{1}{\sqrt{\pi}} \Gamma(m(\gamma-1) + \frac{1}{2}), \quad (27)$$

where $\Gamma(x)$ is the usual Γ function. From this it follows that

$$\langle n \rangle = \frac{6_{in}^b}{6_{in}} \frac{1}{\sqrt{\pi}} \Gamma(\gamma - \frac{1}{2}) \left(\frac{\gamma \beta (\sqrt{2\Delta})^\gamma}{\sqrt{2}} \right)^2 \quad (28)$$

$$\langle n^m \rangle = C_m \langle n \rangle^m \quad (29)$$

with

$$C_m = \left(\frac{6i_n}{6i_n} \right)^{m-1} \frac{\frac{1}{\sqrt{\pi}} \Gamma(m(\gamma-1) + \frac{1}{2})}{\left(\frac{1}{\sqrt{\pi}} \Gamma(\gamma - \frac{1}{2}) \right)^m} \quad (30)$$

For $\gamma = 1$ the ratio C_{m+1}/C_m is independent of m . For $\gamma > 1$ this ratio increases with m . This seems to be the case experimentally.

We mention that for our derivation of Eq. (29) condition a) can be slightly relaxed. Factors of $\ln x$ would not matter apart from giving a somewhat different numerical value for the coefficients C_m . An exponentially increasing $\delta(x)$ would, however, give rise to multiplicity moments which increased with energy more rapidly than $\langle n \rangle^m$.

5. GEOMETRICAL MODELS OF N MODES

The mesons can now be emitted in any one of the $N > 1$ modes $W_j(Y, \vec{B}, y, \vec{q})$, $j = 1, 2, \dots, N$. From the solution given by Eqs. (6) and (8) we obtain that

$$G_{tot} = 2 \int d^2 B \prod_{j=1}^N \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx_j e^{-x_j^2} \right) (1 - \operatorname{Re} e^{2i \delta(x_1 \sqrt{2\Delta_1}, \dots, x_N \sqrt{2\Delta_N})}) \quad (31)$$

$$G_u = \int d^2 B \left| \prod_{j=1}^N \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx_j e^{-x_j^2} \right) (1 - e^{2i \delta(x_1 \sqrt{2\Delta_1}, \dots, x_N \sqrt{2\Delta_N})}) \right|^2 \quad (32)$$

and

$$G_m = \frac{1}{2^n} \int d^2 B \sum_{\lambda_1=0}^{\infty} \dots \sum_{\lambda_N=0}^{\infty} \delta_{m, \sum \lambda} \left| \prod_{j=1}^N \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx_j e^{-x_j^2} \right) \frac{1}{\sqrt{\lambda_j!}} \left(\frac{\partial}{\partial x_j} \right)^{\lambda_j} e^{2i \delta(x_1 \sqrt{2\Delta_1}, \dots, x_N \sqrt{2\Delta_N})} \right|^2 \quad (33)$$

from which we can calculate, by using Eq. (32) of paper I, the multiplicity moments F_n

$$F_n = \frac{1}{2^n} \frac{1}{6in} \sum_{\lambda_1=0}^{\infty} \dots \sum_{\lambda_N=0}^{\infty} \delta_{n, \sum \lambda} \left(\frac{1}{\sqrt{\pi}} \right)^N \int_{-\infty}^{+\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_N e^{-(x_1^2 + \dots + x_N^2)} \left| \frac{\partial^{\lambda_1}}{\partial x_1^{\lambda_1}} \dots \frac{\partial^{\lambda_N}}{\partial x_N^{\lambda_N}} e^{2i \delta(x_1 \sqrt{2\Delta_1}, \dots, x_N \sqrt{2\Delta_N})} \right|^2 \quad (34)$$

We introduce, in the same fashion as before, the characteristic functions $\mathcal{L}_j^b(\vec{B})$ and $\mathcal{L}_j^t(\vec{B})$ for each mode W_j (for simplicity we assume the "grey" region to be absent) and define the characteristic functions for the "black" and "transparent" regions by

$$\mathcal{L}^t(\vec{B}) = \prod_{j=1}^N \mathcal{L}_j^t(\vec{B}) \quad (35)$$

and

$$\mathcal{L}^b(\vec{B}) = 1 - \mathcal{L}^t(\vec{B}) \quad (36)$$

Having discussed the single mode case in detail we can convince ourselves by direct inspection of the relevant equations that even for the N mode case the qualitative behaviour described by the Table is valid. If we in addition assume that $\delta(x_1, \dots, x_N)$ behaves like a power ≥ 1 as any of $x_j \rightarrow \pm \infty$, the multiplicity scaling given by Eq. (26) again follows as before.

Finally, let us, in order to get some idea how the values of C_m are affected by the presence of several modes, discuss a simple example. We put $\delta(x_1, \dots, x_N) = \delta(x_1) + \dots + \delta(x_N)$ and $\Delta_j(Y, \vec{B}) = \Delta(Y)$. We assume that $\Delta(Y) \rightarrow \infty$ as $Y \rightarrow \infty$ and that $\delta(x) \rightarrow (\beta/2) |x|^\gamma$, $\gamma \geq 1$, as $x \rightarrow \pm \infty$. A simple calculation gives that

$$\langle n^m \rangle = C_m \langle n \rangle^m,$$

where now

$$\langle n \rangle = \left(\frac{\gamma \beta (\sqrt{2} \Delta)^\gamma}{\sqrt{2}} \right)^2 \left(\frac{1}{\sqrt{\pi}} \right)^N N \Gamma(\gamma - \frac{1}{2}) \quad (37)$$

and

$$C_m = \frac{\left(\frac{1}{\sqrt{\pi}} \right)^N \sum_{\lambda_1=0}^{\infty} \dots \sum_{\lambda_N=0}^{\infty} \delta_{m, \sum \lambda} \Gamma(\lambda_1(\gamma-1) + \frac{1}{2}) \dots \Gamma(\lambda_N(\gamma-1) + \frac{1}{2})}{\left(\frac{1}{\sqrt{\pi}} \right)^N N \Gamma(\gamma - \frac{1}{2})^m} \quad (38)$$

as $Y \rightarrow \infty$.

Comparing this with Eq. (30) we see that the presence of several modes tends to increase the numerical values of C_m .

6. CONCLUSIONS

We have discussed the high energy properties of the simplest class of unitary eikonal models for multiproduction. Assuming the absence of the "grey" region we obtained the following behaviour; each individual σ_n goes to zero as $Y \rightarrow \infty$, but their sum σ_{in} approaches a constant, different from zero. The mean multiplicity and its moments diverge. If one in addition assumes a positive power (≥ 1) behaviour, the K.N.O. scaling for the multiplicity moments follows in a natural fashion. These qualitative features seems to be in agreement with some of the main experimental trends of multipion production.

We have not made any numerical calculations for the following reason: just by inspecting Eqs. (30) and (38) one sees that the best chance of reproducing the experimental values of C_m in proton-proton scattering is to take a single mode model with a γ slightly greater than 1. In Ref. 2) a detailed study was made of a geometrical single mode model with $\gamma = 1$ and reasonable numerical results were obtained. We cannot improve on this in a simple manner with the class of models discussed here.

The generalization of the formalism to include the effect of charge conservation will be discussed elsewhere.

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The general meaning of the impact parameter in multiparticle processes has been discussed by G. Calucci and R. Jengo "The Meaning of the Impact Parameter in Multiple Production Processes", Trieste preprint, unpublished and Nuovo Cimento Letters 4, 33 (1972). A geometrical model based on a Hamiltonian approach which in our terminology contains an infinity of modes has been presented in G. Calucci, Nuclear Physics B44, 629 (1972).

The author is indebted to Dr. R. Jengo for calling his attention to these references.

T A B L E

	Transparent Region	Grey Region	B l a c k Case I	r e g i o n Case II
ϵ_{tot}	0	constant	constant	constant
ϵ_{el}	0	constant	$\frac{1}{2}\epsilon_{tot}^b$	ϵ_{tot}^b
ϵ_{in}	0	constant	ϵ_{el}^b	0
ϵ_n	0	constant	0	0
F_n	0	constant	diverges	diverges

Asymptotic contributions from the "transparent", "grey",
and "black" regions.

R E F E R E N C E S

- 1) By low energy mesons we shall mean that each c.m. meson rapidity y is bounded by $|y| \leq \frac{1}{2}(1-\epsilon)Y$ and that the mean multiplicity of the produced mesons does not grow faster than $s^{\epsilon/2}$, s being the total c.m. energy squared. This implies that the average energy carried off by the mesons is small compared to $\sqrt{s}$.
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and Nuovo Cimento 6A, 601 (1971).
- 3) R. Aviv, R.L. Sugar and R. Blankenbecler, Phys.Rev. 5D, 3252 (1972).
- 4) L.B. Redei, CERN preprint TH.1642, submitted to Nuclear Physics.
- 5) We neglect spin, isospin and nucleon-antinucleon pair creation.
- 6) Z. Koba, H.B. Nielsen and P. Olesen, Nuclear Phys. B40, 317 (1972).
- 7) S. Auerbach, R. Aviv, R. Sugar and R. Blankenbecler, Phys.Rev. D6,
2216 (1972) and Phys.Rev.Letters 29, 552 (1972);
R.L. Sugar, preprint, Santa Barbara (1972);
J.A. Skard and J.F. Fulco, preprint, Santa Barbara (1973).
- 8) There is no obvious physical reason for the contrary. If it did occur one merely would have to treat the contributions from $x=+\infty$ and $x=-\infty$ separately, giving rise only to a trivial notational complication. We also observe that $\delta(x)$ has to be an infinitely many times differentiable function of x . Otherwise, Eq. (12), for example, would not make any sense.
- 9) P. Slattey, Phys.Rev. 7D, 2073 (1973).
M. Jacob, Plenary Session Report, XVI International Conference on High Energy Physics, Batavia (1972).