ADIABATIC LIMIT IN PERTURBATION THEORY *)

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A B S T R A C T

We show that, with correct mass and wave function renormalization, the time-ordered products for Wick polynomials $T(\mathcal{L}(y_1) \dots \mathcal{L}(y_n))$ constructed by a method outlined in a previous paper ¹⁾ are such that the vectors of the form

$$\int T(\mathcal{L}(y_1) \dots \mathcal{L}(y_n)) g(y_1) \dots g(y_n) \Psi dy_1 \dots dy_n$$

have limits when g tends to a constant, provided Ψ is chosen in a suitable dense domain. It follows that the S matrix has a unitary adiabatic limit as an operator-valued formal power series in Fock space.

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INTRODUCTION

In a report to the Strasbourg Conference of May 1970 ¹⁾ and in a paper to be published ²⁾, we have described an approach to perturbation theory based on an inductive construction of chronological products for Wick polynomials. The existence of Green functions for the interacting fields was proved (at every order of perturbation theory) but there remained to show the existence of an "adiabatic limit" for the S matrix as an operator - valued formal power series in the Hilbert space. This is the purpose of the present paper which can be considered as the sequel to Ref. 1) and summarizes the second part of Ref. 2). The notations are the same as in Ref. 1); we restrict ourselves (as in Section VI of ¹⁾) to the case of a single self-interacting field with an interaction Lagrangian

$$\mathcal{L}^{(0)}(x) = :A(x)^\nu: \quad , \quad \nu \geq 3 .$$

It will be essential, in this paper, to take the mass m different from 0. Reference 1) will be denoted I, and formula (n) of I will be referred to as (I,n) in the following.

1. THE PROBLEM

We wish to show that, when $g \in \mathcal{S}(\mathbb{R}^4)$ and g tends to a constant λ (in some sense to be precised) the operator

$$S_n(g) = \int T^{(0, \dots, 0)}(x_1, \dots, x_n) g(x_1) \dots g(x_n) dx_1 \dots dx_n \quad (1)$$

tends to an operator in Fock space, in some sense, to be precised, strong enough to yield the unitarity of the limiting S matrix considered as a formal power series in λ .

Since the operator (1) is unbounded, the convergence may only be expected to hold when this operator is applied to a vector chosen in a suitable domain D_2 . A natural restriction to impose on such vectors is that they contain a finite number of particles. Moreover,

since the kernels of the operator (1) are well known to have singularities such as poles, etc., one must expect that the wave functions of the admissible vectors should be rather regular. But we wish to be able to multiply two of the limiting operators, hence, they should map D_2 into itself. Now, the simplest example [take $T^{(0)}(x) = \mathcal{L}^{(0)}(x) = :A(x)^\vee :$] shows that a vector with $\mathcal{E}^\infty$ wave functions (in momentum space) is transformed by $\int \mathcal{L}^{(0)}(x) dx$ into a vector whose wave functions are no longer $\mathcal{E}^\infty$ but only Hölder-continuous. Thus we are led to conjecture that D_2 might be formed of vectors with a finite number of non-zero Hölder-continuous wave functions. This turns out to be correct.

As to the sense in which $S_n(g)$ tends to a limit, the simplest conjecture would be that, if $\Psi \in D_2$, the vector valued tempered distribution

$$\int e^{i \sum_{j=1}^n q_j x_j} T(x_1, \dots, x_n) \Psi dx_1 \dots dx_n \quad (2)$$

might actually be a (vector valued) continuous function of $q_1, \dots, q_n$ in a neighbourhood of zero. However, the simplest examples dispel this illusion, so that a more subtle way has to be found for defining the value of (2) at zero. The next section describes a class of distributions for which it is possible to define a value at a certain point. It turns out that the method of evaluation given there can be used to define the "value at 0" of (2) and to find the sense in which $S_n(g)\Psi$ converges to that value as $g(x) \rightarrow 1$.

Outline of the method

Suppose T is a distribution on $\mathbb{R}^N$ and that there exist constants $\delta > 0$, $M \geq 0$, $C \geq 0$, such that for every $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$|\langle T, \varphi \rangle| \leq C \left\{ \left| \int \varphi(q) dq \right| + \sum_{|\alpha| \leq M} \sup_q |q|^{N+\delta+|\alpha|} |D^\alpha \varphi(q)| \right\}$$

"adiabatic property"

Then, it will be seen in Section 2 that the limit

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \varepsilon^{-N} \int T(q) \varphi\left(\frac{q}{\varepsilon}\right) dq$$

exists and is of the form $\text{const.} \int \varphi(q) dq$; we then take this constant to be the "value" or "adiabatic limit" of the distribution T at 0 .

The idea of our method is to prove by induction on n that (2), as a vector valued distribution in $q_1, \dots, q_n$, satisfies such a condition. It will be recalled that the inductive construction of the $T(X)$ operators, or, equivalently, of the advanced, retarded operators etc., proceeds in two steps.

- a) One constructs the "discontinuity", or difference between a totally advanced and totally retarded operator of n "Lagrangians" $\mathcal{L}^{(r_1)}(x_1), \dots, \mathcal{L}^{(r_n)}(x_n)$. This difference is expressed in terms of products of chronological operators with fewer arguments. Thus, one must verify that the product of two operator-valued distributions having the above property also has it. This is essentially due to the fact that the "adiabatic property" for distributions is stable under tensor products. It follows that the "discontinuity" possesses the adiabatic property.
- b) The totally advanced operator of n Lagrangians is then obtained essentially by a dispersion relation in many variables, i.e., by convoluting the discontinuity with a generalized Cauchy kernel; this operation also preserves the adiabatic property.

However, there are technical difficulties:

- i) The convolution is performed over $n-1$ of the n four-vector variables, say, $q_1, \dots, q_{n-1}$; as a consequence the variable $Q = \sum_{j=1}^n q_j$ has to be treated separately: in this variable one must prove the Hölder continuity of the matrix elements of (2), except those corresponding to self-energy kernels.
- ii) The self-energy kernels must have been properly renormalized i.e., correct mass and wave function renormalization must have been performed (See I, Section VIII). This insures that they

possess the adiabatic property, and that their adiabatic limit vanishes twice on the mass shell, and is necessary in order to prove the correct properties of the product of a self-energy kernel with another kernel.

In Section 2 we have assembled the mathematical information regarding distributions having the "adiabatic property". In Section 3, the induction hypothesis is described. Section 4 contains the rather straightforward verifications needed to show that the induction works. Except for the treatment of self-energy kernels, this can be omitted by the reader without great loss of information.

2. SOME MATHEMATICAL PRELIMINARIES

2.1 Notation for Hölder Continuous Functions

(We are interested in functions defined over momentum space, hence, the natural unit of length is the mass m .)

Let f be a function defined on $\mathbb{R}^N$ and taking its values in a Banach space whose norm is denoted B . We denote, for $\delta > 0$:

$$H_{x;B}^{\delta}(f(x)) = \max \left\{ \sup_x B(f(x)), \sup_{\substack{x,h \\ |h| \leq m}} \left| \frac{h}{m} \right|^{-\delta} B(f(x+h) - f(x)) \right\}$$

(3)

In particular, if f is a numerical function, we omit B and denote

$$H_x^{\delta}(f(x)) = \max \left\{ \sup_x |f(x)|, \sup_{\substack{x,h \\ |h| \leq m}} \left| \frac{h}{m} \right|^{-\delta} |f(x+h) - f(x)| \right\}$$

(4)

Let f be a function of two variables $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^M$. It is easy to verify that, if $\rho > 0$, $\sigma > 0$, $\delta > 0$:

$$H_{x; H_{y; B}^\sigma}^\rho (f(x, y)) = H_{y; H_{x; B}^\rho}^\sigma (f(x, y)) \quad (5)$$

$$H_{x; H_{y; B}^\sigma}^\rho (f(x, y)) \leq 2 H_{x, y; B}^{\rho+\sigma} (f(x, y)) \quad (6)$$

$$H_{x, y; B}^\delta (f(x, y)) \leq 2 H_{x; H_{y; B}^\delta}^\delta (f(x, y)) \quad (7)$$

Note that, for a fixed f , $H_x^\delta (f(x))$ is an increasing function of δ .

2.2 "Adiabatic limit" of a Distribution at a Point

Definition 1

We say that a distribution $T \in \mathcal{D}'(\mathbb{R}^N)$ with compact support satisfies an adiabatic norm of order δ at 0 if $\delta > 0$ and if there exist constants $C \geq 0$ and $P \geq 0$ such that, for every $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$| \langle T, \varphi \rangle | \leq C \{ |I(\varphi)| + \|\varphi\|_{\delta, P} \} \quad (8)$$

Here, and in the rest of this section, we use the notation:

$$\|\varphi\|_{\delta, P} = \sum_{0 \leq |\alpha| \leq P} \sup_{x \in \mathbb{R}^N} |x|^{N+\delta+|\alpha|} |\mathcal{D}^\alpha \varphi(x)|$$

$|x|$ being the Euclidean norm in $\mathbb{R}^N$.

$$I(\varphi) \text{ denotes } \int_{\mathbb{R}^N} \varphi(x) dx .$$

Clearly this notion is invariant against linear (non-singular) changes of co-ordinates. Moreover, if T has this property, and if h is a function in $\mathcal{S}(\mathbb{R}^N)$, we have

$$\begin{aligned} | \langle hT, \varphi \rangle | &= | \langle T, h\varphi \rangle | \leq \\ &\leq C \left\{ \left| \int h(x) \varphi(x) dx \right| + \sum_{|\alpha| \leq p} \sum_{\delta \leq \alpha} \frac{\alpha!}{\delta! (\alpha - \delta)!} \sup_x |x|^{N+\delta+|\alpha|} |D^{\alpha-\delta} h(x)| \cdot \right. \\ &\quad \left. \cdot |D^\delta \varphi(x)| \right\} \leq \\ &\leq C' \left\{ \left| \int h(x) \varphi(x) dx \right| + \|\varphi\|_{\delta, p} \right\} \end{aligned}$$

However,

$$\begin{aligned} \left| \int h(x) \varphi(x) dx \right| &= \left| \int [h(x) - h(0)] \varphi(x) dx + h(0) \int \varphi(x) dx \right| \leq \\ &\leq |h(0)| \left| \int \varphi(x) dx \right| + \sup_x |x|^{N+\delta} |\varphi(x)| \cdot \int \frac{|h(y) - h(0)| dy}{|y|^{N+\delta}} \end{aligned}$$

If $0 < \delta < 1$ the last integral exists (since h is continuously differentiable and bounded) and is bounded by $\text{const. } \delta^{-1}(1-\delta)^{-1}$.

Hence

$$| \langle hT, \varphi \rangle | \leq C_\delta'' \{ |I(\varphi)| + \|\varphi\|_{\delta, p} \} \quad \text{if } 0 < \delta < 1$$

The same calculation shows that, under the same condition, and if $h(0) \neq 0$,

$$|I(\varphi)| \leq C''' \left[\left| \int \varphi(x) h(x) dx \right| + \|\varphi\|_{\delta, 0} \right]$$

Hence, in the preceding definition one can replace $I(\varphi)$ by

$\int h(x) \varphi(x) dx$ with any $h \in \mathcal{J}(\mathbb{R}^N)$ such that $h(0) \neq 0$, provided $0 < \delta < 1$.

This gives a meaning to the following extension of definition 1.

Definition 2

We say that a tempered distribution $T \in \mathcal{J}'(\mathbb{R}^N)$ satisfies an adiabatic norm of order δ at 0 if there is a function $h \in \mathcal{D}(\mathbb{R}^N)$ with $h(0) = 1$ such that

$$|\langle hT, \varphi \rangle| \leq C \{ |I(\varphi)| + \|\varphi\|_{\delta, P} \}$$

for some $C \geq 0$, some $P \geq 0$, and all $\varphi \in \mathcal{J}(\mathbb{R}^N)$.

While we have given this definition for completeness, we shall always restrict our attention to the local behaviour of distributions in this paper.

Lemma 1

If a distribution $T \in \mathcal{J}'(\mathbb{R}^N)$ satisfies the adiabatic norm (8) then the "adiabatic limit"

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \langle T(\varepsilon x), \varphi(x) \rangle = \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \langle T, \varphi_\varepsilon \rangle = L I(\varphi) \quad (9)$$

exists for every $\varphi(x) \in \mathcal{J}(\mathbb{R}^N)$ and is a distribution in $\mathcal{V}$ of the form indicated in (9), where L is a constant independent of φ . Here

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^N} \varphi\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0 \quad (10)$$

Proof

In view of the formula:

$$\langle T, \varphi_\varepsilon \rangle = \langle T, \varphi \rangle - \int_\varepsilon^\infty \langle T, \frac{\partial}{\partial \eta} \varphi_\eta \rangle d\eta \quad (11)$$

it is enough to prove that the last integral converges absolutely at $\varepsilon = 0$ in order to establish the existence of the adiabatic limit.

Now we have

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} \varphi_\varepsilon(x) &= -\frac{1}{\varepsilon} \chi_\varepsilon(x) \quad \text{with} \quad \chi_\varepsilon(x) = \sum_{i=1}^N \frac{\partial}{\partial x_i} \{x_i \varphi(x)\} \equiv \partial(x \varphi(x)), \\ \chi_\varepsilon(x) &= \varepsilon^{-N} \chi\left(\frac{x}{\varepsilon}\right) \end{aligned} \quad (12)$$

and, therefore, by inserting in (8)

$$|\langle T, \frac{\partial}{\partial \varepsilon} \varphi_\varepsilon \rangle| \leq \varepsilon^{\delta-1} C \|\partial(x \varphi)\|_{\delta, p} \leq \text{const.} \varepsilon^{\delta-1} \quad (13)$$

Here we have used the fact that

$$I\left(-\frac{1}{\varepsilon} \chi_\varepsilon\right) = -\varepsilon^{-1} I(\chi) = 0$$

(change of integration variables $y = x/\varepsilon$ and Gauss theorem since χ is a divergence) and the formula

$$\sup_{x \in \mathbb{R}^N} |F(x)| = \sup_{y \in \mathbb{R}^N} F(\varepsilon y) \quad \text{for any } \varepsilon > 0 \quad (14)$$

in evaluating the sup appearing in (8). Since $\delta > 0$ the integral (11) is absolutely convergent at $\varepsilon = 0$. Hence, the adiabatic limit (9) exists and is by a well-known theorem again a distribution in $\mathcal{S}'$, say S :

$$\lim_{\varepsilon \rightarrow +0} \langle T, \varphi_\varepsilon \rangle = \langle S, \varphi \rangle$$

Now, by the same argument which led us to (13) we get for any $\varphi \in \mathcal{S}$ and $\varepsilon > 0$:

$$|\langle T, \varphi_\varepsilon \rangle| \leq C \{ |I(\varphi)| + \varepsilon^\delta \|\varphi\|_{\delta, p} \} \quad (15)$$

and upon taking the limit $\varepsilon \rightarrow +0$:

$$|\langle S, \varphi \rangle| \leq C |I(\varphi)|$$

By applying this inequality to $\psi(x) = \varphi(x) - I(\varphi) \cdot \varphi_0(x) \in \mathcal{J}$, where φ is any element $\in \mathcal{J}$ and φ_0 a fixed element $\in \mathcal{J}$ such that $I(\varphi_0) = 1$, we get $\langle S, \psi \rangle = \langle S, \varphi \rangle - \langle S, \varphi_0 \rangle \cdot I(\varphi) = 0$ in view of $I(\psi) = 0$. This proves the lemma if we set $\langle S, \varphi_0 \rangle = L$.

In the next Lemma we shall need the inequality:

$$\begin{aligned} |\langle T, \varphi \rangle - L I(\varphi)| &= \left| \int_0^1 \langle T, \frac{\partial}{\partial \varepsilon} \varphi_\varepsilon \rangle d\varepsilon \right| \leq \\ &\leq \delta^{-1} C \|\partial(x\varphi)\|_{\delta, p} \end{aligned} \quad (16)$$

which is an immediate consequence of the equality (11) with $\varepsilon = 0$ and the inequality (13).

Lemma 2

Suppose for a $T \in \mathcal{J}'(\mathbb{R}^N)$ the adiabatic norm

$$|\int T(x+y) \varphi(x) dx| \leq C \{ |I(\varphi)| + \|\varphi\|_{\delta, p} \} \quad (17)$$

holds uniformly for all y in a closed convex set $\mathcal{K} \subset \mathbb{R}^N$ (i.e., C is a constant independent of $y \in \mathcal{K}$). Then (by the preceding lemma) the adiabatic limit

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \int T(\varepsilon x + y) \varphi(x) dx = \lim_{\varepsilon \rightarrow +0} \int T(x+y) \varphi_\varepsilon(x) dx = L(y) I(\varphi) \quad (18)$$

exists for all $y \in \mathcal{K}$ and $L(y)$ is a uniformly bounded and uniformly Hölder continuous function on $\mathcal{K}$ with the index of continuity δ . More precisely

$$a) |L(y)| \leq C, \quad b) |L(y+h) - L(y)| \leq M|h|^5 \quad (19)$$

for all y and $y+h \in \mathcal{H}$

With C and M , two constants independent of y .

Proof

The inequality (19a) follows immediately from the inequality (15) applied to this case upon passage to the limit $\varepsilon \rightarrow +0$. In order to obtain (19b) we take any fixed $\varphi \in \mathcal{J}$ such that $I(\varphi) = 1$ and write for an $\varepsilon > 0$ to be chosen later:

$$\begin{aligned} |L(y+h) - L(y)| &\leq \left| \int T(x+y+h) \varphi_\varepsilon(x) dx - L(y+h) \right| + \\ &+ \left| \int T(x+y) \varphi_\varepsilon(x) dx - L(y) \right| + \left| \int \{T(x+y+h) - T(x+y)\} \varphi_\varepsilon(x) dx \right| \\ &\leq 2\delta^{-1} C \|\partial(x \varphi_\varepsilon)\|_{\delta, P} + \left| \int T(x+y) \{ \varphi_\varepsilon(x-h) - \varphi_\varepsilon(x) \} dx \right| \end{aligned} \quad (20)$$

The last inequality follows from the inequality (16) in view of $I(\varphi_\varepsilon) = I(\varphi) = 1$. Now, it is easily seen that:

$$\|\partial(x \varphi_\varepsilon)\|_{\delta, P} = \varepsilon^\delta \|\partial(x \varphi)\|_{\delta, P} \quad (21)$$

On the other hand we write:

$$\varphi_\varepsilon(x-h) - \varphi_\varepsilon(x) = \int_0^1 dt \frac{d}{dt} \varphi_\varepsilon(x-th) = - \int_0^1 h \partial \varphi_\varepsilon(x-th) dt \quad (22)$$

with

$$h \partial = \sum_{i=1}^n h_i \frac{\partial}{\partial x_i}$$

so that the last term in the inequality (20) becomes

$$\begin{aligned} |\int \tau(x+y) \{ \varphi_\varepsilon(x-h) - \varphi_\varepsilon(x) \} dx| &= |\int_0^1 dt \int \tau(x+y+th) h \partial \varphi_\varepsilon(x) dx| \\ &\leq C \|h \partial \varphi_\varepsilon(x)\|_{\delta, p} = C \varepsilon^{\delta-1} \|h \partial \varphi\|_{\delta, p} \end{aligned} \quad (23)$$

The inequality in (23) is obtained by application of the Ineq. (17) to the test function $h \partial \varphi_\varepsilon$ in view of the fact that by convexity $y+th \in \mathcal{K}$ for all $0 \leq t \leq 1$. By the theorem of Gauss the integral term does not contribute. As to the last equality, we obtain it by using the trick (14). Combining (21) and (23) we finally arrive at:

$$|L(y+h) - L(y)| \leq C \left[2\delta^{-1} \|\partial(x\varphi)\|_{\delta, p} + \left\| \frac{h}{\varepsilon} \partial \varphi \right\|_{\delta, p} \right] \varepsilon^\delta \quad (24)$$

By choosing $\varepsilon = |h|$ we see that the expression in square brackets remains bounded for all h and hence (19b) is also proven.

Lemma 3

Under the condition of Lemma 2 we have, for any y and $y+h \in \mathcal{K}$ the inequality:

$$\begin{aligned} |\int \{ \tau(x+y+h) - \tau(x+y) \} \varphi(x) dx| &\leq \\ &\leq M \left\{ |h|^\delta |I(\varphi)| + |h|^{(1-\theta)\delta} \|\varphi\|_{(1-\theta)\delta, p+1} \right\} \end{aligned} \quad (25)$$

Here $0 < \delta \leq 1$, $0 \leq \theta < 1$ and M is a constant independent of y , h and φ .

Proof

Choose a $u \in \mathcal{D}(\mathbb{R}^N)$ such that $u(x) = 1$ for $0 \leq |x| \leq \frac{1}{2}$, $u(x) = 0$, for $|x| \geq 1$ and $0 \leq u(x) \leq 1$ everywhere and write:

$$\varphi(x) = u\left(\frac{x}{|h|}\right) \varphi(x) + v\left(\frac{x}{|h|}\right) \varphi(x) \equiv \varphi_1(x) + \varphi_2(x) \quad (26)$$

where $v = 1 - u$.

Note the following support properties of u and v :

$$\begin{aligned} \text{supp. } u\left(\frac{x}{|k|}\right) &\subset \{|x| \leq |k|\}, \quad \text{supp. } v\left(\frac{x}{|k|}\right) \subset \{|x| \geq \frac{|k|}{2}\} \\ \text{supp. } \mathcal{D}^\alpha u\left(\frac{x}{|k|}\right) &= \text{supp. } \mathcal{D}^\alpha v\left(\frac{x}{|k|}\right) \subset \left\{ \frac{|k|}{2} \leq |x| \leq |k| \right\} \end{aligned}$$

for any $\alpha \neq 0$

(27)

With the shorthand notation $\Delta = T(x+y+h) - T(x+y)$ the left-hand side of (25) is upon insertion of (26) bounded by

$$|\langle \Delta, \varphi \rangle| \leq |\langle \Delta, \varphi_1 \rangle| + |\langle \Delta, \varphi_2 \rangle|$$

(28)

The first term we treat as follows:

$$\begin{aligned} |\langle \Delta, \varphi_1 \rangle| &\leq |\langle \Delta - L(y+h) + L(y), \varphi_1 \rangle| + \\ &+ |L(y+h) - L(y)| |I(\varphi_1)| \leq \\ &\leq \text{const.} \left\{ \|\partial(x\varphi_1)\|_{\delta, P} + |k|^\delta |I(\varphi_1)| \right\} \end{aligned} \quad (29)$$

Here L is the adiabatic limit of the distribution T , and we have used the inequality (19b) of Lemma 2 and the inequality (16).

Now we evidently have

$$\begin{aligned} \|\partial(x\varphi_1)\|_{\delta, P} &\leq \text{const} \|\varphi_1\|_{\delta, P+1} = \\ &= \text{const.} \sum_{0 \leq |\alpha| \leq P+1} \sup_{|x| \leq |k|} |x|^{N+\delta+|\alpha|} |\mathcal{D}^\alpha \varphi_1(x)| \end{aligned} \quad (30)$$

We differentiate φ_1 using Leibniz's formula and obtain:

$$|x|^{N+|\alpha|+\delta} |\mathcal{D}^\alpha \varphi_1(x)| \leq \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta! \gamma!} \frac{|x|^{N+\delta+|\beta|+|\gamma|}}{|k|^{|\gamma|}} |\mathcal{D}^\gamma u\left(\frac{x}{|k|}\right)| |\mathcal{D}^\beta \varphi(x)|$$

Since $|\mathcal{D}^\gamma u(x)| \leq \text{const}$ for all $|\gamma| \leq P+1$ and $|x|^\omega \leq |h|^\omega$ for all $\omega \geq 0$ in the sup region, we get:

$$\begin{aligned} \|\partial(x\varphi_2)\|_{\delta, P} &\leq \text{const. } |h|^\theta \sum_{|\alpha| \leq P+1} \sup_{|x| \leq |h|} |x|^{N+(1-\theta)\delta+|\alpha|} |D^\alpha \varphi(x)| \\ &\leq \text{const. } |h|^{\delta\theta} \|\varphi\|_{(1-\theta)\delta, P+1} \end{aligned} \quad (31)$$

for all $0 \leq \theta \leq 1$. As for the integral term in (29) we can bound it as follows:

$$\begin{aligned} |I(\varphi_2)| &\leq |I(\varphi)| + |I(\varphi_2)| \leq |I(\varphi)| + \int_{|x| \geq \frac{|h|}{2}} |\varphi(x)| d^N x \leq \\ &\leq |I(\varphi)| + \sup_{x \in \mathbb{R}^N} |x|^{N+\eta} |\varphi(x)| \int_{|y| \geq \frac{|h|}{2}} \frac{d^N y}{|y|^{N+\eta}} \leq \\ &\leq |I(\varphi)| + \frac{\text{const.}}{\eta} |h|^{-\eta} \sup_x |x|^{N+\eta} |\varphi(x)| \end{aligned} \quad (32)$$

for any $\eta > 0$. By choosing $\eta = (1-\theta)\delta$ with $0 \leq \theta < 1$, and inserting the inequalities (32) and (31) into (29) we see that the contribution $|\langle \Delta, \varphi_1 \rangle|$ satisfies the inequality (25).

The last term in (28) we treat as in formula (23):

$$\begin{aligned} |\langle \Delta, \varphi_2 \rangle| &= \left| \int_0^1 dt \int T(x+y+th) h \partial \varphi_2(x) dx \right| \leq \\ &\leq \text{const } |h| \sum_{1 \leq |\alpha| \leq P+1} \sup_{|x| \geq \frac{|h|}{2}} |x|^{N-1+\delta+|\alpha|} |D^\alpha \varphi_2(x)| \leq \\ &\leq \text{const } |h|^\theta \sum_{1 \leq |\alpha| \leq P+1} \sup_{|x| \geq \frac{|h|}{2}} |x|^{N+(1-\theta)\delta+|\alpha|} |D^\alpha \varphi_2(x)| \end{aligned} \quad (33)$$

In the last step we have written $|h| = |h|^\theta |h|^{1-\theta\delta}$, $0 \leq \theta \leq 1$, and used the fact that $|h|^\omega \leq 2^\omega |x|^\omega$ for all $\omega \geq 0$ in the sup region. Again, by Leibniz's formula and with the abbreviation

$$\begin{aligned} \delta' &= (1-\theta)\delta, \\ \sup_{|x| \geq \frac{|h|}{2}} |x|^{N+\delta'+|x|} |D^\alpha \varphi_2(x)| &\leq \sup_{|x| \geq \frac{|h|}{2}} |x|^{N+\delta'+|x|} |v(\frac{x}{|h|}) D^\alpha \varphi(x)| + \\ &+ \text{const.} \sum_{\substack{\delta+\beta=\alpha \\ \delta \neq 0}} \sup_{\frac{|h|}{2} \leq |x| \leq |h|} |x|^{N+\delta'+|x|+|\beta|} \frac{1}{|h|^{|\beta|}} |(D^\beta v)(\frac{x}{|h|})| |D^\beta \varphi(x)| \end{aligned} \quad (34)$$

[compare (27)]. The remark that $|x|^{|\gamma|} \leq |h|^{|\gamma|}$ in the sup region $|h|/2 \leq |x| \leq |h|$ leads to the conclusion that

$$|K\Delta, \varphi_2| \leq \text{const.} |h|^{\theta\delta} \|\varphi\|_{(1-\theta)\delta, p+1}$$

which completes the proof.

We shall need the following special case.

Lemma 4

Suppose that, for a certain $T \in \mathcal{S}'(\mathbb{R}^N)$ the inequality

$$|\int T(x+y) \varphi(x) dx| \leq C \|\varphi\|_{\delta, p}$$

holds uniformly for all y in a closed convex set $\mathcal{K} \subset \mathbb{R}^N$ (i.e., C is a constant independent of $y \in \mathcal{K}$). Then, we have, for any y and $y+h \in \mathcal{K}$:

$$\begin{aligned} |\int \{T(x+y+h) - T(x+y)\} \varphi(x) dx| &\leq \\ &\leq M |h|^{\theta\delta} \|\varphi\|_{(1-\theta)\delta, p+1} \end{aligned}$$

Here $0 < \delta \leq 1$, $0 \leq \theta < 1$, and M is a constant independent of y , h , and φ .

Proof

This has already been proved in the course of proving Lemma 3.

Lemma 5

Let T be a distribution with compact support which, in an open neighbourhood of 0 , coincides with a Hölder continuous function. Then T satisfies an adiabatic norm at 0 .

Proof

First suppose that T vanishes in a neighbourhood of 0 . Then, it is clear, by a partition of the unit, that there are constants $C \geq 0$, $a > 0$, and $P \geq 0$, such that

$$\begin{aligned} | \langle T, \varphi \rangle | &\leq C \sum_{|\alpha| \leq P} \sup_{\substack{x \\ |x| \geq a}} |D^\alpha \varphi(x)| \leq \\ &\leq C'_\omega \sum_{|\alpha| \leq P} \sup_{\substack{x \\ |x| \geq a}} a^{\omega + |\alpha|} |D^\alpha \varphi(x)| \leq \\ &\leq C'_\omega \sum_{|\alpha| \leq P} \sup_x |x|^{\omega + |\alpha|} |D^\alpha \varphi(x)| \end{aligned}$$

for any $\omega \geq 0$.

It suffices now to prove the lemma in case T is a Hölder continuous function with compact support. Then T satisfies

$$|T(x) - T(0)| < C |x|^\delta \quad \text{with } 0 \leq \delta < 1$$

and

$$\begin{aligned} \int T(x) \varphi(x) dx &= T(0) \int \varphi(x) dx + \int [T(x) - T(0)] \varphi(x) dx, \\ | \int [T(x) - T(0)] \varphi(x) dx | &\leq C \left\{ \int_{|x| < R} |x|^\delta |\varphi(x)| dx + |T(0)| \int_{|x| \geq R} |\varphi(x)| dx \right\} \\ &\leq C' \left\{ \sup_x |x|^{N+\delta-\varepsilon} |\varphi(x)| \frac{R^\varepsilon}{\varepsilon} + |T(0)| \sup_x |x|^{N+p} |\varphi(x)| \frac{1}{\rho R^p} \right\} \end{aligned}$$

for any $\varepsilon > 0$ and $\rho > 0$.

In particular, choosing $\varepsilon = (1 - \theta)\delta$, with $0 < \theta < 1$ and $\rho = \theta\delta$ we have

$$|\int T(x) \varphi(x) dx| \leq C_{\theta\delta} \left\{ \sup_x |x|^{N+\theta\delta} |\varphi(x)| + |I(\varphi)| \right\}$$

Thus, we see that T satisfies an adiabatic norm of order $\theta\delta$ where $0 < \theta < 1$.

Tensor Product Rule

Lemma 6

Let $T \in \mathcal{J}'(\mathbb{R}^N)$ and $S \in \mathcal{J}'(\mathbb{R}^M)$ be two distributions with compact support. Assume that there exist numbers $\omega > 0$, $P \geq 0$, $C \geq 0$, $\lambda > 0$, $Q \geq 0$, $D \geq 0$ such that, for any $\varphi \in \mathcal{J}(\mathbb{R}^N)$ and $\psi \in \mathcal{J}(\mathbb{R}^M)$,

$$|\langle T, \varphi \rangle| \leq C \left\{ \left| \int_{\mathbb{R}^N} \varphi(x) dx \right| + \sum_{|\alpha| \leq P} \sup_{x \in \mathbb{R}^N} |x|^{\omega+|\alpha|} |D_x^\alpha \varphi(x)| \right\}$$

$$|\langle S, \psi \rangle| \leq D \left\{ \left| \int_{\mathbb{R}^M} \psi(y) dy \right| + \sum_{|\beta| \leq Q} \sup_{y \in \mathbb{R}^M} |y|^{\lambda+|\beta|} |D_y^\beta \psi(y)| \right\}$$

Then, if $\lambda = M + \delta_2$, $\omega = N + \delta_1$, $0 < \delta_2 < 1$, $0 < \delta_1 < 1$, $\delta = \min(\delta_1, \delta_2)$, there exists for each $\varepsilon > 0$ with $\varepsilon \leq N$, $\varepsilon \leq M$, a constant K_ε such that, for any $\chi \in \mathcal{D}(\mathbb{R}^N \times \mathbb{R}^M)$

$$|\langle T \otimes S, \chi \rangle| \leq K_\varepsilon \left\{ \left| \int \chi(x, y) dx dy \right| + \right.$$

$$\left. + \sum_{|\alpha| \leq P+Q} \sup_{x, y} |x, y|^{N+M+\delta+|\alpha|-\varepsilon} |D_{x, y}^\alpha \chi(x, y)| \right\}$$

Proof

Let $\chi \in \mathcal{D}(\mathbb{R}^N \times \mathbb{R}^M)$ with support in an open bounded ball containing $\text{supp. } T \times \text{supp. } S$.

$$\begin{aligned}
 & \left| \int T(x) S(y) \chi(x, y) dx dy \right| = \left| \int T(x) dx \int S(y) \varphi(x, y) dy \right| \\
 & \leq C \left\{ \left| \int S(y) dy \int \chi(x, y) dx \right| + \right. \\
 & \quad \left. + \sum_{|\alpha| \leq P} \sup_{x \in \mathbb{R}^N} |x|^{\omega+|\alpha|} \left| \int S(y) D_x^\alpha \chi(x, y) dy \right| \right\} \\
 & \leq C D \left\{ \left| \int \chi(x, y) dx dy \right| + \sum_{|\beta| \leq Q} \sup_{y \in \mathbb{R}^M} |y|^{\lambda+|\beta|} \left| \int D_y^\beta \chi(x, y) dx \right| \right. \\
 & \quad \left. + \sum_{|\alpha| \leq P} \sup_{x \in \mathbb{R}^N} |x|^{\omega+|\alpha|} \left| \int D_x^\alpha \chi(x, y) dy \right| + \right. \\
 & \quad \left. + \sum_{\substack{|\alpha| \leq P \\ |\beta| \leq Q}} \sup_{\substack{x \in \mathbb{R}^N \\ y \in \mathbb{R}^M}} |x|^{\omega+|\alpha|} |y|^{\lambda+|\beta|} |D_x^\alpha D_y^\beta \chi(x, y)| \right\};
 \end{aligned}$$

$$\sup_{y \in \mathbb{R}^M} |y|^{\lambda+|\beta|} \left| \int D_y^\beta \chi(x, y) dx \right| \leq$$

$$\leq \text{const. } \varepsilon^{-1} \sup_{\substack{y \in \mathbb{R}^M \\ x \in \mathbb{R}^N}} |y|^{\lambda+|\beta|} |x|^{N-\varepsilon} |D_y^\beta \chi(x, y)|$$

$$\leq \text{const. } \varepsilon^{-1} \sup_{\substack{y \in \mathbb{R}^M \\ x \in \mathbb{R}^N}} |x, y|^{\lambda+N-\varepsilon+|\beta|} |D^\beta \chi(x, y)|$$

for any ε such that $0 < \varepsilon \leq N$. The terms

$$\sup_{x \in \mathbb{R}^N} |x|^{\omega+|\alpha|} \left| \int \mathcal{D}_x^\alpha \chi(x, y) dy \right|$$

can be treated similarly, and the result follows.

The result can be extended to the case $\chi \in \mathcal{S}(\mathbb{R}^N \times \mathbb{R}^M)$ by noticing that $\langle T \otimes S, \chi \rangle = \langle T \otimes S, h\chi \rangle$ for a fixed $h \in \mathcal{D}(\mathbb{R}^{N+M})$ and using the remarks at the beginning of Section 2.2.

2.3 Study of $\mathcal{D}^\alpha \tilde{\omega} * \varphi$

In this subsection, $\tilde{\omega}$ is a tempered distribution over $\mathbb{R}^N$, $N \geq 2$ with the following property:

$$\tilde{\omega} = \sum_{|\beta|=1} \mathcal{D}^\beta \chi_\beta$$

where, for each β (with $|\beta|=1$), χ_β is the distribution defined by a function over $\mathbb{R}^N$ (also denoted by χ_β); this function is assumed to be $\mathcal{C}^\infty$ everywhere except at 0, and homogeneous of degree $-N+1$, i.e.,

$$\chi_\beta(\rho q) = \rho^{-N+1} \chi_\beta(q) \quad \text{for all } q \neq 0 \text{ and } \rho > 0$$

As a consequence, in the complement of the origin,

$$|\mathcal{D}^\alpha \chi_\beta(q)| < \frac{C_\alpha}{|q|^{N+|\alpha|-1}},$$

$$|\mathcal{D}^\alpha \tilde{\omega}(q)| < \frac{B_\alpha}{|q|^{N+|\alpha|}}$$

This implies, in particular, that χ_β is locally absolutely integrable.

We wish to investigate, for $\varphi \in \mathcal{S}(\mathbb{R}^N)$, the $(\mathcal{C}^\infty)$ function obtained by the convolution $\mathcal{D}^\alpha \tilde{\omega} * \varphi(q) = \tilde{\omega} * \mathcal{D}^\alpha \varphi(q)$

$$\begin{aligned}
 &= \sum_{|\beta|=1} \chi_{\beta} * D^{\alpha+\beta} \varphi(q) = \\
 &= \int \sum_{|\beta|=1} \chi_{\beta}(q-q') D^{\alpha+\beta} \varphi(q') dq' \quad (35)
 \end{aligned}$$

and we assume, in the following, $q \neq 0$.

The integral (35) can be broken into two terms by dividing the range of integration into $\{q' : |q'| \leq \frac{1}{2}|q|\}$ and $\{q' : |q'| > \frac{1}{2}|q|\}$. In the first of these regions we have:

$$|q'| \leq \frac{1}{2}|q|, \quad |q - q'| \geq \frac{1}{2}|q| > 0$$

Hence,

$$\begin{aligned}
 &\int_{|q'| \leq \frac{1}{2}|q|} \sum_{|\beta|=1} \chi_{\beta}(q-q') D^{\alpha+\beta} \varphi(q') dq' = \\
 &= \int_{|q'| \leq \frac{1}{2}|q|} D^{\alpha} \tilde{W}(q-q') \varphi(q') dq' + \\
 &+ \int_{|q'| = \frac{1}{2}|q|} d\sigma(q') \sum_{|\beta|=1} \frac{q'^{\beta}}{|q'|} \chi_{\beta}(q-q') D^{\alpha} \varphi(q') + \\
 &+ \sum'_{\substack{\rho+\sigma+\gamma=\alpha \\ |\rho|=1}} \int_{|q'| = \frac{1}{2}|q|} d\sigma(q') \frac{q'^{\rho}}{|q'|} D^{\sigma} \tilde{W}(q-q') D^{\gamma} \varphi(q')
 \end{aligned}$$

[The symbol $\sum'$ means that not all values of ρ, σ, γ such that $\rho + \sigma + \gamma = \alpha$, $|\rho| = 1$ are actually present in the sum; $d\sigma(q')$ is the surface element of the sphere $\{q' : |q'| = \frac{1}{2}|q|\}$.]
We shall first deal with the surface terms:

$$\left| \int_{|q'| = \frac{1}{2}|q|} d\sigma(q') \frac{q'^{\rho}}{|q'|} D^{\sigma} \tilde{W}(q-q') D^{\gamma} \varphi(q') \right| \leq$$

$$\begin{aligned}
 & \leq \int_{|q'| = \frac{1}{2}|q|} d\sigma(q') |q - q'|^{-N-|\alpha|} |D^{\alpha} \varphi(q')| \leq \\
 & \leq \text{Const. } |q|^{-|\alpha|-2-1} (1+|q|)^{-1} \sup_{q'} |q'|^2 (1+|q'|)^4 |D^{\alpha} \varphi(q')| = \\
 & = \text{Const. } |q|^{-2-|\alpha|-1} (1+|q|)^{-1} \times \\
 & \times \sup_{q'} |q'|^2 (1+|q'|)^4 |D^{\alpha} \varphi(q')|
 \end{aligned}$$

for all $s \geq 0$ and any real r ; and, similarly

$$\begin{aligned}
 & \left| \int_{|q'| = \frac{1}{2}|q|} d\sigma(q') \sum_{|\beta|=1} \frac{q'^{\beta}}{|q'|} \chi_{\beta}(q - q') D^{\alpha} \varphi(q') \right| \leq \\
 & \leq \text{Const. } |q|^{-2} (1+|q|)^{-1} \sup_{q'} |q'|^2 (1+|q'|)^4 |D^{\alpha} \varphi(q')|
 \end{aligned}$$

for all $s \geq 0$ and any real r . The term

$$\int_{|q'| \leq \frac{1}{2}|q|} D^{\alpha} \tilde{W}(q - q') \varphi(q') dq'$$

may be written

$$\begin{aligned}
 & D^{\alpha} \tilde{W}(q) \int_{|q'| \leq \frac{1}{2}|q|} \varphi(q') dq' + \\
 & + \int_{|q'| \leq \frac{1}{2}|q|} [D^{\alpha} \tilde{W}(q - q') - D^{\alpha} \tilde{W}(q)] \varphi(q') dq'
 \end{aligned}$$

Here, the first term is bounded in modulus by

$$|q|^{-N-|\alpha|} \left| \int_{|q'| \leq \frac{1}{2}|q|} \varphi(q') dq' \right|$$

This, in turn, is majorized by

$$\begin{aligned}
 & |q|^{-N-|\alpha|} \left| \int \varphi(q') dq' \right| + |q|^{-N-|\alpha|} \int_{|q'| \geq \frac{1}{2}|q|} |\varphi(q')| dq' \leq \\
 & \leq |q|^{-N-|\alpha|} \left| \int \varphi(q') dq' \right| + |q|^{-N-|\alpha|} \left[\sup_{q'} |q'|^2 |\varphi(q')| \right] \times \\
 & \times \int_{|q'| \geq \frac{1}{2}|q|} \frac{dq'}{|q'|^2} \leq \frac{C}{2-N} |q|^{-2-|\alpha|} \sup_{q'} |q'|^2 |\varphi(q')| + \\
 & + |q|^{-N-|\alpha|} \left| \int \varphi(q') dq' \right|
 \end{aligned}$$

for any $r > N$.

The modulus of the second term is bounded by

$$\begin{aligned}
 & \left| \int_0^1 dt \int_{|q'| \leq \frac{1}{2}|q|} \sum_{|\beta|=1} (-q')^\beta D^{\alpha+\beta} \tilde{U}(q-tq') \varphi(q') dq' \right| \leq \\
 & \leq C \int_0^1 dt \int_{|q'| \leq \frac{1}{2}|q|} |q'| |q-tq'|^{-N-|\alpha|-1} |\varphi(q')| dq' \leq \\
 & \leq C \int_{|q'| \leq \frac{1}{2}|q|} |q|^{-N-|\alpha|-1} |q'| |\varphi(q')| dq' \leq \\
 & \leq C |q|^{-N-|\alpha|-1} \left[\int_{|q'| \leq \frac{1}{2}|q|} dq' |q'|^{\varepsilon-N} \right] \sup_{q''} |q''|^{N+1-\varepsilon} |\varphi(q'')|
 \end{aligned}$$

for any $\varepsilon > 0$,

$$\leq \frac{C}{1-\delta} |q|^{-N-|\alpha|-\delta} \sup_{q'} |q'|^{N+\delta} |\varphi(q')|$$

for any $\delta < 1$.

On the other hand, if $r < N$,

$$\begin{aligned}
 \left| \int_{|q'| \leq \frac{1}{2}|q|} \mathcal{D}^\alpha \tilde{W}(q-q') \varphi(q') dq' \right| &\leq C \int_{|q'| \leq \frac{1}{2}|q|} |q|^{-N-|\alpha|} \frac{|q'|^{\alpha_2}}{|q'|^{\alpha_2}} |\varphi(q')| dq' \\
 &\leq C |q|^{-N-|\alpha|} \left[\sup_{q''} |q''|^{\alpha_2} |\varphi(q'')| \right] \int_{|q'| \leq \frac{1}{2}|q|} |q'|^{-\alpha_2} dq' \\
 &\leq \frac{\text{Const}}{N-\alpha_2} |q|^{-\alpha_2-|\alpha|} \sup_{q'} |q'|^{\alpha_2} |\varphi(q')|
 \end{aligned}$$

We now turn to the contributions of the region $\{q' : |q'| > \frac{1}{2}|q|\}$:

$$\begin{aligned}
 \left| \int_{|q'| > \frac{1}{2}|q|} \chi_\beta(q-q') \mathcal{D}^{\alpha+\beta} \varphi(q') dq' \right| &\leq \\
 &\leq C \int_{|q'| > \frac{1}{2}|q|} |q-q'|^{-N+1} |\mathcal{D}^{\alpha+\beta} \varphi(q')| dq' \leq \\
 &\leq C \int_{|q'| > \frac{1}{2}|q|} |q-q'|^{-N+1} |q'|^{-\alpha_2-1} |q'|^{\alpha_2+1} |\mathcal{D}^{\alpha+\beta} \varphi(q')| dq' \leq \\
 &\leq C \sup_{q''} |q''|^{\alpha_2+1} |\mathcal{D}^{\alpha+\beta} \varphi(q'')| \int_{|q'| > \frac{1}{2}|q|} dq' |q-q'|^{-N+1} |q'|^{-\alpha_2-1},
 \end{aligned}$$

for any $r > 0$.

Note that since $|q'| > \frac{1}{2}|q| > 0$ in the range of integration of the last integral, this integral is well defined; by homogeneity it is equal to $\text{const.} |q|^{-r}$.

A more precise evaluation shows that this integral is bounded by $\frac{C}{r} |q|^{-r}$ where $C > 0$ is independent of r . Thus, the quantity we consider is bounded by

$$\leq \frac{C}{r} |q|^{-r} \sup_{q'} |q'|^{r+1} |D^{\alpha+\beta} \varphi(q')| \quad \text{for any } r > 0.$$

We can now reassemble all the pieces of information we have gathered as follows:

1) if $N+1 > r \geq N$, we have:

$$\begin{aligned} & |q|^{r+|\alpha|} |D^{\alpha} \tilde{U} * \varphi(q)| \leq \\ & \leq \frac{C}{N+1-r} \sum_{|\delta| \leq |\alpha|+1} \sup_{q'} |q'|^{r+|\delta|} |D^{\delta} \varphi(q')| \\ & + C |q|^{r-N} \left| \int_{|q'| \leq \frac{1}{2}|q|} \varphi(q') dq' \right| \end{aligned} \tag{36}$$

2) if $0 < r < N$, we have

$$\begin{aligned} & |q|^{r+|\alpha|} |D^{\alpha} \tilde{U} * \varphi(q)| \leq \\ & \leq \frac{C}{r(N-r)} \sum_{|\delta| \leq |\alpha|+1} \sup_{q'} |q'|^{r+|\delta|} |D^{\delta} \varphi(q')| \end{aligned} \tag{37}$$

3) if $r > N$, say $r = N + \delta$ with $0 < \delta < 1$, (36) yields:

$$\begin{aligned} & |q|^{N+\delta+|\alpha|} |D^{\alpha} \tilde{U} * \varphi(q)| \leq C \left| \int \varphi(q') dq' \right| + \\ & + \frac{C}{\delta(1-\delta)} \sum_{|\delta| \leq |\alpha|+1} \sup_{q'} |q'|^{N+\delta+|\delta|} |D^{\delta} \varphi(q')|. \end{aligned}$$

Finally, we note that if $h \in \mathcal{S}(\mathbb{R}^N)$ we have

$$\begin{aligned} \int h(q) \tilde{\omega} * \varphi(q) dq &= \\ &= \int \varphi(q') \psi(q') dq' \end{aligned}$$

where $\psi(q') = \int \tilde{\omega}(q - q') h(q) dq$ is a $\mathcal{C}^\infty$ bounded function over $\mathbb{R}^N$.

Let now F be a tempered distribution over $\mathbb{R}^N$, which, for simplicity, we shall take with compact support; let F have the adiabatic property at 0, i.e., let us assume that, for some $P \geq 0$, $1 > \delta > 0$, $C > 0$, and for all $\varphi \in \mathcal{S}$:

$$| \langle F, \varphi \rangle | \leq C \left\{ \left| \int \varphi(q) dq \right| + \| \varphi \|_{\delta, P} \right\} \quad (38)$$

Then $\tilde{\omega} * F$ also has the adiabatic property at 0. Indeed, first note that $\langle F, \varphi \rangle = \langle F, h\varphi \rangle$ where $h \in \mathcal{D}(\mathbb{R}^N)$ is equal to 1 on the support of F and, hence

$$| \langle F, \varphi \rangle | \leq C' \left\{ \left| \int \varphi(q) h(q) dq \right| + \| \varphi \|_{\delta, P} \right\} \quad (39)$$

Applying the preceding inequalities, in the case $r = N + \delta$, $[\tilde{\omega}(q)]$ being replaced there by $\check{\omega}(q) = \tilde{\omega}(-q)$, which has the same properties] we obtain:

Lemma 7

Let F be a distribution over $\mathbb{R}^N$ satisfying the bound (39) with $0 < \delta < 1$ [for some $h \in \mathcal{S}(\mathbb{R}^N)$]. Then, there exists a constant K_δ (depending on δ) such that

$$|\langle \tilde{W} * F, \varphi \rangle| = |\langle F, \tilde{W} * \varphi \rangle| \leq \\ \leq K_\delta \left[\left| \int \varphi(q) dq \right| + \|\varphi\|_{\delta, p+1} \right]$$

2.4 Adiabatic Limits of Derivatives

We first consider a distribution F with compact support over $\mathbb{R}^N$ such that

$$|\langle F, \varphi \rangle| \leq C \sum_{|\alpha| \leq M} \sup_q |q|^{N+1+\delta+|\alpha|} |D^\alpha \varphi(q)|, \\ \delta > 0. \quad (40)$$

Clearly, this implies that the adiabatic limit of F at 0 is 0. But it also implies that, for $|\beta| = 1$

$$|\langle D^\beta F, \varphi \rangle| = |\langle F, D^\beta \varphi \rangle| \leq \\ \leq C \sum_{|\alpha| \leq M} \sup_q |q|^{N+1+\delta+|\alpha|} |D^{\alpha+\beta} \varphi(q)| \\ \leq C \sum_{|\alpha| \leq M+1} \sup_q |q|^{N+\delta+|\alpha|} |D^\alpha \varphi(q)|$$

Thus $D^\beta F$ also has a zero adiabatic limit at 0. Conversely suppose $\{T^\beta : |\beta| = 1\}$ is a family of distributions having adiabatic limits at 0 (which we may assume to be 0) and satisfying

$$D^\gamma T^\beta - D^\beta T^\gamma = 0 \text{ for any } \beta \text{ and } \gamma \text{ with } |\beta| = |\gamma| = 1,$$

$$|\langle T^\beta, \varphi \rangle| \leq C \sum_{|\alpha| \leq M} \sup_q |q|^{N+\delta+|\alpha|} |D^\alpha \varphi(q)|$$

(41)

We wish to show that

$$T = \int_0^1 dt \sum_{|\beta|=1} q^\beta T^\beta(tq)$$

makes sense as a distribution, i.e., that, for any $\varphi \in \mathcal{S}$,

$$\psi(q) = \int_0^1 \frac{dt}{t^{N+1}} q^\beta \varphi\left(\frac{q}{t}\right) \quad (|\beta|=1)$$

is an admissible test function for T .

Indeed, we have, for $q \neq 0$

$$\begin{aligned} |q|^{N+\delta+|\alpha|} |D^\alpha \psi(q)| &= \left| \int_0^1 \frac{dt}{t^{N+1}} \left[\alpha_j (D^{\alpha-\beta} \varphi)\left(\frac{q}{t}\right) t^{-|\alpha|+1} + \right. \right. \\ &\quad \left. \left. + q^\beta (D^\alpha \varphi)\left(\frac{q}{t}\right) t^{-|\alpha|} \right] |q|^{N+\delta+|\alpha|} \right| \leq \\ &\leq \int_0^1 dt \left[\left| \frac{q}{t} \right|^{N+|\alpha|+\delta} t^{|\alpha|} t^\delta |(D^{\alpha-\beta} \varphi)\left(\frac{q}{t}\right)| + \right. \\ &\quad \left. + \left| \frac{q}{t} \right|^{N+|\alpha|+\delta+1} t^\delta |(D^\alpha \varphi)\left(\frac{q}{t}\right)| \right] \leq \\ &\leq |\alpha| \sup_q |q|^{N+|\alpha-\beta|+1+\delta} |D^{\alpha-\beta} \varphi(q)| + \\ &\quad + \sup_q |q|^{N+|\alpha|+\delta+1} |D^\alpha \varphi(q)| \end{aligned}$$

As a consequence, T exists and satisfies:

$$|\langle T, \varphi \rangle| \leq C' \sum_{|\alpha| \leq M} \sup_q |q|^{N+1+\delta+|\alpha|} |D^\alpha \varphi(q)|$$

Now let T' be any solution of

$$D^\beta T' = T^\beta \quad \text{for all } \beta \text{ with } |\beta| = 1$$

then $T' - T$ is a constant. Hence T' also has an adiabatic limit $T'(0)$ at 0, which is precisely equal to $T' - T$. Finally, the result of this discussion can be expressed as follows:

Let T' be a distribution whose first derivatives $D^\beta T' = T^\beta$ have a zero adiabatic limit at 0, i.e., satisfy (41). Then

- 1) T' has an adiabatic limit $T'(0)$ at 0;
- 2) $T' - T'(0)$ satisfies:

$$|\langle T' - T'(0), \varphi \rangle| \leq C' \sum_{|\alpha| \leq M} \sup_q |q|^{N+1+\delta+|\alpha|} |D^\alpha \varphi(q)|$$

Of course, if instead of assuming that the T^β have a zero adiabatic limit at 0 we assume that they have non-vanishing adiabatic limits, denoted $T^\beta(0)$, we find:

$$\begin{aligned} |\langle T' - T'(0) - \sum_{|\beta|=1} q^\beta T^\beta(0), \varphi \rangle| &\leq \\ &\leq C' \sum_{|\alpha| \leq M} \sup_q |q|^{N+1+\delta+|\alpha|} |D^\alpha \varphi(q)| \end{aligned}$$

Conversely, suppose

$$\begin{aligned} |\langle T, \varphi \rangle| &\leq C \left[\left| \int \varphi(q) dq \right| + \sum_{|\beta|=1} \left| \int q^\beta \varphi(q) dq \right| + \right. \\ &\quad \left. + \sum_{|\alpha| \leq M} \sup_q |q|^{N+1+\delta+|\alpha|} |D^\alpha \varphi(q)| \right] \end{aligned}$$

Then, for $|\gamma| = 1$

$$\begin{aligned}
 | \langle \mathcal{D}^\gamma T, \varphi \rangle | &= | \langle T, \mathcal{D}^\gamma \varphi \rangle | \leq \\
 &\leq C \left[\sum_{|\beta|=1} \left| \int q^\beta \mathcal{D}^\gamma \varphi(q) dq \right| + \right. \\
 &\quad \left. + \sum_{|\alpha| \leq M} \sup_q |q|^{N+1+\delta+|\alpha|} | \mathcal{D}^{\alpha+\gamma} \varphi(q) | \right] \leq \\
 &\leq C' \left[\left| \int \varphi(q) dq \right| + \sum_{|\alpha| \leq M+1} \sup_q |q|^{N+\delta+|\alpha|} | \mathcal{D}^\alpha \varphi(q) | \right]
 \end{aligned}$$

3. INDUCTION HYPOTHESIS

We recall that the "chronological products" $T^{r_1 \dots r_n}(x_1, \dots, x_n)$ of $\mathcal{L}^{(r_1)}(x_1), \dots, \mathcal{L}^{(r_n)}(x_n)$ have been constructed in I by induction on n . The first step is to obtain the difference

$$A(Y; j) - R(Y; j) = D(Y; j)$$

(where $j \in \{1, \dots, n\}$ and $Y = \{1, \dots, j-1, j+1, \dots, n\}$) in terms of already constructed quantities. Here we shall need the fact that $D(Y; j)$ can be expressed in terms of already known "connected" operators, namely the Steinmann monomials for a smaller number of fields $\mathcal{L}^{(r)}(x)$. Indeed, from the Steinmann arrow calculus (see I, Section VIII) we have, for example if $Y = \{1, \dots, n-1\}$, $j = n$,

$$\begin{aligned}
 D(Y; n) &= A(Y; n) - R(Y; n) \equiv Y \uparrow n - Y \downarrow n = \\
 &= 1 \uparrow 2 \uparrow \dots \uparrow n - 1 \downarrow 2 \downarrow \dots \downarrow n = 1 \uparrow 2 \uparrow 3 \uparrow \dots \uparrow n - 1 \downarrow 2 \uparrow 3 \uparrow \dots \uparrow n \\
 &+ 1 \downarrow 2 \uparrow 3 \uparrow \dots \uparrow n - 1 \downarrow 2 \downarrow 3 \downarrow \dots \downarrow n = [1, 2 \uparrow \dots \uparrow n] + \\
 &+ 1 \downarrow [2 \uparrow \dots \uparrow n - 2 \downarrow \dots \downarrow n] = \\
 &= \sum_{p=1}^{n-1} \sum_{\substack{I \cup J = \{1, \dots, p-1\} \\ I \cap J = \emptyset}} [I \downarrow p, J \downarrow (p+1) \uparrow \dots \uparrow n]
 \end{aligned}$$

(42)

If we insert intermediate states in these products and take vacuum expectation values, we see that the vacuum state contribution vanishes and we only have to deal with "connected" products of "connected" operators.

For any GRP, or chronological, or antichronological, product of the fields $\mathcal{L}^{(r_1)}(x_1), \dots, \mathcal{L}^{(r_n)}(x_n)$, a formula of the type [I, (35)] holds. For example, if we denote (as in I, Section VIII) $\mathcal{A}^S(\mathcal{L}^{(r_1)}(x_1), \dots, \mathcal{L}^{(r_n)}(x_n))$ the GRP corresponding to the cell S we have

$$\begin{aligned} & \mathcal{A}^S(\mathcal{L}^{(r_1)}(x_1) \dots \mathcal{L}^{(r_n)}(x_n)) = \\ & = \sum_n (\Omega, \mathcal{A}^S(\mathcal{L}^{(r_1)}(x_1), \dots, \mathcal{L}^{(r_n)}(x_n)) \Omega) : \frac{A(X)^n}{n!} : \end{aligned} \quad (43)$$

Thus, the operator on the left-hand side is given by the various kernels appearing on the right-hand side. Denote

$$\begin{aligned} & K(q_1, \dots, q_n) = \\ & = \int e^{i \sum_{j=1}^n q_j x_j} (\Omega, \mathcal{A}^S(\mathcal{L}^{(r_1)}(x_1), \dots, \mathcal{L}^{(r_n)}(x_n)) \Omega) dx_1 \dots dx_n \end{aligned} \quad (44)$$

The expression $K(q_1 + p_{I_1} - p'_{J_1}, \dots, q_n + p_{I_n} - p'_{J_n})$ will be called a (connected) kernel with n vertices, ℓ' incoming external lines, ℓ outgoing external lines: here

$$\begin{aligned} & I_1, \dots, I_n \text{ is a partition of } \{1, \dots, \ell\} \\ & J_1, \dots, J_n \text{ is a partition of } \{1, \dots, \ell'\} \\ & |I_k| + |J_k| = r_k \text{ for } k = 1, \dots, n, \\ & p_I = \sum_{j \in I} p_j, \quad p'_J = \sum_{j \in J} p'_j, \\ & \ell + \ell' = \sum_{k=1}^n r_k \end{aligned}$$

[Of course $K(q_1, \dots, q_n)$ is of the form $\delta(q_1 + \dots + q_n) F(q_1, \dots, q_{n-1})$.]

We shall say that the kernel

$$\delta(q_1 + \dots + q_n + p_1 + \dots + p_\ell - p'_1 - \dots - p'_{\ell'}) F(q_1 + p_{I_1} - p'_{J_1}, \dots, q_{n-1} + p_{I_{n-1}} - p'_{J_{n-1}})$$

possesses the property A with omitted variable q_n if the following conditions are satisfied.

There exist:

- an open neighbourhood V_1 of the set

$$\{q_1, \dots, q_{n-1}, p'_1, \dots, p'_{\ell'} : q_1 = q_2 = \dots = q_{n-1} = 0,$$

$$p_j'^2 = m^2, \quad p_j'^0 > 0, \quad 1 \leq j \leq \ell'\} ;$$

- an open neighbourhood V_2 of

$$\{Q \in \mathbb{R}^4, p_1, \dots, p_\ell : Q = 0, p_j^2 = m^2, p_j^0 > 0, 1 \leq j \leq \ell\}$$

positive numbers $\lambda > 0$, $M \geq 0$ and, for each $R > 0$, and sufficiently small $\delta > 0$, a constant $K_\delta > 0$ such that:

if ψ is a $\mathcal{C}^\infty$ function of $q_1, \dots, q_{n-1}, p'_1, \dots, p'_{\ell'}$, with support in $V_1 \cap \{q, p' : |q_j| < R, 1 \leq j \leq n-1, |p'_k| < R, 1 \leq k \leq \ell'\}$ then

$$E(Q, p) = \int \delta(Q + \sum_{j=1}^{\ell} p_j - \sum_{k=1}^{\ell'} p'_k) F(q + \hat{p} - \hat{p}') \prod_{j=1}^{\ell'} d\sigma(p'_j) \cdot \psi(q_1, \dots, q_{n-1}; p'_1, \dots, p'_{\ell'}) dq_1 \dots dq_{n-1}$$

is Hölder continuous in Q and $p = (p_1, \dots, p_\ell)$ in V_2 [here we have used the abbreviations:

$$q = q_1, \dots, q_{n-1}$$

$$\hat{p} = \hat{p}_1, \dots, \hat{p}_{n-1}, \quad \hat{p}_j = p_{I_j}$$

$$\hat{p}' = \hat{p}'_1, \dots, \hat{p}'_{n-1}, \quad \hat{p}'_j = p_{j_j}$$

$$d\sigma(k) = \delta(k^2 - m^2) \theta(k^0) d^4 k.$$

Moreover, (for sufficiently small $\delta > 0$) and $|Q| < R$,
 $|p_j| < R$

$$H_{Q, p}^{\lambda \delta}(E(Q, p)) \leq K_\delta \left[H_{p'}^\delta \left(\int \psi(q; p') dq \right) + \right. \\ \left. + \sum_{\substack{\alpha \in \mathbb{N} \\ |\alpha| \leq M}} \sup_q |q|^{4(n-1) + |\alpha| + \lambda \delta} H_{p'}^\delta (D_q^\alpha \psi(q; p')) \right] \quad (46)$$

Note that here q stands for $(q_1, \dots, q_{n-1})$ and $|q|^2 = \sum_{j=1}^{n-1} |q_j|^2$.

We shall say that a kernel $K(q_1 + p_{I_1} - p'_{J_1}, \dots, q_n + p_{I_n} - p'_{J_n})$ [where $K(q_1, \dots, q_n) = \delta(q_1 + \dots + q_n) F(q_1, \dots, q_{n-1})$] has the property B if there exist: an open neighbourhood V_1 of

$$\{q_1, \dots, q_n, p'_1, \dots, p'_{\ell'} : q_1 = \dots = q_n = 0, p'_j{}^2 = m^2, p'_j{}^0 > 0, 1 \leq j \leq \ell'\}$$

a neighbourhood V_2 of

$$\{p_1, \dots, p_{\ell'} : p_j{}^2 = m^2, p_j{}^0 > 0, 1 \leq j \leq \ell'\}$$

numbers $\lambda > 0$ and $M \geq 0$, and for each $R > 0$ and sufficiently small $\delta > 0$, a constant C_δ such that if $\psi(q_1, \dots, q_n; p'_1, \dots, p'_{\ell'})$ is a $\mathcal{C}^\infty$ function with support in

$$V_1 \cap \{q, p' : |q_j| < R, 1 \leq j \leq n, |p'_k| < R, 1 \leq k \leq \ell'\}$$

then

$$E'(p) = \int K(q_1 + p_{I_1} - p'_{J_1}, \dots, q_n + p_{I_n} - p'_{J_n}) \prod_{j=1}^{\ell'} d\sigma(p'_j) \\ \psi(q_1, \dots, q_n; p'_1, \dots, p'_{\ell'}) dq_1 \dots dq_n$$

is a Hölder continuous function of p in V_2 , satisfying:

$$H_p^{\lambda\delta}(E'(p)) \leq C_\delta \left[H_{p'}^\delta \left(\int \psi(q; p') dq_1 \dots dq_n \right) + \right. \\ \left. + \sum_{|\alpha| \leq M} \sup_q |q|^{4n + \lambda\delta + |\alpha|} H_{p'}^\delta \left(\mathcal{D}_q^\alpha \psi(q; p') \right) \right] \quad (47)$$

Note that here q stands for $(q_1, \dots, q_n)$ and $|q|^2 = \sum_{j=1}^n |q_j|^2$.

We intend to prove (by induction on the number n of vertices) that the kernels defined by (43) all have the property B; the kernels having exactly one incoming and one outgoing external line ($\ell = \ell' = 1$) will be called self-energy kernels; we shall also prove by induction that all the kernels defined by (43), except self-energy kernels possess the property A. As for the self-energy kernels $K_2(p, p', q_1, \dots, q_n)$ they may be either of the form:

$$K(q_2 + p, q_2, \dots, q_{n-1}, q_n - p') = \delta \left(\sum_{j=1}^n q_j + p - p' \right) S(q_2 + p, q_2, \dots, q_{n-1}) \quad (48)$$

(calling q_1 the vertex carrying the outgoing line p and q_n the vertex carrying the ingoing line p') or of the form

$$K(q_2, \dots, q_{n-1}, q_n + p - p') = \delta \left(\sum_{j=1}^n q_j + p - p' \right) S(q_2, \dots, q_{n-1}) \quad (49)$$

(in case the two external lines are attached to the same vertex q_n). For such kernels we shall prove (by induction on n) that, provided correct mass and wave function renormalizations have been performed, the distributions S have the following property (C).

1) Property C_1

In the first case, there exist, a neighbourhood V_1 of $\{q_1, \dots, q_{n-1} = 0\}$, a positive number $M \geq 0$, and given any $\delta > 0$ sufficiently small, any $R > 0$ and any $p \in \mathbb{R}^4$ with $p^2 < 4m^2$, positive numbers $A(p) \geq 0$, $B(p) \geq 0$, $K_\delta(p) \geq 0$ such that if ψ is a $\mathcal{C}^\infty$ function of $q_1, \dots, q_{n-1}$ with support

in $V_1 \cap \{q : |q_j| \leq R, 1 \leq j \leq n-1\}$, then

$$E(p) = \int S(q_1 + p, q_2, \dots, q_{n-1}) \psi(q_2, \dots, q_{n-1}) dq_2 \dots dq_{n-1}$$

satisfies

$$\begin{aligned} |E(p)| &\leq A(p) \left| \int \psi(q) dq \right| + B(p) \sum_{\substack{\beta \\ |\beta|=1}} \left| \int q^\beta \psi(q) dq \right| + \\ &+ K_\delta(p) \sum_{|\alpha| \leq M} \sup_q |q|^{4(n-1) + \delta + 1 + |\alpha|} |D^\alpha \psi(q)|. \end{aligned} \quad (50)$$

($q = (q_2, \dots, q_{n-1})$)

2) Property D

In the same case, $A(p)$ and $B(p)$ vanish on the mass hyperboloid, i.e., for $p^2 = m^2$, $p^0 > 0$: in other words, if $p^2 = m^2$, $p^0 > 0$ then

$$|E(p)| \leq K_\delta(p) \sum_{|\alpha| \leq M} \sup_q |q|^{4(n-1) + \delta + 1 + |\alpha|} |D^\alpha \psi(q)|$$

Note that, according to Lemma 3, (50) implies the Hölder continuity of $E(p)$ in $\{p : p^2 < m^2\}$.

3) Property C₂

In the second case, there exist similar objects V_1 , M , K_δ such that, if ψ is $\mathcal{C}^\infty$ with support in

$$V_1 \cap \{q, |q_j| < R, \text{ all } j \leq n-1\},$$

$$\begin{aligned} \left| \int S(q_1, \dots, q_{n-1}) dq_2 \dots dq_{n-1} \psi(q_2, \dots, q_{n-1}) \right| &\leq \\ &\leq K_\delta \sum_{|\alpha| \leq M} \sup_q |q|^{4(n-1) + 1 + \delta + |\alpha|} |D^\alpha \psi(q)| \end{aligned}$$

(Actually, the proof shows, and needs, the fact that all these inequalities can be extended, in the obvious way, to Banach-space-valued test functions. We neglect this point here for notational reasons.)

Properties D and C_2 clearly express the correct mass and wave function renormalizations necessary for the existence of the adiabatic limit.

We shall frequently appeal in the following to Lemma 8.

Lemma 8

Let $S(q_1, \dots, q_{n-1})$ be a tempered distribution on $\mathbb{R}^{4(n-1)}$. Assume that there exist a neighbourhood V_1 of the origin in $\mathbb{R}^{4(n-1)}$, positive numbers $R > 0$, $M \geq 0$, $\delta > 0$, ($0 < \delta < 1$), and K_δ such that, for every $p \in \mathbb{R}^4$ with $p^2 = m^2$, $p^0 > 0$, $|p| < 2R$ and for every $\psi \in \mathcal{J}(\mathbb{R}^{4(n-1)})$ with support in $V_1 \cap \{q = q_1, \dots, q_{n-1} : |q| < R\}$, denoting

$$E(p; \psi) = \int S(p + q_1, q_2, \dots, q_{n-1}) \psi(q_2, \dots, q_{n-1}) dq_2 \dots dq_{n-1},$$

$$|E(p; \psi)| \leq K_\delta \sum_{|k| \leq M} \sup_q |q|^{4(n-1) + 1 + \delta + |k|} |D^k \psi(q)|$$

Let

$$F(p; \chi) = \int \delta((p - Q)^2 - m^2) \theta(p^0 - Q^0) S(p - Q + q_1, q_2, \dots, q_{n-1}) \chi(q, Q) dq d^4 Q$$

[where $q = (q_1, \dots, q_{n-1})$]. Suppose χ is a $\mathcal{C}^\infty$ function of q and Q with support in

$$\{q, Q : q \in V_1, |q| < R, |Q| < R\}$$

Then $F(p; \chi)$ is Hölder continuous in p in the open set

$$\{p : |p| < R\}$$

where it satisfies

$$H_p^{\tau, \delta}(F(p, \chi)) \leq C_\delta \sum_{|\alpha| \leq M+1} \sup_{Q, q} |Q, q|^{4n + \lambda\delta + |\alpha|} |D^\alpha \chi(q, Q)|$$

(Here C_δ is a positive constant depending on δ , τ and λ ; τ and λ are any two numbers such that $\tau + \lambda < 1$, $\tau \geq 0$, $\lambda > 0$.)

Proof

$$F(p; \chi) = \int \delta((p-Q)^2 - m^2) \theta(p^0 - Q^0) dQ V(p-Q, Q),$$

$$V(z, Q) = \int S(z + q_1, \dots, q_{n-1}) \chi(q, Q) dq$$

$$|F(p; \chi)| \leq \sup_{\substack{Q, z \\ |Q| < R \\ |z| < 2R \\ z^2 = m^2, z^0 > 0}} |Q|^{3-\varepsilon\delta} |V(z, Q)| \times$$

$$\times \int_{|Q| < R} \frac{d^3 \vec{Q}}{2m |\vec{Q}|^{3-\varepsilon\delta}}$$

The last integral is $(2\pi/\varepsilon m) R^{\varepsilon\delta}$. But, according to our hypotheses

$$|V(z, Q)| \leq K_\delta \sum_{|\alpha| \leq M} \sup_q |q|^{4(n-1)+1+\delta+|\alpha|} |D^\alpha \chi(q, Q)|,$$

$$|F(p; \chi)| \leq \frac{K'_\delta}{\varepsilon \delta} \sum_{|\alpha| \leq M} \sup_{\substack{q, Q \\ |Q| < R \\ |q| < R}} |q|^{4(n-1)+1+\delta+|\alpha|} |Q|^{3-\varepsilon\delta} |D_q^\alpha \chi(q, Q)|$$

$$|F(p; \chi)| \leq \frac{K'_\delta}{\varepsilon \delta} \sum_{|\alpha| \leq M} \sup_{q, Q} |q, Q|^{4n-\varepsilon\delta+\delta+|\alpha|} |D_q^\alpha \chi(q, Q)|$$

(0 < \varepsilon < 1)

Since this inequality holds in the convex set $\{p : |p| < R\}$, and since

$$F(p+h; \chi) = F(p; \chi_h), \quad \chi_h(q, Q) = \chi(q, Q-h)$$

we can apply Lemma 4 and obtain

$$H_p^{\tau\delta}(F(p; \chi)) \leq C_\delta \left\{ \sum_{|\alpha| \leq M+1} \sup_{q, Q} |q, Q|^{4n+\lambda\delta+|\alpha|} |D_q^\alpha \chi(q, Q)| \right\}$$

where $\tau + \lambda = 1 - 2\varepsilon$, $\lambda > 0$.

Lemma 9

Let $S(q_1, \dots, q_{n-1})$ be a tempered distribution on $\mathbb{R}^{4(n-1)}$. Assume that there exists a neighbourhood V_1 of 0 in $\mathbb{R}^{4(n-1)}$, positive numbers $R > 0$, $M \geq 0$, $\delta > 0$, ($0 < \delta < 1$), and K_δ such that, for every $\psi \in \mathcal{S}(\mathbb{R}^{4(n-1)})$ with support in $V_1 \cap \{q : |q| < R\}$,

$$\begin{aligned} & \left| \int S(q_1, \dots, q_{n-1}) \psi(q_1, \dots, q_{n-1}) dq_1 \dots dq_{n-1} \right| \leq \\ & \leq K_\delta \sum_{|\alpha| \leq M} \sup_q |q|^{4(n-1)+1+\delta+|\alpha|} |D_q^\alpha \psi(q)| \end{aligned}$$

Let

$$F(p; \chi) = \int \delta((p-Q)^2 - m^2) \theta(p^0 - Q^0) S(q_1, \dots, q_{n-1}) \\ \chi(q_1, \dots, q_{n-1}, Q) dq_1 \dots dq_{n-1} d^4 Q$$

Suppose χ is a $\mathcal{C}^\infty$ function of q and Q with support in

$$\{q, Q : q \in V_1, |q| < R, |Q| < R\}$$

Then $F(p; \chi)$ is Hölder continuous in p in $\{p : |p| < R\}$, where it satisfies

$$H_p^{\tau\delta}(F(p; \chi)) \leq C_\delta \sum_{|\alpha| \leq M+1} \sup_{Q, q} |Q, q|^{4n + \lambda\delta + |\alpha|} |D^\alpha \chi(q, Q)|$$

Here $C_\delta > 0$ depends on δ, τ, λ ; τ and λ are any two numbers such that $\tau + \lambda < 1$, $\tau \geq 0$, $\lambda > 0$.

Proof

The proof is a simpler version of the proof of the preceding lemma.

4. PASSAGE FROM $n-1$ TO n

The inductive proof involves several steps. The first is

Step 1

Lemma 10 (Phase space lemma)

Let φ be a function defined on $\mathbb{R}^{4\ell}$ in the neighbourhood of

$$\{p = \{p_1, \dots, p_\ell\} : p_j^2 = m^2, p_j^0 > 0, 1 \leq j \leq \ell\}$$

with $\ell \geq 2$, and taking its values in a Banach space. The norm of a vector u in this Banach space will be denoted $B(u)$. Let

$$F_\ell(Q; \varphi) = \int \delta(Q - \sum_{j=1}^{\ell} p_j) \left\{ \prod_{j=1}^{\ell} \delta(p_j^2 - m^2) \theta(p_j^0) d^4 p_j \right\} \varphi(p_1, \dots, p_\ell)$$

Then, for all $Q \in \mathbb{R}^4$ and every $h \in \mathbb{R}^4$ with $|h| \leq m$, and for $0 < \delta \leq \frac{1}{2}$,

$$B(F_\ell(Q+h; \varphi) - F_\ell(Q; \varphi)) \leq C_\ell |Q|^{2(\ell-1)} \left| \frac{h}{m} \right|^\delta H_{p; B}^\delta(\varphi(p))$$

(52)

The proof is straightforward and tedious and will not be given here.

This lemma shows our assertion is true for $n=1$. Indeed, in that case $(\Omega, T^r(x)\Omega) = (\Omega, \mathcal{L}^{(r)}(x)\Omega)$ is only different from 0 if $r = \nu$; in this case it is equal to $\nu!$. Thus, for ℓ outgoing and ℓ' incoming external lines, $E(Q, p)$ simply becomes:

$$\begin{aligned} E(Q, p) &= \nu! \int \delta(Q + \sum_{j=1}^{\ell} p_j - \sum_{k=1}^{\ell'} p'_k) \psi(p') \prod_{k=1}^{\ell'} d\sigma(p'_k) = \\ &= F_{\ell'}(Q+p; \psi), \quad p = \sum_{j=1}^{\ell} p_j. \end{aligned}$$

We always assume $\nu = \ell + \ell' \geq 3$; if $\ell' < 2$, $E(Q, p)$ is then zero in a neighbourhood of $\{Q, p : Q=0, p_j^2 = m^2, p_j^0 > 0\}$ by energy momentum conservation. For $\ell' \geq 2$, Lemma 10 shows that property A holds.

Step 2

As a second step we shall verify that if two kernels have the property A, then a "connected product" of these two kernels also possesses this property. The q variables attached to the first kernel (resp. the second kernel) are denoted $(q_1, \dots, q_n)$ [resp. $(q_{n+1}, \dots, q_{n+m})$]. If a vertex of the product kernel has an external line, then this vertex also has an external line in one of

the original kernels and can be an "omitted variable" for this kernel. We can always assume that this variable, the omitted variable for the product kernel is either q_n or q_{n+m} . (See Fig. 1.) The proof will only be outlined; in particular we omit the (straightforward) considerations concerning the supports and the neighbourhoods V_1 , V_2 , etc., which occur in this proof.

First case:

In the first case (omitted variable = q_n) we have to study

$$\begin{aligned}
 E(Q, P) &= \int \delta(Q + P - P') F(q_2 + p_{I_2} - p'_{J_2} - k_{L_2}, \dots, q_{n-1} + p_{I_{n-1}} - p'_{J_{n-1}} - k_{L_{n-1}}) \\
 &\quad \prod_{j=1}^1 d\sigma(k_j) \delta(q_{n+1} + \dots + q_{n+m} + P_2 - P'_2 + \sum_{j=1}^1 k_j) \\
 &\quad G(q_{n+1} + p_{I_{n+1}} + k_{L'_1} - p'_{J_{n+1}}, \dots, q_{n+m-1} + p_{I_{n+m-1}} + k_{L'_{m-1}} - p'_{J_{n+m-1}}) \\
 &\quad \prod_{j=1}^{\ell'} d\sigma(p'_j) \psi(q_2, \dots, q_{n-1}, q_{n+1}, \dots, q_{n+m}; p'_1, \dots, p'_{\ell'}) \\
 &\quad dq_2 \dots dq_{n-1} dq_{n+1} \dots dq_{n+m}
 \end{aligned}$$

Here $I_1, \dots, I_n, I_{n+1}, \dots, I_{n+m}$ are disjoint sets of indices whose union is $\{1, \dots, \ell\}$; $J_1, \dots, J_n, J_{n+1}, \dots, J_{n+m}$ are disjoint sets of indices whose union is $\{1, \dots, \ell'\}$. $L_1, \dots, L_n$ (resp. $L'_1, \dots, L'_m$) are disjoint sets of indices whose union is $\{1, \dots, s\}$. The abbreviations

$$\begin{aligned}
 P_I &= \sum_{j \in I} p_j, \quad P'_J = \sum_{j \in J} p'_j, \quad k_L = \sum_{j \in L} k_j, \quad P = \sum_{j=1}^{\ell} p_j, \\
 P' &= \sum_{j=1}^{\ell'} p'_j
 \end{aligned}$$

have been used;

$$\mathcal{P}_2 = \sum_{j=n+1}^{n+m} p_{I_j} \quad ; \quad \mathcal{P}'_2 = \sum_{j=n+1}^{n+m} p'_{J_j} \quad ; \quad \mathcal{P}_1 = \mathcal{P} - \mathcal{P}_2 \quad ,$$

$$\mathcal{P}'_1 = \mathcal{P}' - \mathcal{P}'_2 .$$

To simplify the notation we shall now denote

$$\begin{aligned} q &= (q_1, \dots, q_{n-1}) \\ q' &= (q_{n+1}, \dots, q_{n+m-1}) \\ p &= (p_1, \dots, p_\ell) \\ \hat{p} &= (p_{I_1}, \dots, p_{I_{n-1}}), & \check{p} &= (p_{I_{n+1}}, \dots, p_{I_{n+m-1}}) \\ \hat{p}' &= (p'_{J_1}, \dots, p'_{J_{n-1}}), & \check{p}' &= (p'_{J_{n+1}}, \dots, p'_{J_{n+m-1}}) \\ \hat{k} &= (k_{L_1}, \dots, k_{L_{n-1}}), & \check{k} &= (k_{L'_1}, \dots, k_{L'_{m-1}}) \end{aligned}$$

(Note that $\hat{k}$ and $\check{k}$ are not independent!). Furthermore, let

$$I_1 \cup \dots \cup I_n = \{1, \dots, t\} \quad , \quad I_{n+1} \cup \dots \cup I_{n+m} = \{t+1, \dots, t'\}$$

we denote

$$p'' = \{p'_1, \dots, p'_t\} \quad , \quad p''' = \{p'_{t+1}, \dots, p'_{t'}\}$$

We now make a change of variables by replacing the variable q_{n+m} by

$$Q_2 = \sum_{j=n+1}^{n+m-1} q_j$$

and denote

$$\chi(q, q', Q_2; p') = \psi(q, q', Q_2 - \sum_{j=n+1}^{n+m-1} q_j; p')$$

Then,

$$E(Q, p) = \int \delta(Q + \mathcal{P} - \mathcal{P}') F(q + \hat{p} - \hat{p}' - \hat{k}) \prod_{j=1}^t d\sigma(k_j)$$

$$\delta(Q_2 + \mathcal{P}_2 - \mathcal{P}'_2 + \sum_{j=1}^t k_j) G(q' + \check{p} - \check{p}' + \check{k}) \prod_{j=1}^{t'} d\sigma(p'_j)$$

$$\chi(q, q', Q_2; p') dq dq' dQ_2$$

$$E(Q, p) = \int \delta(Q - Q_2 + P_1 - P'_1 - \sum_{j=1}^1 k_j) F(q + \hat{p} - \hat{p}' - \hat{k})$$

$$\left[\prod_{j=1}^1 d\sigma(k_j) \right] \left[\prod_{j=1}^t d\sigma(p'_j) \right] dq_1 \dots dq_{n-1} dQ_2$$

$$U(Q_2; q, Q_2; p'', k; p)$$

where

$$U(Q_2; q, Q_3; p'', k; p) =$$

$$= \int \delta(Q_2 + P_2 + \sum_{j=1}^1 k_j - P'_2) G(q' + \check{p} - \check{p}' - \check{k}) \prod_{j=t+1}^{\ell'} d\sigma(p'_j)$$

$$\chi(q, q', Q_3; p') dq'$$

(53)

$U(Q_2; q, Q_3; p'', k; p)$ is Hölder continuous in Q_2 , p , k , p'' , and since the second kernel has the property A (with q_{n+m} omitted), for all sufficiently small $\delta > 0$,

$$H_{Q_2, p'', p, k}^{\kappa \delta} (D_q^\beta U(Q_2; q, Q_3; p'', k; p)) \leq$$

$$\leq \kappa_\delta \left[H_{p'}^\delta \left(\int D_q^\beta \chi(q, q', Q_3; p') dq' \right) + \right.$$

$$\left. + \sum_{|\alpha| \leq M_2} \sup_{q'} |q'|^{4(m-1) + \kappa \delta + |\alpha|} H_{p'}^\delta (D_{q'}^\alpha D_q^\beta \chi(q, q', Q_3; p')) \right]$$

where κ is a certain constant, $0 < \kappa < 1$. When χ is $\mathcal{C}^\infty$, U is $\mathcal{C}^\infty$ in q , Q_3 . Note also that

$$\int \mathcal{D}_q^\beta V(Q_2; q, Q_3; p'', k; p) dQ_3$$

is obtained by replacing in (53), $\chi(q, q', Q_3; p')$ by $\int \mathcal{D}_q^\beta \chi(q, q', Q_3; p')$ dQ_3 . As a consequence, for all sufficiently small δ

$$\begin{aligned} & H_{Q_2, p'', p, k}^{\kappa\delta} \left(\int \mathcal{D}_q^\beta V(Q_2; q, Q_3; p'', k; p) dQ_3 \right) \leq \\ & \leq \kappa_\delta \left[H_{p'}^\delta \left(\int \mathcal{D}_q^\beta \chi(q, q', Q_3; p') dq' dQ_3 \right) + \right. \\ & \left. + \sum_{|\alpha| \leq M_2} \sup_{q'} |q'|^{4(m-1) + \kappa\delta + |\alpha|} H_{p'}^\delta \left(\int \mathcal{D}_{q'}^\alpha \mathcal{D}_q^\beta \chi(q, q', Q_3; p') dQ_3 \right) \right] \end{aligned}$$

Let

$$V(p, q; Q_2; Q_3) = \int \delta(Q - Q_2 + P_2 - P_1' - \sum_{j=1}^1 k_j)$$

$$F(q + \hat{p} - \hat{p}' - \hat{k}) \left[\prod_{j=1}^t d\sigma(p_j') \right] \left[\prod_{j=1}^1 d\sigma(k_j) \right]$$

$$V(Q_2; q, Q_3; p'', k; p)$$

This is a Hölder continuous function of p, Q, Q_2 , and there is a constant τ ($0 < \tau < \kappa$) and, for each sufficiently small $\delta > 0$, a constant K'_δ such that

$$\begin{aligned} & H_{Q_2, p, Q}^{\tau\delta} (V(p, q; Q_2; Q_3)) \leq \\ & \leq K'_\delta \left[H_{Q_2, p'', p, k}^{\kappa\delta} \left(\int V(Q_2; q, Q_3; p'', k; p) dq \right) + \right. \\ & \left. + \sum_{|\beta| \leq M} \sup_q |q|^{4(n-1) + \tau\delta + |\beta|} H_{Q_2, p'', p, k}^{\kappa\delta} \left(\mathcal{D}_q^\beta V(Q_2; q, Q_3; p'', k; p) \right) \right] \end{aligned}$$

Note again that $\int U(Q_2; q, Q_3; p'', k, p) dq$ is obtained by replacing in (53), $\chi(q, q', Q_3; p')$ by $\int \chi(q, q', Q_3; p') dq$. From the preceding estimates it follows that

$$\begin{aligned}
 & H_{Q_2, p, Q}^{\tau\delta} (V(p, Q; Q_2; Q_3)) \leq \\
 & \leq \kappa'' [H_{p'}^{\delta} (\int \chi(q, q', Q_3; p') dq dq') + \\
 & + \sum_{|\alpha| \leq M_2} \sup_{q'} |q'|^{4(m-1) + \kappa\delta + |\alpha|} H_{p'}^{\delta} (\int \mathcal{D}_q^{\alpha} \chi(q, q', Q_3; p') dq) \\
 & + \sum_{|\beta| \leq M_3} \sup_q |q|^{4(n-1) + \tau\delta + |\beta|} H_{p'}^{\delta} (\int \mathcal{D}_q^{\beta} \chi(q, q', Q_3; p') dq') \\
 & + \sum_{\substack{|\alpha| \leq M_2 \\ |\beta| \leq M_1}} \sup_{q, q'} |q'|^{4(m-1) + \kappa\delta + |\alpha|} |q|^{4(n-1) + \tau\delta + |\beta|} \\
 & H_{p'}^{\delta} (\mathcal{D}_q^{\beta} \mathcal{D}_{q'}^{\alpha} \chi(q, q', Q_3; p'))] = \\
 & = B(Q_3)
 \end{aligned}$$

(54)

A similar calculation shows that

$$H_{p, Q}^{\tau\delta} (\int V(p, Q; 0; Q_3) dQ_3)$$

is bounded by the quantity obtained by replacing, in (54), $\chi(q, q', Q_3; p')$ by $\int \chi(q, q', Q_3; p') dQ_3$. We have

$$E(Q, p) = \int V(p, Q; Q_2; Q_3) dQ_2 =$$

$$= \int [V(p, Q; Q_2; Q_2) - V(p, Q; 0; Q_2)] dQ_2 \\ + \int V(p, Q; 0; Q_3) dQ_3$$

A bound has already been found for the last integral. Denoting

$$W(p, Q; Q_2, Q_3) = V(p, Q; Q_2; Q_3) - V(p, Q; 0; Q_3)$$

the Hölder continuity of $V(p, Q; Q_2; Q_3)$ in p, Q, Q_2 yields:

$$H_{p, Q}^{\frac{\tau}{2}\delta}(W(p, Q; Q_2; Q_3)) \leq |Q_2|^{\frac{\tau}{2}\delta} B(Q_3)$$

[B being defined by (54)]. Hence

$$H_{p, Q}^{\frac{\tau}{2}\delta}(\int W(p, Q; Q_2; Q_2) dQ_2) \leq \\ \leq \int |Q_2|^{\frac{\tau}{2}\delta} B(Q_2) dQ_2 \leq \frac{C}{\varepsilon} \sup_{Q_2} |Q_2|^{4+\frac{\tau}{2}\delta-\varepsilon} B(Q_2)$$

We choose $\varepsilon = (\tau/4)\delta$. Finally, putting together all the bounds, we obtain:

$$H_{p, Q}^{\frac{\tau}{2}\delta}(E(Q, p)) \leq K_\delta''' \left\{ \sup_{Q_2} |Q_2|^{4+\frac{\tau}{4}\delta} B(Q_2) + \right. \\ + H_{p'}^\delta \left(\int X(q, q', Q_2; p') dq dq' dQ_2 \right) \\ + \sum_{|\alpha| \leq M_2} \sup_{q'} |q'|^{4(m-1)+\tau\delta+|\alpha|} \\ \left. H_{p'}^\delta \left(\int \mathcal{D}_{q'}^\alpha X(q, q', Q_2; p') dQ_2 dq \right) + \right.$$

$$\begin{aligned}
 & + \sum_{|\beta| \leq M_1} \sup_{\mathbf{q}} |\mathbf{q}|^{4(n-1)+2\delta+|\beta|} H_{\beta'}^{\delta} \left(\int \mathcal{D}_{\mathbf{q}}^{\beta} \chi(\mathbf{q}, \mathbf{q}', \mathbf{Q}_2; \mathbf{p}') d\mathbf{Q}_2 d\mathbf{q}' \right) \\
 & + \sum_{\substack{|\alpha| \leq M_2 \\ |\beta| \leq M_1}} \sup_{\mathbf{q}, \mathbf{q}'} |\mathbf{q}'|^{4(m-1)+2\delta+|\alpha|} |\mathbf{q}|^{4(n-1)+2\delta+|\beta|} \\
 & \quad H_{\beta'}^{\delta} \left(\int \mathcal{D}_{\mathbf{q}}^{\alpha} \mathcal{D}_{\mathbf{q}'}^{\beta} \chi(\mathbf{q}, \mathbf{q}', \mathbf{Q}_2; \mathbf{p}') d\mathbf{Q}_2 \right) \}
 \end{aligned}$$

We recall, from the proof of the tensor product rule, that if f is a function of $\mathbf{x} \in \mathbb{R}^N$ and $\mathbf{y} \in \mathbb{R}^M$, with compact support, we have

$$\begin{aligned}
 & \sup_{\mathbf{x}} |\mathbf{x}|^{N+|\alpha|+\delta} \left| \int \mathcal{D}_{\mathbf{x}}^{\alpha} f(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right| \leq \\
 & \leq \text{Const} \frac{1}{\varepsilon} \sup_{\mathbf{x}, \mathbf{y}} |\mathbf{x}, \mathbf{y}|^{N+M+\delta-\varepsilon+|\alpha|} |\mathcal{D}_{\mathbf{x}}^{\alpha} f(\mathbf{x}, \mathbf{y})|
 \end{aligned}$$

A repeated use of this device shows that there exists a constant λ with $0 < \lambda < 1$ such that for sufficiently small $\delta > 0$,

$$\begin{aligned}
 & H_{\mathbf{p}, \mathbf{Q}}^{\lambda \delta} (E(\mathbf{Q}, \mathbf{p})) \leq A_{\delta} \left\{ H_{\mathbf{p}'}^{\delta} \left(\int \chi(\mathbf{q}, \mathbf{q}', \mathbf{Q}_2; \mathbf{p}') d\mathbf{q} d\mathbf{q}' d\mathbf{Q}_2 \right) \right. \\
 & + \sum_{|\alpha| \leq M_1 + M_2} \sup_{\mathbf{q}, \mathbf{q}', \mathbf{Q}_2} |\mathbf{q}, \mathbf{q}', \mathbf{Q}_2|^{4(m+n-1)+\lambda\delta+|\alpha|} \\
 & \quad \left. H_{\mathbf{p}'}^{\delta} \left(\mathcal{D}_{\mathbf{q}, \mathbf{q}'}^{\alpha} \chi(\mathbf{q}, \mathbf{q}', \mathbf{Q}_2; \mathbf{p}') \right) \right\}
 \end{aligned}$$

This, expressed in terms of ψ , is just the result required.

Second case: the omitted variable is $\mathbf{q}_{n+m}$

With the same notations as in the previous case, we have to study

$$E(Q, p) = \int \delta(Q + P + P') F(q + \hat{p} - \hat{p}' - \hat{k}) \delta(q_1 + \dots + q_n + P_1 - P'_1 - \sum_{j=1}^1 k_j) \\ G(q' + \check{p} - \check{p}' + \check{k}) \prod_{j=1}^1 d\sigma(k_j) \prod_{k=1}^{\ell'} d\sigma(p'_k) \\ \psi(q, q_n, q'; p') dq dq_n dq'$$

We make the change of variable

$$q_n = Q_1 - \sum_{j=1}^{n-1} q_j$$

and denote

$$\chi(q, q', Q_1; p') = \psi(q, Q_1 - \sum_{j=1}^{n-1} q_j, q'; p')$$

Then,

$$E(Q, p) = \int \delta(Q_1 + P_1 - P'_1 - \sum_{j=1}^1 k_j) F(q + \hat{p} - \hat{p}' - \hat{k}) \prod_{j=1}^1 d\sigma(k_j) \prod_{k=1}^{\ell'} d\sigma(p'_k) \\ dq dQ_1 \delta(Q - Q_1 + P_2 + \sum_{j=1}^1 k_j - P'_2) G(q' + \check{p} - \check{p}' + \check{k}) \\ \prod_{k=t+1}^{\ell'} d\sigma(p'_k) \chi(q, q', Q_1; p') dq'$$

The principle is the same: we define

$$U(Q - Q_1, p; q, Q_3; p'', k) = \\ = \int \delta(Q - Q_1 + P_2 + \sum_{j=1}^1 k_j - P'_2) G(q' + \check{p} - \check{p}' + \check{k}) \\ \chi(q, q', Q_3; p') dq' \prod_{j=t+1}^{\ell'} d\sigma(p'_j)$$

$$\begin{aligned}
 V(p, Q; Q_1, Q_3) &= \\
 &= \int \delta(Q_1 + p_1 - p'_1 - \sum_{j=1}^1 k_j) F(q + \hat{p} - \hat{p}' - \hat{k}) \prod_{j=1}^1 d\sigma(k_j) \\
 &\quad \prod_{k=1}^t d\sigma(p'_k) \mathcal{U}(Q - Q_1, p; q, Q_3; p'', k) dq, \\
 W(p, Q; Q_1, Q_3) &= V(p, Q; Q_1, Q_3) - V(p, Q; 0, Q_3) \quad (56)
 \end{aligned}$$

so that

$$E(Q, p) = \int W(p, Q; Q_1, Q_3) dQ_1 + \int V(p, Q; 0, Q_3) dQ_3$$

We note that $V(p, Q; Q_1, Q_3)$ is Hölder continuous in p , Q and Q_1 . The rest of the proof is quite similar to that of the preceding case.

Step 3

Now, let K_1 be a kernel (not of the self-energy type) having the property A (with respect to the omission of any q_j carrying an external line). Let $K_2(k; p'; q_{n+1}, \dots, q_{n+m})$ be a self-energy kernel [with ingoing (resp. outgoing) external line p' (resp. k)], of the form either:

$$\delta(k - p' + \sum_{j=n+1}^{n+m} q_j) S(q_{n+1}, k, q_{n+2}, \dots, q_{n+m-1}) \quad (57)$$

or

$$\delta(k - p' + \sum_{j=n+1}^{n+m} q_j) S(q_{n+1}, \dots, q_{n+m-1}) \quad (58)$$

Assume that, in either case, K_2 possesses the property C. We wish to show that a product kernel obtained by connecting K_1 and K_2 by one line again possesses the property A. We have to consider two cases, according to which kernel comes first.

First case: K_1 comes first

Note that (Fig. 2) the connecting line between the two kernels is an external line for the first kernel; it is attached to the vertex q_1 , for example, which is thus an omissible variable for K_1 . If q_1 carries another external line (which then remains an external line for the product kernel), it may be the omitted variable for the product kernel. Thus we have to distinguish three cases.

a) The omitted variable for the product is not q_1 but belongs to K_1

Suppose q_n is the omitted variable

$$E(Q, p) = \int \delta(Q + P - P') F(q_1 + p_{I_1} - p'_{J_1} - k, \dots, q_{n-1} + p_{I_{n-1}} - p'_{J_{n-1}}) \\ d\sigma(k) \prod_{j=1}^{\ell'} d\sigma(p'_j) K_2(k; p'_\ell; q_{n+1}, \dots, q_{n+m}) \\ \psi(q_1, \dots, q_{n-1}, q_{n+1}, \dots, q_{n+m}; p') dq_1 \dots dq_{n-1} dq_{n+1} \dots dq_{n+m}$$

Because K_2 contains

$$\delta(k - p' + \sum_{j=n+1}^{n+m} q_j)$$

this can be rewritten, using the abbreviation

$$Q_2 = \sum_{j=n+1}^{n+m} q_j$$

as

$$E(Q, p) = \int \delta(Q + P - P') F(q_1 + Q_2 + p_{I_1} - p'_{J_1} - p'_\ell, \dots, q_{n-1} + p_{I_{n-1}} - p'_{J_{n-1}}) \\ \prod_{j=1}^{\ell'} d\sigma(p'_j) K_2(k; p'_\ell; q_{n+1}, \dots, q_{n+m}) d\sigma(k) \\ \psi(q_1, \dots, q_{n-1}, q_{n+1}, \dots, q_{n+m}; p') dq_1 \dots dq_{n-1} dq_{n+1} \dots dq_{n+m}$$

Denote

$$\begin{aligned} & \chi(q_1, \dots, q_{n-1}, q_{n+1}, \dots, q_{n+m}; p') = \\ & = \psi(q_1 - Q_2, \dots, q_{n-1}, q_{n+1}, \dots, q_{n+m}; p') \end{aligned}$$

we see that

$$\begin{aligned} E(Q, p) &= \int \delta(Q - P - P') F(q_1 + p_{I_1} - p'_{I_1} - p'_l, \dots, q_{n-1} + p_{I_{n-1}} - p'_{I_{n-1}} - p'_{J_{n-1}}) \\ & \prod_{j=1}^{l'} d\sigma(p'_j) dq_1 \dots dq_{n-1} \int K_2(k; p'_l; q_{n+1}, \dots, q_{n+m}) d\sigma(k) \\ & \chi(q_1, \dots, q_{n-1}, q_{n+1}, \dots, q_{n+m}; p') dq_{n+1} \dots dq_{n+m} \end{aligned}$$

If K_2 is of the form (57), the second integral can be written

$$W(q; p') = V(q; p'_l; p')$$

where for $r \in \mathbb{R}^4$, we define

$$\begin{aligned} V(q; r; p') &= \int dQ_2 \delta((r - Q_2)^2 - m^2) \theta(r^0 - Q_2^0) \times \\ & \times \int S(r - Q_2 + q_{n+1}, q_{n+2}, \dots, q_{n+m-1}) \\ & \chi_1(q, q', Q_2; p') dq' \end{aligned}$$

Here

$$\begin{aligned} q &= (q_1, \dots, q_{n-1}), \quad q' = (q_{n+1}, \dots, q_{n+m-1}), \quad p' = (p'_1, \dots, p'_l), \\ \chi_1(q, q', Q_2; p') &= \chi(q_1, \dots, q_{n-1}, q_{n+1}, \dots, q_{n+m-1}, Q_2 - \sum_{j=n+1}^{n+m-1} q_j; p') \end{aligned}$$

By applying Lemma 8, we find that $V(q; r; p')$ is actually Hölder continuous in r in a neighbourhood of $\{r: r^2 = m^2, r^0 > 0\}$,

where it satisfies:

$$H_{p', n}^{\kappa \delta}(V(q; n; p')) \leq C_{\delta} \sum_{|n| \leq M+1} \sup_{q', Q_2} |q', Q_2|^{4m + \kappa \delta + |n|}$$

$$H_{p'}^{\delta} \left(D_{q', Q_2}^{\kappa} \chi_1(q, q', Q_2; p') \right)$$

As a consequence the same inequality holds for $H_{p'}^{\kappa \delta}(W(q; p'))$, since $W(q; p') = V(q; p'_l; p')$. Now

$$E(Q, p) = \int \delta(Q + p - p') F(q + \hat{p} - \hat{p}') \prod_{j=1}^{\ell'} d\sigma(p'_j) dq W(q; p')$$

and the result follows from property A of K_1 .

If K_2 is of the form (58), the second integral can be written:

$$W(q; p') = \int dQ_2 \delta((p'_l - Q_2)^2 - m^2) \theta(p'_l \cdot Q_2) \times \\ \times \int S(q_{n+1}, \dots, q_{n+m-1}) \chi_1(q, q', Q_2; p') dq'$$

where χ_1 is defined as in the preceding case, and the conclusion is the same (using Lemma 9).

b) The omitted variable for the product is q_1

We can then represent $K_1(q_1, \dots, q_n)$ as

$$\delta\left(\sum_{j=1}^n q_j\right) F(q_2, \dots, q_n)$$

Then,

$$E(Q, p) = \int \delta(Q + p - p') F(q_2 + p_{I_2} - p'_{J_2}, \dots, q_n + p_{I_n} - p'_{J_n}) \prod_{j=1}^{\ell'} d\sigma(p'_j) \\ dq_2 \dots dq_n \int d\sigma(k) \kappa_2(k; p'_l; q_{n+1}, \dots, q_{n+m}) \\ \psi(q_2, \dots, q_n, q_{n+1}, \dots, q_{n+m}; p') dq_{n+1} \dots dq_{n+m}$$

From this point on, the proof is identical to that of case a) (with χ replaced by ψ).

c) The omitted variable for the product is q_{n+m}

We represent K_1 as in case b) so that, if K_2 is of the form (57)

$$E(Q, p) = \int \delta(Q + P - P') F(q_2 + p_{I_2} - p'_{J_2}, \dots, q_n + p_{I_n} - p'_{J_n}) \\ \delta(P - P' - k + \sum_{j=1}^n q_j) S(k + q_{n+1}, \dots, q_{n+m-1}) \prod_{j=1}^{\ell'} d\sigma(p'_j) d\sigma(k) \\ \psi(q_1, \dots, q_n, q_{n+1}, \dots, q_{n+m-1}; p') dq_1 \dots dq_{n+m-1}$$

Let us perform the change of variable

$$q_1 = Q_1 - \sum_{j=2}^n q_j$$

and denote

$$q = (q_2, \dots, q_n), \quad \hat{p} = (p_{I_2}, \dots, p_{I_n}), \quad \hat{p}' = (p'_{J_2}, \dots, p'_{J_n}),$$

$$q' = (q_{n+1}, \dots, q_{n+m-1})$$

and

$$\chi(q, q', Q_1; p') = \psi(Q_1 - \sum_{j=2}^n q_j, q_2, \dots, q_n, q_{n+1}, \dots, q_{n+m-1}; p')$$

Then

$$E(Q, p) = \int \delta(Q + P - P') F(q + \hat{p} - \hat{p}') \prod_{j=1}^{\ell'} d\sigma(p'_j) dq \times \\ \times \int dQ_1 \delta((p'_{I_1} - Q + Q_1)^2 - m^2) \theta(p'_{I_1}{}^0 - Q^0 + Q_1^0) \times$$

$$\times \int S(p'_1, -Q + Q_2 + q_{n+1}, \dots, q_{n+m-1}) \\ \chi(q, q', Q_2; p') dq'$$

Defining

$$V(q; r; p') = \int dQ_2 \delta((r + Q_2)^2 - m^2) \theta(r^0 + Q_2^0) \\ \int S(r + Q_2 + q_{n+1}, \dots, q_{n+m-1}) \chi(q, q', Q_2; p') dq'$$

we see, as in case a), that $V(q; r; p')$ is Hölder continuous in r . As a consequence (by property A of K_1)

$$E(Q, Q', p) = \int \delta(Q + P - P') F(q + \hat{p} - \hat{p}') \prod_{j=1}^{\ell'} d\sigma(p'_j) \times \\ \times V(q; p'_1, -Q'; p')$$

is Hölder continuous in p , Q and Q' . Setting $Q' = Q$ we finally obtain the required bound for $H_{p,Q}^{\lambda\delta}(E(Q,p))$. The proof is similar if K_2 is of the form (58).

Second case: K_2 comes first

The omitted variable may be (see Fig. 3) q_1 , q_{n+1} , or a variable belonging to K_1 and different from q_1 , say q_n .

a) The omitted variable for the product is $q_n \neq q_1$

$$E(Q, p) = \int S(p_1 + q_{n+1}, \dots, q_{n+m-1}) \delta((p_1 + \sum_{j=n+1}^{n+m} q_j)^2 - m^2) \\ \theta(p_1^0 + \sum_{j=n+1}^{n+m} q_j^0) \delta(Q + P - P') F(q_1 + p_1 + \sum_{j=n+1}^{n+m} q_j + p_{I_2} - p'_{J_2}, \dots, \\ q_{n-1} + p_{I_{n-1}} - p'_{J_{n-1}}) \prod_{j=1}^{\ell'} d\sigma(p'_j) dq_1 \dots dq_{n-1} dq_{n+1} \dots dq_{n+m} \\ \psi(q_1, q_2, \dots, q_{n-1}, q_{n+1}, \dots, q_{n+m}; p')$$

$$\begin{aligned}
 &= \int S(p_1 + Q_2 + q_{n+1}, q_{n+2}, \dots, q_{n+m-1}) \delta((p_1 + Q_2)^2 - m^2) \\
 &\theta(p_1^0 + Q_2^0) dQ_2 dq' \int \delta(Q + P - P') \\
 &\quad F(q_1 + p_1 + p_{I_1} - p'_{J_1}, \dots, q_{n-1} + p_{I_{n-1}} - p'_{J_{n-1}}) \prod_{j=1}^{\ell'} d\sigma(p'_j) \\
 &\quad \psi(q_1 - Q_2, q_2, \dots, q_{n-1}, q_{n+1} + Q_2, q_{n+2}, \dots, -\sum_{j=n+1}^{n+m-1} q_j; p') \\
 &\quad dq_1 \dots dq_{n-1}
 \end{aligned}$$

Calling $\chi(q, q', Q_2; p') =$

$$\psi(q_1 - Q_2, \dots, q_{n-1}, q_{n+1} + Q_2, \dots, q_{n+m-1}, -\sum_{j=n+1}^{n+m-1} q_j; p')$$

where

$$q = (q_1, \dots, q_{n-1}), \quad q' = (q_{n+1}, \dots, q_{n+m-1})$$

and

$$\begin{aligned}
 L(Q, p; q', Q_2) &= \int \delta(Q + P - P') \\
 &\quad F(q_1 + p_1 + p_{I_1} - p'_{J_1}, \dots, q_{n-1} + p_{I_{n-1}} - p'_{J_{n-1}}) \\
 &\quad \prod_{j=1}^{\ell'} d\sigma(p'_j) dq_1 \dots dq_{n-1} \chi(q, q', Q_2; p')
 \end{aligned}$$

we can estimate the Hölder continuity of $L(Q, p; q', Q_2)$ by using the property A of K_1 . Then, applying Lemma 8 to the expression

$$\int \delta((p_1 + Q_2)^2 - m^2) \theta(p_1^0 + Q_2^0) \\ S(q_{n+1} + p_1 + Q_2, q_{n+2}, \dots, q_{n+m-1}) \\ \mathcal{L}(Q, p; q', Q_2) dq' dQ_2$$

we easily find that the product kernel does have the property A with q_n omitted. The case when K_2 is of the form (58) is treated similarly, using Lemma 9.

b) The omitted variable for the product is q_1

In this case, we find [if K_2 is of the form (57)]

$$E(Q, p) = \int \delta((p_1 + Q_2)^2 - m^2) \theta(p_1^0 + Q_2^0) dQ_2 dq' \\ S(p_1 + Q_2 + q_{n+1}, q_{n+2}, \dots, q_{n+m-1}) \int \delta(Q + P - P') \\ F(q_2 + p_{I_2} - p'_{J_2}, \dots, q_n + p_{I_n} - p'_{J_n}) \prod_{j=1}^{l'} d\sigma(p'_j) \\ \mathcal{X}(q, q', Q_2; p') dq$$

where

$$q = (q_2, \dots, q_n), \quad q' = (q_{n+1}, \dots, q_{n+m-1})$$

and

$$\mathcal{X}(q, q', Q_2; p') = \\ = \psi(q_2, \dots, q_n, q_{n+1} + Q_2, q_{n+2}, \dots, -\sum_{j=n+1}^{n+m-1} q_j; p')$$

and the end of the argument is similar to the preceding one.

Again the treatment is similar for the case when K_2 is of the form (58).

c) The omitted variable for the product is q_{n+1}

If K_2 is of the form (57)

$$\begin{aligned}
 E(Q, p) &= \int S(p, + Q - Q_1, - \sum_{j=n+2}^{n+m} q_j, q_{n+2}, \dots, q_{n+m-1}) \\
 &\quad \delta((p, + Q - Q_1)^2 - m^2) \theta(p_0^0 + Q^0 - Q_1^0) dQ_1 dq_{n+2} \dots dq_{n+m} \\
 &\quad \int \delta(Q + P - P') F(q_2 + p_{I_2} - p'_{J_2}, \dots, q_n + p_{I_n} - p'_{J_n}) \prod_{j=1}^{\ell'} d\sigma(p'_j) \\
 &\quad \psi(Q_1 - \sum_{j=2}^n q_j, q_2, \dots, q_n, q_{n+2}, \dots, q_{n+m}) dq_2 \dots dq_n = \\
 &= \int S(p, + Q - Q_1 + q_{n+1}, q_{n+2}, \dots, q_{n+m-1}) \\
 &\quad \delta((p, + Q - Q_1)^2 - m^2) \theta(p_0^0 + Q^0 - Q_1^0) dQ_1 dq_{n+1} \dots dq_{n+m-1} \\
 &\quad \int F(q_2 + p_{I_2} - p'_{J_2}, \dots, q_n + p_{I_n} - p'_{J_n}) \prod_{j=1}^{\ell'} d\sigma(p'_j) \\
 &\quad dq_2 \dots dq_n \chi(q, q', Q_1; p')
 \end{aligned}$$

where $q = q_2, \dots, q_n$, $q' = q_{n+1}, \dots, q_{n+m-1}$, and

$$\begin{aligned}
 \chi(q, q', Q_1; p') &= \\
 &= \psi(Q_1 - \sum_{j=2}^n q_j, q_2, \dots, q_n, q_{n+2}, \dots, q_{n+m-1}, - \sum_{j=n+1}^{n+m-1} q_j; p')
 \end{aligned}$$

The rest of the proof continues as in case b).

If K_2 is of the form

$$K_2(p; p'; q_{n+1}, \dots, q_{n+m}) = \\ = \delta(p - p' + \sum_{j=n+1}^{n+m} q_j) S(q_{n+2}, \dots, q_{n+m})$$

[Note that we have changed the names of the variables from (58) in order to call q_{n+1} the variable carrying the external outgoing line], then

$$E(Q, p) = \int S(q_{n+2}, \dots, q_{n+m}) \delta((p_1 + Q - Q_1)^2 - m^2)$$

$$\theta(p_1^0 + Q^0 - Q_1^0) dq_{n+2} \dots dq_{n+m} dQ_1 \int \delta(Q + P - P')$$

$$F(q_2 + p_{I_2} - p'_{J_2}, \dots, q_n + p_{I_n} - p'_{J_n}) \prod_{j=1}^{e'} d\sigma(p'_j)$$

$$\psi(Q_1 - \sum_{j=2}^n q_j, q_2, \dots, q_n, q_{n+2}, \dots, q_{n+m}; p') \\ dq_2 \dots dq_n$$

Owing to the property A of K_1 and to Lemma 9 we see that, calling $p_1 + Q = r$, $E(Q, p)$ is Hölder continuous in Q , r , $p_2, \dots, p_\ell$ with norms as required by the property A.

Step 4

We now verify that a connected product of two self-energy kernels possessing the properties C and D again possesses these properties.

First case

The two kernels are of type (48). (See Fig. 4.) Then, the product kernel is also of type (48), i.e.,

$$\delta(\sum_{j=1}^{n+m} q_j + p - p') S(q_1 + p, q_2, \dots, q_n, q_{n+1}, \dots, q_{n+m-1})$$

with

$$\begin{aligned} S(q_1 + p, q_2, \dots, q_{n+m-1}) &= \\ &= S_1(q_1 + p, q_2, \dots, q_{n-1}) \delta((p + \sum_{j=1}^n q_j)^2 - m^2) \theta(p^0 + \sum_{j=1}^n q_j^0) \\ &\quad S_2(p + \sum_{j=1}^n q_j + q_{n+1}, q_{n+2}, \dots, q_{n+m-1}) \end{aligned}$$

Thus

$$\begin{aligned} E(p) &= \int S(q_1 + p, q_2, \dots, q_{n+m-1}) \psi(q_1, \dots, q_{n+m-1}) dq_1, \dots \\ &\quad \dots dq_{n+m-1} \\ &= \int \delta((p+Q)^2 - m^2) \theta(p^0 + Q^0) S_1((p+Q) + q_1, q_2, \dots, q_{n-1}) \\ &\quad S_2((p+Q) + q_{n+1}, q_{n+2}, \dots, q_{n+m-1}) \chi(q, q', Q) dq dq' dQ \end{aligned}$$

where $q = (q_1, \dots, q_{n-1})$, $q' = (q_{n+1}, \dots, q_{n+m-1})$ and

$$\begin{aligned} \chi(q, q', Q) &= \\ &= \psi(q_1 + Q, q_2, \dots, q_{n-1}, -\sum_{j=1}^{n-1} q_j, q_{n+1}, \dots, q_{n+m-1}) \end{aligned}$$

From here on, the proof is analogous to that of Lemma 8: denote

$$\begin{aligned} W(k, Q) &= \int S_1(k + q_1, \dots, q_{n-1}) S_2(k + q_{n+1}, \dots, q_{n+m-1}) \\ &\quad \chi(q, q', Q) dq dq' \end{aligned}$$

Then

$$\begin{aligned}
 |E(p)| &= \left| \int \delta((p+Q)^2 - m^2) \theta(p^0 + Q^0) W(p+Q, Q) dQ \right| \leq \\
 &\leq \int \frac{d^3 \vec{Q}}{2m} \sup_{\substack{k \\ k^2 = m^2 \\ k^0 > 0}} |W(k, Q)| \\
 &\leq \frac{C}{\varepsilon \delta} \sup_{\substack{k, Q \\ k^2 = m^2, k^0 > 0}} |Q|^{3-\varepsilon \delta} |W(k, Q)|
 \end{aligned}$$

But, by property D of S_1 and S_2 , we have, for a certain $\delta > 0$

$$\begin{aligned}
 |E(p)| &\leq \frac{C}{\varepsilon \delta} K_\delta \sum_{\substack{|\alpha| \leq M \\ |\alpha'| \leq M'}} \sup_{q, q', Q} |Q|^{3-\varepsilon \delta} |q|^{4(n-1)+\delta+1+|\alpha|} \\
 &\quad |q'|^{4(m-1)+\delta+1+|\beta|} |D_q^\alpha D_{q'}^\beta \chi(q, q', Q)| \\
 &\leq K'_\delta \sum_{|\alpha| \leq M+M'} \sup_{q, q', Q} |q, q', Q|^{4(n+m-1)+(2-\varepsilon)\delta+1+|\alpha|} \\
 &\quad |D^\alpha \chi(q, q', Q)|
 \end{aligned}$$

This shows that S has the property C_1 with $A(p) = B(p) = 0$, and, in particular, property D.

Second case

The first kernel is of type (57), the second of type (58).
(See Fig. 5.) In this case

$$\begin{aligned}
 & S(p+q_1, q_2, \dots, q_n, q_{n+1}, \dots, q_{n+m-1}) = \\
 & = S_1(q_1+p, q_2, \dots, q_{n-1}) \delta((p+\sum_{j=1}^n q_j)^2 - m^2) \theta(p^0 + \sum_{j=1}^n q_j^0) \times \\
 & \quad \times S_2(q_{n+1}, \dots, q_{n+m-1}),
 \end{aligned}$$

$$\begin{aligned}
 E(p) &= \int S(p+q_1, \dots, q_{n+m-1}) \psi(q_1, \dots, q_{n+m-1}) dq_1 \dots dq_{n+m-1} \\
 &= \int \delta((p+Q)^2 - m^2) \theta(p^0 + Q^0) W(p+Q, Q) dQ,
 \end{aligned}$$

with

$$W(k, Q) = \int S_1(k+q_1, q_2, \dots, q_{n-1})$$

$$S_2(q_{n+1}, \dots, q_{n+m-1}) \chi(q, q', Q) dq dq',$$

$$q = \{q_1, \dots, q_{n-1}\}, \quad q' = \{q_{n+1}, \dots, q_{n+m-1}\},$$

$$\chi(q, q', Q) =$$

$$= \psi(q_1+Q, q_2, \dots, q_{n-1}, -\sum_{j=1}^{n-1} q_j, q_{n+1}, \dots, q_{n+m-1})$$

and the rest of the proof follows the same line as in the first case, with the same conclusion $\boxed{A(p) = B(p) = 0}$.

Third case

K_1 is of type (49), K_2 of type (48). (See Fig. 6.)

$$\begin{aligned} S(p+q_n, q_1, \dots, q_{n-1}, q_{n+1}, q_{n+2}, \dots, q_{n+m-1}) = \\ = S_1(q_1, \dots, q_{n-1}) \delta((p + \sum_{j=1}^n q_j)^2 - m^2) \theta(p^0 + \sum_{j=1}^n q_j^0) \times \\ \times S_2(p + \sum_{j=1}^n q_j + q_{n+1}, \dots, q_{n+m-1}), \end{aligned}$$

$$E(p) = \int \delta((p+Q)^2 - m^2) \theta(p^0 + Q^0) dQ W((p+Q), Q)$$

$$\begin{aligned} W(k, Q) = \int S_1(q_1, \dots, q_{n-1}) S_2(k + q_{n+1}, q_{n+2}, \dots, q_{n+m-1}) \\ \chi(q, q', Q) dq dq', \end{aligned}$$

$$q = (q_1, \dots, q_{n-1}), \quad q' = (q_{n+1}, \dots, q_{n+m-1})$$

$$\chi(q, q', Q) =$$

$$= \psi(q_1, \dots, q_{n-1}, Q - \sum_{j=1}^{n-1} q_j, q_{n+1}, \dots, q_{n+m-1})$$

and the rest of the proof is similar to that of the first case, with the same conclusion $[A(p) = B(p) = 0]$.

Fourth case

K_1 and K_2 are both of type (49). (See Fig. 7.)

$$E(p) = \int \delta((p+Q)^2 - m^2) \theta(p^0 + Q^0) dQ W(Q),$$

$$W(Q) = \int S_1(q_1, \dots, q_{n-1}) S_2(q_{n+1}, \dots, q_{n+m-1})$$

$$\psi(q_1, \dots, q_{n-1}, Q - \sum_{j=1}^{n-1} q_j, q_{n+1}, \dots, q_{n+m-1})$$

$$dq_1 \dots dq_{n-1} dq_{n+1} \dots dq_{n+m-1}$$

The conclusion is the same as in the first three cases $\boxed{A(p) = B(p) = 0}$.

Step 5

The result of steps 2, 3, 4 is that if the kernels defined by (43) satisfy the induction hypothesis for a number of vertices $\leq n-1$, then the kernels corresponding to $D(Y;n)$ [see (42)] possess the same property. It has been seen in I that $A(Y;n)$ can be obtained from $D(Y;n)$ as follows. Denote

$$\begin{aligned} (\Omega, A^n(x_1, \dots, x_{n-1}; x_n) \Omega) &= \\ &= \int e^{-i \sum_{j=1}^n x_j q_j} \delta(\sum_{j=1}^n q_j) a_n^n(q_1, \dots, q_{n-1}) dq_1 \dots dq_n \end{aligned}$$

$$\begin{aligned} (\Omega, D^n(x_1, \dots, x_{n-1}; x_n) \Omega) &= \\ &= \int e^{-i \sum_{j=1}^n x_j q_j} \delta(\sum_{j=1}^n q_j) d_n^n(q_1, \dots, q_{n-1}) dq_1 \dots dq_n \end{aligned} \tag{59}$$

Then, $a_n^r(q_1, \dots, q_{n-1})$ is given, up to a polynomial in $q_1, \dots, q_{n-1}$ of degree ω , by

$$\begin{aligned} a_n^{\wedge}(q) &= \\ &= \int dt (1-t)^{\omega} \sum_{|\alpha|= \omega+1} \frac{\omega+1}{\alpha!} q^{\alpha} D^{\alpha} \tilde{W}(tq-q') d_n^{\wedge}(q') dq' \end{aligned} \quad (60)$$

where $\tilde{W}(q)$ is a certain distribution over $\mathbb{R}^{4(n-1)}$, homogeneous of degree $-4(n-1)$, and $\mathcal{C}^{\infty}$ except at the origin. Moreover,

$$\tilde{W}(q) = \sum_{|\beta|=1} D^{\beta} \chi_{\beta}(q)$$

where χ_{β} is, for each β (with $|\beta|=1$) a homogeneous distribution of degree $-4(n-1)+1$, $\mathcal{C}^{\infty}$ except at 0. The formula (60) makes sense because $d_n^{\wedge}(q)$ is $\mathcal{C}^{\infty}$ (in fact vanishes) in a neighbourhood of 0.

Step 5 consists of examining how this operation preserves the properties postulated in the induction hypothesis. Let F be a tempered distribution over $\mathbb{R}^N$ vanishing in

$$\{ q \in \mathbb{R}^N, |q| < R \}, \quad (R > 0).$$

We wish to study the tempered distribution T symbolically defined by

$$T(q) = \int_0^1 dt (1-t)^{\omega} \int \sum_{|\alpha|= \omega+1} \frac{\omega+1}{\alpha!} q^{\alpha} D^{\alpha} \tilde{W}(tq-q') F(q') dq' \quad (61)$$

It can be shown that this formula makes sense if F vanishes in a neighbourhood of 0 and has a degree of growth at infinity $\leq \omega$ in the sense of Definition 2 of I (Section VI). Moreover

$$D^{\alpha} T(q) = \int D^{\alpha} \tilde{W}(q-q') F(q') dq'$$

for all α with $|\alpha| \geq \omega+1$. (Note that $|D^{\alpha} \tilde{W}(q)| < C|q|^{-N-|\alpha|}$ for $q \neq 0$.) Let w be a $\mathcal{C}^{\infty}$ function such that:

$$0 \leq w \leq 1; \quad w(q) = 1 \text{ for } |q| > R - \varepsilon R; \quad w(q) = 0 \text{ for } |q| < R - 2\varepsilon R$$

(here $0 < 2\varepsilon < 1$). Then

$$T(q) = \int_0^1 dt (1-t)^\omega \sum_{|\alpha|=\omega+1} \int \frac{\omega+1}{\alpha!} q^\alpha D^\alpha \tilde{W}(tq-q') w(q') F(q') dq'$$

If $|q| < R - 2\varepsilon R$, $D^\alpha \tilde{W}(tq-q') w(q')$ is a $\mathcal{C}^\infty$ function of q' and $T(q)$ is the result of applying the distribution F to the test function

$$\psi(q') = \int_0^1 dt (1-t)^\omega \sum_{|\alpha|=\omega+1} \frac{\omega+1}{\alpha!} q^\alpha D^\alpha \tilde{W}(tq-q') w(q')$$

$D_q^\gamma D_{q'}^\beta \psi(q')$ is given by a sum of terms of the form:

$$\text{Const.} \int_0^1 dt (1-t)^\omega t^{|\sigma|} q^{\alpha-\rho} D^{\alpha+\sigma+\tau} \tilde{W}(tq-q') D^\lambda w(q')$$

where $\rho + \sigma = \gamma$ and $\rho \leq \alpha$, $\tau + \lambda = \beta$.

If $|q| < R(1 - 2\varepsilon)$ this expression is bounded in modulus by

$$\text{Const.} \int_0^1 dt \frac{|q|^{|\alpha| - |\rho|} |D^\lambda w(q')|}{|tq - q'|^{N + |\alpha| + |\sigma| + |\tau|}} \leq$$

$$\leq \text{const.} \frac{|q|^{|\alpha| - |\rho|} |D^\lambda w(q')|}{[|q'| - |q|]^{N + |\alpha| + |\sigma| + |\tau|}}$$

Moreover, if $|\lambda| \geq 1$, $D^\lambda w(q')$ vanishes for $|q'| > R(1 - \varepsilon)$; if $|q| < R(1 - 3\varepsilon)$, we have

$$|q'| - |q| > \varepsilon R, \quad |q'| - |q| \geq \left(1 - \frac{1-3\varepsilon}{1-2\varepsilon}\right) |q'| = \frac{\varepsilon}{1-2\varepsilon} |q'|$$

Hence, $|q'| - |q| > \text{const.} (1 + |q'|)$, and for all values of λ , the above expression is bounded by:

$$\frac{C_{\beta} |q|^{\alpha-1}}{(1+|q'|)^{N+\alpha+1+\beta}} \leq \frac{C'_{\beta} |q|^{(\alpha-1)^+}}{(1+|q'|)^{N+\alpha+1+\beta}} =$$

$$= \frac{C'_{\beta} |q|^{(\omega+1-1\delta)^+}}{(1+|q'|)^{N+\omega+1+\beta}}$$

(here a^+ means $\max \{a, 0\}$).

Now, for all $\varphi \in \mathcal{F}$, and for every sufficiently small $\varepsilon > 0$

$$|\langle F, \varphi \rangle| \leq C(\varepsilon) \sum_{|\beta| \leq p} \sup_{q'} (1+|q'|)^{N+\omega+1+\beta+\varepsilon} |D^{\beta} \varphi(q')|$$

Applying this to $\varphi = \psi$, we find, for $|q| < R(1-3\varepsilon)$:

$$|D^{\gamma} T(q)| \leq \text{Const. } |q|^{(\omega+1-1\delta)^+}$$

Suppose now that F vanishes in an open ball $\{q : |q-a| < R\}$ centered at some point $a \in \mathbb{R}^N$. Then

$$\int_0^1 dt (1-t)^{\omega} \int \sum_{|\alpha|=\omega+1} (q-a)^{\alpha} \frac{(\omega+1)}{\alpha!} D^{\alpha} \tilde{W}(t(q-a)+q'-a) F(q') dq'$$

is (by the previous argument) $\mathcal{C}^{\infty}$ in the open ball $\{q : |q-a| < R\}$ and has the same derivatives of order $\omega+1$ as $T(q)$. Therefore, it differs from $T(q)$ by a polynomial of degree ω ; hence $T(q)$ is $\mathcal{C}^{\infty}$ in the ball $\{q : |q-a| < R\}$. We conclude that: if F vanishes in an open set $\mathcal{O}$, then T is $\mathcal{C}^{\infty}$ in $\mathcal{O}$. (This is a natural generalization of an obvious property of the Cauchy kernel.) Since it is easy to see that if F is $\mathcal{C}^{\infty}$ with compact support, T is $\mathcal{C}^{\infty}$ everywhere, it follows that, in general T is $\mathcal{C}^{\infty}$ in any open set where F is $\mathcal{C}^{\infty}$.

Now let h be a $\mathcal{C}^\infty$ function with compact support equal to one in a (bounded) open set $\mathcal{O}$. If we write F as $(1-h)F + hF$, the contribution of $(1-h)F$ to (61) is $\mathcal{C}^\infty$ in $\mathcal{O}$ by the preceding argument. As to the contribution of hF it can be written:

$$G(q) = \int [\tilde{W}(q-q') - \sum_{|\alpha| \leq \omega} \frac{q^\alpha}{\alpha!} \mathcal{D}^\alpha \tilde{W}(-q')] F(q') h(q') dq'$$

(Indeed all the terms of the above expression make sense because hF has compact support and vanishes in a neighbourhood of 0; moreover it has the same derivatives of order $\omega+1$ as G , as well as zero derivatives at 0 of order $\leq \omega$; hence, it coincides with G .)

We note that

$$\begin{aligned} & \frac{q^\alpha}{\alpha!} \int \mathcal{D}^\alpha \tilde{W}(-q') F(q') h(q') dq' = \\ & = \frac{q^\alpha}{\alpha!} \int \mathcal{D}^\alpha \tilde{W}(-q') F(q') u(q') h(q') dq' \end{aligned} \quad (62)$$

where u is $\mathcal{C}^\infty$ and vanishes in a neighbourhood of 0. Since $u(q')h(q')\mathcal{D}^\alpha \tilde{W}(-q')$ is a $\mathcal{C}^\infty$ function with compact support (62) is of the form Cq^α .

In conclusion we see that

$$T(q) = S(q) + \int \tilde{W}(q-q') h(q') F(q') dq' \quad (63)$$

where $S(q)$ is a tempered distribution which is $\mathcal{C}^\infty$ in the bounded open set $\mathcal{O}$ where $h(q')=1$.

We now assume that the kernel

$$\delta\left(\sum_{j=1}^n q_j + \sum_{j=1}^{\ell} p_j - \sum_{j=1}^{\ell'} p'_j\right) F(q_1 + p_{I_1} - p'_{J_1}, \dots, q_{n-1} + p_{I_{n-1}} - p'_{J_{n-1}})$$

has the property A (the omitted variable being q_n); furthermore, we suppose that $F(q) = 0$ for $|q| < m$ $[q = (q_1, \dots, q_{n-1})]$, and define T by (61), so that (63) holds for suitable h . The expression

$$\int \delta(q + p - p') T(q + \hat{p} - \hat{p}') \prod_{j=1}^{\ell'} d\sigma(p_j') \psi(q; p') dq \quad (64)$$

has two terms corresponding to the decomposition (63). The most interesting is:

$$\int \delta(q + p - p') F(q' + \hat{p} - \hat{p}') h(q' + \hat{p} - \hat{p}') \chi(q'; p') \prod_{j=1}^{\ell'} d\sigma(p_j') dq',$$

$$\chi(q'; p') = \int \tilde{W}(q - q') \psi(q; p') dq \quad (65)$$

Suppose, for example, that the support of χ is contained in $\{p' : |\hat{p}' - \hat{p}'_0| < r\}$ and that p is chosen such that $|\hat{p} - \hat{p}'_0| < r$; let h have support in $\{q : |q - \hat{p}'_0 + \hat{p}'_0| < 3r\}$ and let h take the value 1 in $\{q : |q - \hat{p}'_0 + \hat{p}'_0| < 2r\}$. Then, the support of $h(q' + \hat{p} - \hat{p}') \chi(q'; p')$ is contained in $\{q' : |q'| < 5r\}$; moreover, if $\text{supp. } \psi \subset \{q, p' : |q| < r/2, |\hat{p}'_0 - \hat{p}'| < r/2\}$ it is contained in the region $\mathcal{O} = \{q : |q + \hat{p} - \hat{p}' - \hat{p}'_0 + \hat{p}'_0| < 2r\}$ where $h(q + \hat{p} - \hat{p}') = 1$ and where $S(q + \hat{p} - \hat{p}')$ is $\mathcal{E}^\infty$.

Considering χ as a function of q' with values in the Hölder continuous functions of p' , $h(q' + \hat{p} - \hat{p}')$ as a $\mathcal{E}^\infty$ function of q' with values in the Hölder continuous functions of p and p' and using the fact that (see Section 2.3) convolution with $\tilde{W}$ preserves the adiabatic property, we find (with evident conditions on the support of ψ):

$$\begin{aligned}
 & H_{p,p'}^{\lambda\delta} \left(\int h(q' + \hat{p} - \hat{p}') \chi(q'; p') dq' \right) + \\
 & + \sum_{|\lambda| \leq M} \sup_{q'} |q'|^{4(n-1) + \lambda\delta + |\lambda|} H_{p,p'}^{\lambda\delta} \left(\mathcal{D}_q^\lambda [h(q' + \hat{p} - \hat{p}') \chi(q'; p')] \right) \\
 & \leq C_\delta \left\{ H_{p,p'}^\delta \left(\int \psi(q; p') dq \right) + \right. \\
 & \left. + \sum_{|\lambda| \leq M+1} \sup_q |q|^{4(n-1) + \delta + |\lambda|} H_{p,p'}^\delta \left(\mathcal{D}_q^\lambda \psi(q; p') \right) \right\}
 \end{aligned}$$

which proves that this part of the kernel defined by T has the property A with the omitted variable q_n .

Let us consider now the contribution of $S(q)$ to (64):

$$\int S(q + \hat{p} - \hat{p}') \psi(q; p') dq \prod_{j=1}^{e'} d\sigma(p'_j)$$

If, as we suppose, the support of ψ is contained in $\{q : |q| < R\}$, we may consider $\psi(q; p')$ as a $\mathcal{C}^\infty$ function of q with values in the Hölder continuous functions of p' , and $S(q + \hat{p} - \hat{p}')$ as a $\mathcal{C}^\infty$ function of q with values in the Hölder continuous functions of p and p' . We have

$$\begin{aligned}
 & H_{p,p'}^\delta \left(\int S(q + \hat{p} - \hat{p}') \psi(q; p') dq \right) \leq \\
 & \leq C_\delta \left\{ H_{p,p'}^\delta \left(\int \psi(q; p') dq \right) + \sup_q |q|^{4(n-1) + \delta} \right. \\
 & \quad \left. H_{p,p'}^\delta (\psi(q; p')) \right\}
 \end{aligned}$$

(same calculation as in 2.2).

Applying now Lemma 10 (phase space Lemma) we find that the kernel defined by T has the property A with omitted variable q_n

provided the number ℓ' of its incoming external lines is at least 2. However, if $\ell' \leq 1$ and if T (and hence F) is not a self energy kernel, it automatically has the property A (because it vanishes in a neighbourhood of the mass shell and $q=0$). We now reach the following conclusion.

Let K be a kernel, not of the self-energy type, arising from $a_n^r(1, \dots, n-1; n)$. The induction hypothesis for kernels with at most $n-1$ vertices implies that K has the property A with omitted variable q_n , provided q_n carries an external line. But, if q_j is another variable carrying an external line, K can be written as

$$K' + \sum_{k=1}^1 R_k$$

where K' is a kernel (with the same vertices and external lines) arising from $a_n^r(i_1, \dots, i_{n-1}; j)$ (hence, having the property A with q_j as the omitted variable), and, for each k , R_k is a connected product of two kernels with at most $n-1$ vertices, having the property A by steps 2 and 3. Hence, K has the property A with the omitted variable being any variable q_j carrying at least one external line. The same argument yields the same conclusion for the vacuum expectation values of all the G.R.P. with n vertices.

The case of self-energy kernels

We have seen in Step 4 that if a self-energy type kernel K is the connected product of two self-energy kernels [with at most $(n-1)$ vertices each], it follows from the induction hypothesis that it satisfies property C_1 [with $A(p)=B(p)=0$] and D . If, on the other hand, K is the connected product of any two kernels not of the self-energy type, it vanishes in a neighbourhood of $p^2 < 4m^2$, $q=0$, hence, it again satisfies C_1 and D [$A(p)=B(p)=0$] or C_2 . Hence, the kernels of the self-energy type arising from $(\Omega, D^r(Y; j)\Omega)$ either possess property C_1 , with $A(p)=B(p)=0$, and, hence, property D , or property C_2 (here $|Y| + 1 = n$).

Now let

$$T(p+q_1, \dots, q_{n-1}) \delta(p + \sum_{j=1}^n q_j - p')$$

be a kernel of type (48) arising from $(\Omega, A^r(Y; n)\Omega)$, (where $Y = \{1, \dots, n-1\}$). According to the above remarks

$$\begin{aligned} T(q_1, \dots, q_{n-1}) &\equiv T(q) = \\ &= \int_0^1 dt (1-t)^\omega \sum_{|\alpha|=\omega+1} \frac{\omega+1}{\alpha!} D^\alpha \tilde{W}(tq - q') F(q') dq' + \quad (66) \\ &\quad + \text{polynomial of degree } \omega \end{aligned}$$

where $F(p+q_1, \dots, q_{n-1})$ satisfies the property C_1 with $A(p)=B(p)=0$. The same is, of course true for $F(q)h(q)$ if $h \in \mathcal{D}(\mathbb{R}^{4(n-1)})$.

Hence, if we perform the decomposition (53)

$$T(q) = S(q) + \int \tilde{W}(q - q') h(q') F(q') dq',$$

$h \in \mathcal{D}$, $h(q) = 0$ for $|q| > 2R$, $h(q) = 1$ for $|q| < \frac{3}{2}R$,

[possible because $F(q)$ vanishes for $|q| < m$], we see that:

1) $S(p+q_1, \dots, q_{n-1})$ is infinitely differentiable for $|q| < R/2$, $|p| < R$. Hence, S and its derivatives of all orders have adiabatic limits in this region.

2) For $|\beta| = 1$, $D_q^\beta [h(p+q_1, q_2, \dots, q_{n-1}) F(p+q_1, q_2, \dots, q_{n-1})]$ has an adiabatic limit at $q=0$ for every p such that $p^2 < 4m^2$, uniformly in p in every compact there. It follows from our study of the convolution by $\tilde{W}$, that

$$D^\beta \tilde{W} * (hF)(p+q_1, \dots, q_{n-1})$$

also has an adiabatic limit at $q=0$, uniformly in p in every compact in $\{p : p^2 < 4m^2\}$.

Hence, the same is true for $D^{\beta} T(p+q_1, q_2, \dots, q_{n-1}) (|\beta| \leq 1)$.
On the other hand, we know that, if $\varphi \in \mathcal{S}(\mathbb{R}^4)$, β arbitrary

$$\int D^{\beta} T(p+q_1, q_2, \dots, q_{n-1}) \varphi(p) d^4 p \quad (67)$$

is an infinitely differentiable function of $q_1, \dots, q_{n-1}$ in a neighbourhood of zero. The value at $q=0$ of this expression is of the form:

$$\int G_+^{\beta}(p) \varphi(p) dp$$

Here $G_+^{\beta}(p)$ is a tempered distribution, the boundary value from the forward tube of a function $L^{\beta}(k)$, holomorphic in

$$\{k \in \mathbb{C}^4 : k^2 \neq m^2 \text{ and } k^2 \notin 4m^2 + \mathbb{R}^+\}$$

Clearly, for $|\beta| \leq 1$, since $\int G_+^{\beta}(p) dp \varphi(p)$ can be obtained by taking the adiabatic limit of (67) at $q=0$, we have: $G_+^{\beta}(p) =$ adiabatic limit of $D^{\beta} T(p+q_1, q_2, \dots, q_{n-1})$ for $p^2 < 4m^2$ (hence G_+^{β} is Hölder continuous there).

The same argument holds for the corresponding kernel associated with $(\Omega, R(Y; n) \Omega)$ and its first derivatives; their adiabatic limits coincide with the boundary values $G_-^{\beta}(p)$ of $L^{\beta}(k)$ from the backward tube in the region $p^2 < 4m^2$. However, the two kernels we have considered differ by a sum of connected products of kernels of fewer vertices; hence, their adiabatic limits and those of their first derivatives coincide in the region $p^2 < 4m^2$. By the edge-of-the-wedge theorem, $L^{\beta}(k)$ is actually holomorphic in

$$\{k \in \mathbb{C}^4 : k^2 \notin 4m^2 + \mathbb{R}^+\}$$

and, in view of its polynomial boundedness

$$L^{\beta}(k) = \sum_{|\alpha| \leq N} c_{\alpha}^{\beta}(k^2) k^{\alpha}$$

where the $c_{\alpha}^{\beta}(z)$ are holomorphic in $\{z \in \mathbb{C} : z \notin 4m^2 + \mathbb{R}^+\}$ (we have applied the Bogoliubov-Vladimirov theorem ³⁾; see also ⁴⁾).

Let $P(p+q_1, q_2, \dots, q_{n-1})$ be the polynomial:

$$\sum_{|\alpha| \leq N} \left[c_{\alpha}^0(m^2)(p+q_1)^{\alpha} + c_{\alpha}'(m^2)[(p+q_1)^2 - m^2](p+q_1)^{\alpha} \right] + \sum_{\substack{|\beta|=1 \\ \beta_{1\mu}=0 \forall \mu}} c_{\alpha}^{\beta}(m^2)(p+q_1)^{\alpha} q^{\beta}$$

where $c_{\alpha}'(z) = (d/dz)c_{\alpha}^0(z)$.

At $q=0$, this expression takes the value

$$\sum_{|\alpha| \leq N} c_{\alpha}^0(m^2) + 2 c_{\alpha}'(m^2)(p^2 - m^2)p^{\alpha}$$

and, for $p^2 = m^2$, this coincides with $L(p)$.

If γ is a multi-index with $|\gamma| = 1$ and $\gamma_{1\mu} = 0$ for all μ , (i.e., γ refers only to $q_2, \dots, q_{n-1}$)

$$D^{\gamma} P(p+q_1, q_2, \dots, q_{n-1}) \big|_{q=0} = \sum_{\alpha} c_{\alpha}^{\gamma}(m^2) p^{\alpha}$$

and this coincides with $L^{\gamma}(p)$ if $p^2 = m^2$.

If $|\gamma| = 1$ and γ refers to q_1 only,

$$\begin{aligned} & D^{\gamma} P(p+q_1, q_2, \dots, q_{n-1}) \big|_{q_2=\dots=q_{n-1}=0} = \\ & = \sum_{\alpha} \left[c_{\alpha}^0(m^2) D^{\gamma}(p+q_1)^{\alpha} + 2 c_{\alpha}'(m^2)(p+q_1)^{\gamma}(p+q_1)^{\alpha} \right. \\ & \quad \left. + c_{\alpha}'(m^2)[(p+q_1)^2 - m^2] D^{\gamma}(p+q_1)^{\alpha} \right] \end{aligned}$$

At $q_1=0$, $p^2=m^2$, this becomes:

$$\sum_{\alpha} c_{\alpha}^0(m^2) D_p^{\alpha} p^{\alpha} + 2 c_{\alpha}'(m^2) p^{\alpha} p^{\alpha} = \\ = \sum_{\alpha} D_{q_1}^{\alpha} [c_{\alpha}^0((p+q_1)^2) (p+q_1)^{\alpha}] = D_{q_1}^{\alpha} L(p+q_1) |_{q_1=0}$$

Hence, we have found a polynomial $P(q_1, \dots, q_{n-1})$ such that $D^{\beta} T(p+q_1, q_2, \dots, q_{n-1}) - D^{\beta} P(p+q_1, q_2, \dots, q_{n-1})$ has adiabatic limit zero at $q=0$ for all p such that $p^2=m^2$ and all β such that $|\beta| \leq 1$. We can thus redefine T , by subtracting P from it, in such a way that the new kernel has the properties C_1 and D . It will follow that all the kernels with the same vertices and external lines attached to them, arising from the other G.R.P. have the property D : indeed, such a kernel differs from T by a sum of connected products of kernels with fewer vertices; such products have, as we have seen, the property C_1 with $A(p) = B(p) = 0$, and hence the property D .

An unsatisfactory feature of the above argument is that the polynomial P as it is defined here could have degree >2 , and this, in general clashes with the minimality of renormalization. To avoid this trouble we may first assume that renormalization is being performed so as to preserve Lorentz invariance. In this case we have

$$L^0(k) = c(k^2) \\ L^{\beta}(k) \equiv L_{j\mu}(k) = c_j(k^2) k_{\mu} \quad \text{if } |\beta| = \beta_{j\mu} = 1$$

The polynomial P then becomes

$$c_0(m^2) + c_0'(m^2)(q_1^2 - m^2) + \sum_{j=2}^{n-1} c_j(m^2)(q_1, q_j)$$

and has degree ≤ 2 .

More generally, we see that, if the two-point function $H(p+q_1, q_2, \dots, q_{n-1})$ has been defined so that it vanishes at $q=0$ for any p such that $p^2 = m^2$, together with all its first derivatives in $q_1, \dots, q_{n-1}$, the corresponding kernels possess the properties C_1 and D .

Suppose now that $S(q_1, q_2, \dots, q_{n-1}) \delta(p - p' + \sum_{j=1}^n q_j)$ is a self-energy kernel of type (15) arising from some G.R.F. Then, as it is well known, it coincides in a real neighbourhood of zero with function analytic in a complex neighbourhood of zero. By subtracting from S a polynomial of degree 1, we can make it vanish twice at 0. The kernel thus obtained will clearly possess the property C_2 .

Finally, we have proved that all connected kernels possessing external lines and n vertices satisfy the properties expressed in the induction hypothesis. As to the kernels with no external lines (which play no role in the inductive construction of connected kernels with external lines), we also wish them to have adiabatic limits such that the condition

$$(\Omega, S(q) \Omega) \xrightarrow[g \rightarrow \lambda]{} 1$$

be satisfied.

Let $\delta(q_1 + \dots + q_n) F(q_1, \dots, q_{n-1})$ be such a connected vacuum kernel. $F(q)$ is analytic in some complex open neighbourhood of 0.

$$\begin{aligned} & \int \delta(q_1 + \dots + q_n) F(q_1, \dots, q_{n-1}) \psi(q_1, \dots, q_n) dq_1 \dots dq_n = \\ & = \int \delta(Q) F(q) \chi(q, Q) dq dQ = \int F(q) \chi(q, 0) dq, \end{aligned}$$

where

$$q = (q_1, \dots, q_{n-1}) \text{ and } \chi(q, Q) = \psi(q_1, \dots, q_{n-1}, Q - \sum_{j=1}^{n-1} q_j)$$

If we now replace χ by χ_ε given by

$$\chi_\varepsilon(q, Q) = \varepsilon^{-4n} \chi\left(\frac{q}{\varepsilon}, \frac{Q}{\varepsilon}\right)$$

we find: $\varepsilon^{-4} \int F(\varepsilon q) \chi(q, 0) dq$. This, in general, diverges like ε^{-4} unless $F(q)$ vanishes sufficiently strongly at 0. If we redefine F , by subtracting a polynomial of the 4th degree, so that it takes the form

$$F(q) = \sum_{|\alpha|=5} q^\alpha F_\alpha(q)$$

(the F_α being analytic at 0) we find:

$$\begin{aligned} \left| \int F(q) \chi(q, 0) dq \right| &\leq \frac{C}{\varepsilon} \sup_q |q|^{4(n-1)+5-\varepsilon} |\chi(q, 0)| \\ &\leq \frac{C}{\varepsilon} \sup_{q, Q} |q, Q|^{4n+1-\varepsilon} |\chi(q, 0)| \end{aligned}$$

Thus, the kernel does have a zero adiabatic limit at 0.

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Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

Figure 6

Figure 7