ON THE COVARIANT DESCRIPTION OF SPONTANEOUSLY BROKEN SYMMETRY
IN GENERAL FIELD THEORY

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A B S T R A C T

Reducible fields $A(x)$ with degenerate vacuum, which allow the unitary symmetry transformation $U^{-1}(c)A(x)U(c) = A(x) + c$ are analyzed. We describe the mathematical properties of the "charge integral" related to the conserved current of this spontaneously broken symmetry. Further we discuss the structure of the S matrix theory in such a generalized field theory as a guide-line for the treatment of more complex examples of spontaneously broken symmetries.

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1. - INTRODUCTION

The idea of spontaneously broken symmetry plays an important rôle in the attempts ¹⁾ of understanding elementary particle physics : spontaneously broken $SU(2) \times SU(2)$ symmetry, with pions as Goldstone bosons, seems to be realized approximately in Nature ²⁾. Spontaneous symmetry breaking plays an essential rôle in the construction of renormalizable unified theories of weak and electromagnetic interactions ³⁾. Long range forces related to spontaneously broken gauge invariance might be responsible for quark confinement ⁴⁾. These few physical examples should be enough to justify our interest in a new formalism for the treatment of spontaneous symmetry breaking in the framework of relativistic quantum field theory.

The physics of spontaneous symmetry breaking (SSB) is most transparent in the case of a field theory with an explicitly given Lagrangian. It describes the situation in which a symmetric Lagrangian leads to an unsymmetric vacuum (ground state), whereas the symmetric vacuum is unstable. Inspection of the "classical potential" of the Lagrangian ⁵⁾ often gives a clue for the appearance of SSB. Non-vanishing vacuum expectation values of fields indicate the unsymmetric character of this state. As an example we mention the non-linear Lagrangians describing spontaneously broken chiral symmetry ⁶⁾. Since in Lagrangian field theory the description of SSB is intrinsically connected with the dynamical problems, results following directly from symmetry considerations - we mean spontaneously broken symmetry - are often difficult to derive. But, on the other hand, we know how simply consequences of unbroken symmetry can be discussed in "General Field Theory" (GFT) ⁷⁾, a formalism describing only those properties of quantized fields which are independent of the special dynamics. Therefore, one would expect that in the framework of GFT also the general results of spontaneously broken symmetry can be studied formally in a simple way. Such an analysis of spontaneously broken symmetry is the aim of this paper.

In order to explain our intentions, we have to discuss first the abstract features of SSB ⁸⁾. For this we consider $A(x)$, a quantized, Poincaré covariant, local field (with suppressed tensor - and other indices), and assume that the defining relations of $A(x)$, i.e., the field equations and commutation relations, are invariant under a group $\mathcal{G}$ of transformations $\sigma : A(x) \rightarrow A^\sigma(x)$. The existence of such a group of automorphisms of the field algebra generated by $\{A(x)\}$ may lead either to a quantum mechanical symmetry (α), or to a spontaneously broken symmetry (β). These two cases are distinguished by the following properties.

- (α) - There is a quantum mechanical symmetry (QMF) iff in an irreducible representation of $A(x)$ by operators of a Hilbert space $\mathcal{H}$ the transformations σ can be implemented by a unitary representation $U(\sigma)$ of $\mathcal{G}$:

$$A^{\sigma}(x) = U^{-1}(\sigma) A(x) U(\sigma) \quad (1.1)$$

or if an infinitesimal version with $U(\sigma) = e^{-iQs}$:

$$\delta_Q A(x) = i [Q, A(x)] \quad (1.2)$$

The irreducibility assumption implies that $U(\sigma)$ and Q are limits of expressions formed from $A(x)$. Space-time translations, infinitesimally implemented by the momentum and Hamilton operators, are well-known examples of such quantum mechanical symmetries.

- (β) - There is a spontaneously broken symmetry iff in irreducible representations of $A(x)$ such an $U(\sigma)$ does not exist, i.e., the action of $A(x) \rightarrow A^{\sigma}(x)$ on an irreducible representation of the field algebra generates an inequivalent representation. We want to mention here the simplest example of such a spontaneously broken symmetry⁹⁾. The field equation and the commutation relation of a massless free field

$$\square A(x) = 0, \quad [A(x), A(y)] = i D(x-y) \quad (1.3)$$

are invariant under the transformation

$$A^c(x) = A(x) + c, \quad \text{or} \quad \delta A(x) = \delta c \quad (1.4)$$

It is well known⁹⁾, and we discuss it later, that for an irreducible representation $A(x)$ there is no unitary operator $U(c)$ (self-adjoint operator Q) which satisfies

$$U^{-1}(c) A(x) U(c) = A(x) + c$$

or

$$i[Q, A(x)] = 1 \quad (1.5)$$

Therefore, the symmetry described by Eq. (1.4) is spontaneously broken.

In canonical quantum field theory, a continuous automorphism group is related to conserved currents: $j_{\mu}^{(k)}(x)$, $k=1, \dots, n = \dim \mathcal{G}$; $\partial_{\mu} j_{\mu}^{(k)}(x) = 0$. Since discussions of QMS and SSB are for the most part based on the existence of these currents, we want to review shortly the well-known situation for the two cases:

(α') - a gap in the energy spectrum between the energy of the non-degenerate vacuum state and that of the particle states implies the existence of QMS¹⁰⁾ implemented by charge operators $Q^{(k)}$; the matrix element of the integral of the charge density between localized states ψ and ϕ converges^{10), 11)}:

$$\lim_{V \rightarrow \infty} \left(\psi, \int_V j_0^{(k)}(x) d\vec{x} \phi \right) = (\psi, Q^{(k)} \phi) \quad (1.6)$$

$x_0 = t$

(β') - in the case of SSB there is no energy gap; this implies the existence of mass zero particles for manifestly relativistic covariant local theories ("Goldstone theorem"^{8), 12)}); for gauge theories the appearance of physical Goldstone particles might be absent because of the Higgs-Kibble mechanism¹³⁾; in irreducible representations, nothing can be said in general on the existence or meaning of $\int j_0(x) d\vec{x}$.

Conserved currents for which the charge integral cannot be related to the generators $Q^{(k)}$ of a quantum mechanical symmetry may still be used to describe a spontaneously broken symmetry. Thus current algebra is the classical method to treat spontaneously broken chiral symmetry¹⁴⁾. But one should bear in mind that there might be SSB in cases where no currents are related to the automorphism group $\mathcal{G}$. Spontaneous breaking of the discrete T symmetry recently proposed by Lee¹⁵⁾ is such an example.

In our analysis of spontaneously broken symmetry, we do not want to abandon relations of type Eqs. (1.1), (1.2):

$$A^{\epsilon}(x) = U^{-1}(\epsilon) A(x) U(\epsilon) \quad \text{or} \quad \delta_Q A(x) = i [Q, A(x)] \quad (1.7)$$

because most of the quantum mechanical symmetry considerations are based on such equations which define the fields as covariant tensor operators with respect to the group G . Fields covariant with respect to a spontaneously broken symmetry group cannot be represented irreducibly by operators in a Hilbert space $\mathcal{H}$, rather we have to consider the direct integral (sum) of the irreducible representations generated by the action of the group G . This is evident from the characteristics of SSB.

Giving up the irreducibility condition has serious consequences. The fields, i.e., the dynamical variables, no longer generate a complete operator system in $\mathcal{H}$, therefore the relation between observables and operators in $\mathcal{H}$ gives rise to questions similar to those in the discussion of superselection rules¹⁶⁾. In particular the relation of the symmetry generator $Q^{(k)}$ to the local "charge densities" $j_0^{(k)}(x)$ becomes even more intricate than in a case of QMS. The reducibility of the fields $\{A(x)\}$ implies further a degeneracy of the vacuum¹⁷⁾ which means a change in the usual axioms of GFT. In the following we have to deal with all these problems. But the essential point is to show that these modifications of the quantum mechanical scheme does not affect the S matrix interpretation of quantum field theory.

Seeing all these complications, why do we still believe that it is worth while to study fields which are covariant with respect to a spontaneously broken symmetry group? We already mentioned that such fields allow the application of the group theoretical techniques generally used in physics. With help of these methods the group theoretical structure of the consequences of SSB becomes more transparent. Furthermore, this structure shows often a new feature, called "dynamical rearrangement of symmetry" by Umezawa¹⁸⁾, which requires flexible group theoretical methods for its investigation. We shall illustrate this notion for the example of chiral $SU(2) \times SU(2)$ symmetry realized by non-linear transformations of the pion field $\vec{\pi}(x)$. In this example the infinitesimal isospin and chiral transformations have the form¹⁹⁾

$$\begin{aligned}\delta_V \vec{\pi}(x) &= \vec{\pi} \times \delta \vec{\omega}_A \\ \delta_A \vec{\pi}(x) &= \delta \vec{\omega}_A + F(\vec{\pi}^2) \left((\vec{\pi} \cdot \delta \vec{\omega}_A) \vec{\pi} - \vec{\pi}^2 \delta \vec{\omega}_A \right)\end{aligned}\quad (1.8)$$

with $F(\vec{\pi}^2) = (\vec{\pi}^2)^{-1} (1 - |\vec{\pi}| \operatorname{ctg} |\vec{\pi}|)$ in the exponential model. They satisfy the commutation relations

$$\begin{aligned}[\delta_{V_i}, \delta_{V_k}] &= -\varepsilon_{ik\ell} \delta_{V_\ell} & [\delta_{V_i}, \delta_{A_k}] &= -\varepsilon_{ik\ell} \delta_{A_\ell} \\ [\delta_{A_i}, \delta_{A_k}] &= -\varepsilon_{ik\ell} \delta_{V_\ell}\end{aligned}\quad (1.9)$$

In the LSZ limit ²⁰⁾ these transformation laws change to those of the free asymptotic fields for massless particles :

$$\delta_V \vec{\pi}_{\text{ex}}(x) = \vec{\pi}_{\text{ex}} \times \delta \vec{\omega}_V, \quad \delta_A \vec{\pi}_{\text{ex}}(x) = \delta \vec{\omega}_A \quad (1.10)$$

In contrast to Eq. (1.9), these infinitesimal transformations have the commutation relations

$$\begin{aligned}[\delta_{V_i}, \delta_{V_k}] &= -\varepsilon_{ik\ell} \delta_{V_\ell} & [\delta_{V_i}, \delta_{A_k}] &= -\varepsilon_{ik\ell} \delta_{A_\ell} \\ [\delta_{A_i}, \delta_{A_k}] &= 0\end{aligned}\quad (1.11)$$

which are the commutation relations of a group isomorphic to the three-dimensional Euclidean group $\text{ISO}(3)$. This "contraction" ²¹⁾ of the automorphism group $\text{SU}(2) \times \text{SU}(2)$ of the interacting fields $\vec{\pi}(x)$ to that of the massless free pion fields $\vec{\pi}_{\text{ex}}(x)$, induced by the LSZ limit, is called "dynamical rearrangement of symmetry". The only systematic approach to questions raised by this new phenomenon, e.g., in a chiral invariant perturbation theory ²²⁾, are based on functional methods. We hope that the use of covariant representations of fields, together with the group theoretical technique of contraction and deformation ²³⁾ of representations, might give a starting point for an operator approach to these problems. All these more mathematical questions are closely related to practical physical problems. Thus the incorporation of spontaneously broken chiral symmetry in the phenomenology of elementary particles based on the quark model is not yet performed satisfactorily. But it is evident from the

attempts of this problem up to now ²⁴⁾, that a systematic study of SSB in GFT, particularly with respect to the bound state problem, would be very helpful for an understanding of this question of "current and constituent quarks".

After this rather lengthy sketch of questions raised by the problem of spontaneous symmetry breaking in general field theory, the content of this paper appears relatively modest, and it represents only a beginning. In the following chapter we describe completely the covariant fields of the non-interacting model defined by Eqs. (1.3)-(1.5). This is recommended by simplicity, but also by the importance of such fields in the scattering theory. Besides the clarification of the conceptual questions, the precise description of massless particles in a theory with degenerate vacuum forms an important part of this chapter. In addition, we discuss the properties of the "charge" related to the current of this spontaneously broken pion gauge symmetry. In the other chapter, we discuss the interacting fields and the scattering theory in the case of such a pion gauge invariance. The main result will be to show how certain zeros of the scattering amplitude originate in the spontaneous broken symmetry. These correspond to the well-known Adler zeros ²⁵⁾ in chiral symmetry. Since we assume the same symmetry for the interacting and asymptotic fields, we do not yet treat dynamical rearrangement of symmetry in this paper. But we consider our study mainly as a guide-line to the consideration of more complex examples.

This work is very much influenced by the recent discussion of chiral symmetry by Lehmann ²⁶⁾, in which he based many of his arguments directly on the scattering theories in GFT. Under methodical aspects we consider our work as a continuation of an investigation done by Kramer and Palmer ²⁷⁾. However, there are two important points in which we do not follow these authors. We show that it is unnecessary to describe covariant fields in non-separable Hilbert spaces ²⁸⁾, and we shall stay in the conventional one. Furthermore, our discussion is strictly based on the conceptual framework of GFT. This leads to an improvement of the mathematical and conceptual framework for the treatment of quantum fields covariant with respect to a spontaneously broken symmetry.

2. - FREE FIELDS WITH COVARIANT SCALAR GAUGE SYMMETRY

2.1. - Definitions

We shall investigate with appropriate mathematical rigour the free massless quantum field $A(x)$ which is covariant with respect to scalar gauge transformations. This field theoretical model is defined by Eqs. (1.3)-(1.5) which get from the "smeared" field operators : $A(f) = \int A(x)f(x)dx$, the following form :

$$A(\Box f) = 0 \quad (2.1)$$

$$[A(f), A(g)] = i(f, g) \quad (2.1')$$

$$\begin{aligned} U^{-1}(c) A(f) U(c) &= A(f) + c \int f(x) dx \\ &= A(f) + c (2\pi)^{3/2} \tilde{f}(0) \end{aligned} \quad (2.1'')$$

$f \in \mathcal{X}$ are infinitely differentiable, fast decreasing test functions ²⁹⁾ :

$$f(x) = (2\pi)^{-5/2} \int \tilde{f}(p) e^{-ipx} dp \quad (2.2)$$

$$\begin{aligned} (f, g) &= \iint dx dy f(x) D(x-y) g(y) \\ &= -i \int dp \tilde{f}(-p) \tilde{g}(p) \epsilon(p_0) \delta(p^2) \end{aligned} \quad (2.3)$$

$$D(x) = -\frac{1}{2\pi} \epsilon(x_0) \delta(x^2) \quad (2.4)$$

Our aim is to describe the representations of $\{A(f)\}$ by operators in a separable Hilbert space $\mathcal{H}$, restricted by the postulates of GFT ⁷⁾ :

(I) $A(f)$ is defined on a dense set $\mathcal{D} \subset \mathcal{H}$ with $A(f)\mathcal{D} \subset \mathcal{D}$, and with

$$(\phi, A(f)\psi) = (A(f^*)\phi, \psi) \quad \text{for } \phi, \psi \in \mathcal{D}$$

- (II) there is a unitary representation in $\mathcal{H}$ of the Poincaré group $U(\Lambda, a)$ with an energy momentum spectrum in the forward cone V^+ :
 $P_\mu P^\mu \geq 0$, $P_0 \geq 0$, and there is at least one vacuum state :

$$U(\Lambda, a) \Omega_0 = \Omega_0, \quad \Omega_0 \in \mathcal{D}$$

- (III) the fields transform covariantly with respect to $U(\Lambda, a)$:

$$U^{-1}(\Lambda, a) A(x) U(\Lambda, a) = A(\Lambda x + a)$$

Because we treat fields covariant with respect to a spontaneously broken symmetry, we would run into a contradiction, if we would assume in (II) the uniqueness of the vacuum. On the contrary, we have to expect an infinite dimensional subspace $\mathcal{H}_0$ of vectors Ω_i , each invariant under $U(\Lambda, a)$. Let us first clarify this aspect of the problem.

2.2. - The vacuum subspace

In order to avoid complications in our discussion, which originate in the consideration of physically implausible situations, we make the additional assumption ³⁰⁾

$$[U(c), U(\Lambda, a)] = 0 \quad (2.5)$$

This has the immediate consequence that with Ω_0 , from (2.1) (II), also $\Omega_c = U(c)\Omega_0$ is a Poincaré invariant state : $U(\Lambda, a)U(c)\Omega_0 = U(c)U(\Lambda, a)\Omega_0 = U(c)\Omega_0$. Since the expectation values of the covariant field $A(x)$ are different in these states : $(\Omega_c, A(x)\Omega_c) = (\Omega_0, A(x)\Omega_0) + c$, there is a multi-dimensional subspace $\mathcal{H}_0$ of Poincaré invariant states, the "vacuum subspace". It contains a subspace $\hat{\mathcal{H}}_0$ generated by $U(c)$ from Ω_0 as a cyclic vector

$$\Omega \in \hat{\mathcal{H}}_0 : \quad \Omega = \int g(c) U(c) \Omega_0 dc$$

(2.6)

$g(c)$ sufficiently decreasing

It is sufficient for our purpose to consider cases where $\overline{\mathcal{H}}_0 = \mathcal{H}_0$, and we assume this always - if not otherwise stated. Because of

$$\|A(f)\Omega\|^2 = \iint dc \, dc' \, g(c') g^*(c).$$

$$\begin{aligned} & \cdot (U(c-c')\Omega_0, (A^+(f) + (2\pi)^{3/2} c' \tilde{f}^*(0)) (A(f) + (2\pi)^{3/2} c \tilde{f}(0)) \Omega_0) \\ & \leq |G_0|^2 \|A^+(f) A(f)\Omega_0\| + 2 \cdot (2\pi)^{3/2} |G_0 G_1 \tilde{f}(0)| \cdot \|A(f)\Omega_0\| + \\ & + (2\pi)^3 |G_0 G_2| \cdot |\tilde{f}(0)|^2 \end{aligned}$$

$$G_i = \int g(c) c^i \, dc \quad i = 0, 1, 2$$

$A(f)$ can be naturally defined on a dense set in $\mathcal{H}_0$. Therefore we assume generally for the domain $\mathcal{D}$ of $A(f)$: $U(c)\mathcal{D} \subset \mathcal{D}$.

We want to construct an orthonormal base in $\mathcal{H}_0$ ($\equiv \overline{\mathcal{H}}_0$) related to $A(x)$. For this we have to consider certain R_3 -means of $A(x)$

$$C = \lim_{V \rightarrow \mathbb{R}_3} \frac{1}{V} \int_V d\vec{x} \, A(\vec{x}, x_0) \quad (2.7)$$

This operator is defined by the following procedure ³¹⁾ : instead of the mean value of $A(x)$, we first consider that of a smeared field $A_f(\vec{x}) = U(-\vec{x})A(f)U(\vec{x})$. The averaging is performed with help of a family of functions $\chi_V(\vec{x})$, called an M filter ³¹⁾, with the properties :

- i) $\chi_V(\vec{x}) \geq 0$
- ii) $\int \chi_V(\vec{x}) \, d\vec{x} = 1$
- iii) $\lim_V \int d\vec{x} \, |\chi_V(\vec{x}) - \chi_V(\vec{x} + \vec{y})| = 0 \quad \text{for all } \vec{y} \in \mathbb{R}^3$

An example is given by the characteristic function $\chi_V(x)$ of spheres with volume V , which may be smoothed if necessary. Then C is defined on vacua $\Omega \in \mathcal{H}_0 \cap \mathcal{D}$ as the limit

$$\begin{aligned}
 \lim_V \int d\vec{x} \chi_V(\vec{x}) A_f(\vec{x}) \Omega &= \\
 &= \lim_V \int d\vec{x} \chi_V(\vec{x}) U(-\vec{x}) A(f) \Omega = \\
 &= E(0) A(f) \Omega = \\
 &= (2\pi)^{3/2} \tilde{f}(0) \mathbb{C} \Omega
 \end{aligned} \tag{2.8}$$

$E(0)$ projection on $\mathcal{H}_0$.

As a consequence of locality, expressed by Eq. (1.3) or (2.1'), we have for $f_i(x) = f_i''(\vec{x}) f_i'(x_0)$, $f_i'(x_0)$ with compact support :

$$\| [A(f_i), A_f(\vec{x})] \phi \| \rightarrow 0 \quad \text{for} \quad |\vec{x}| \rightarrow \infty$$

stronger than any power in $|\vec{x}|^{-1}$, and therefore

$$\begin{aligned}
 \lim_V \int d\vec{x} \chi_V(\vec{x}) [A_f(\vec{x}), A(f_1)] \phi &= 0 \\
 \text{for } \phi \in \mathcal{D}
 \end{aligned} \tag{2.9}$$

With help of this property, it is easy to show that $\mathbb{C}$ can be defined as strong limit on states $\phi = A(f_1) \dots A(f_n) \Omega$, $\Omega \in \mathcal{D} \cap \mathcal{H}$. We get

$$\begin{aligned}
 s\text{-}\lim_V \int d\vec{x} \chi_V(\vec{x}) A_f(\vec{x}) A(f_1) \dots A(f_n) \Omega &= \\
 &= (2\pi)^{3/2} \tilde{f}(0) \mathbb{C} A(f_1) \dots A(f_n) \Omega = \\
 &= (2\pi)^{3/2} \tilde{f}(0) A(f_1) \dots A(f_n) \mathbb{C} \Omega
 \end{aligned} \tag{2.10}$$

It follows from the wave equation (2.1) that compactness in x_0 of the support of the f_i means no restriction on the states ϕ . Equations (2.8) and (2.10) give a precise meaning to the "mean" in (2.7) and define $\mathbb{C}$ on a dense set of $\mathcal{H}$.

The following properties of C are contained in this derivation :

- i) $[C, A(f)] = 0$ from Eq. (2.9)
- ii) $[C, U(\Lambda, a)] = 0$ from Eq. (2.10)
- iii) $U^{-1}(c) C U(c) = C + c$ from Eq. (2.1", 7)
- iv) $C \mathcal{D} \cap \mathcal{H}_0 \subset \mathcal{H}_0$ from Eq. (2.8) (2.11)

Equation iii) states that the generator Q of $U(c) = e^{-iQc}$ and C satisfy canonical commutation relations

$$[Q, C] = 1 \quad (2.12)$$

Since Q and C leave $\mathcal{H}_0$ invariant, standard procedures of quantum mechanics allow the introduction of an (improper) complete base of $\mathcal{H}_0$ ($\equiv \widehat{\mathcal{H}}_0$), in which C is diagonal :

$$C |c\rangle = c |c\rangle, \quad \langle c | c'\rangle = \delta(c - c') \quad (2.13)$$

$$\Omega \in \mathcal{H}_0 : \quad \Omega = \int_{-\infty}^{+\infty} G(c) |c\rangle dc \quad \text{iff}$$

$$\|\Omega\|^2 = \int_{-\infty}^{+\infty} dc |G(c)|^2 < \infty \quad (2.14)$$

The base $|c\rangle$ transforms under $U(c)$ like

$$U(c') |c\rangle = |c + c'\rangle \quad (2.15)$$

if the relative phases of the vacua are chosen appropriately. It is this base which allows a very simple discussion of the covariant free fields.

2.3. - Vacuum expectation values

The structure of the covariant field $A(x)$ can be exhibited with help of their vacuum expectation values (VEV) :

$$(\Omega_1, A(x_1) \cdots A(x_n) \Omega_2) = \iint dc dc' G_1^*(c) G_2(c') \cdot \langle c | A(x_1) \cdots A(x_n) | c' \rangle \quad (2.16)$$

$$\Omega_i = \int dc G_i(c) |c\rangle \in \mathcal{H}_0$$

Since in the base $\{|c\rangle\}$ the central element C is diagonal, it follows from Eq. (2.11,i), i.e., $[C, A(x)] = 0$:

$$\langle c | A(x_1) \cdots A(x_n) | c' \rangle = W_c^n(x_1, \dots, x_n) \delta(c - c') \quad (2.17)$$

These generalized "Wightman functions" $W_c^n(x_1, \dots, x_n)$ have to be considered as tempered distributions in x_i according to GFT 29), and also with respect to their c dependence, because of the possible restrictions of the domain of $A(f_1) \dots A(f_n)$. We mention the distribution theoretic aspects of our formal manipulations only, when this seems to be necessary.

Now the usual procedure gives the symmetry properties of the Wightman functions related to the gauge group

$$\begin{aligned} W_{c+d}^n(x_1, \dots, x_n) \delta(c - c') &= \langle c | U(-d) A(x_1) \cdots A(x_n) U(d) | c' \rangle = \\ &= \langle c | (A(x_1) + d) \cdots (A(x_n) + d) | c' \rangle = \\ &= \delta(c - c') \left\{ W_c^n(x_1, \dots, x_n) + d \sum_r W_c^{n-1}(x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_n) + \right. \\ &\quad + d^2 \sum_{r,s} W_c^{n-2}(x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_{s-1}, x_{s+1}, \dots, x_n) + \\ &\quad \left. + \dots + d^n \right\} \quad (2.18) \end{aligned}$$

This formula expresses the W_c^n as function of c with help of the Wightman functions for a given value $c = c_0$, e.g., $c_0 = 0$. We repeat this expression in distribution form, compactly written :

$$W_c^n(f_1, \dots, f_n) = \sum_{v=0}^n (2\pi)^{3v/2} c^v \sum_{\substack{\text{partitions} \\ i_1, \dots, i_n}} \tilde{f}_{i_1}(0) \dots \tilde{f}_{i_v}(0) W_0^{n-v}(f_{i_1}, \dots, f_{i_n}) \quad (2.19)$$

The indices of the arguments of W_0 must have the natural order. The Wightman functions of lowest order are

$$\begin{aligned} W_c^1(x) &= c \\ W_c^2(x_1, x_2) &= W_0^2(x_1 - x_2) + c^2 \end{aligned} \quad (2.20)$$

In our case of a free massless field,

$$\begin{aligned} W_0^2(x) &\text{ satisfies the wave equation } \square W_0^2(x) = 0 ; \\ &\text{ is Lorentz invariant ;} \\ &\text{ has a Fourier transform with support on } \{p | p^2 = 0, p_0 \geq 0\} ; \\ &\text{ has a } R_3\text{-mean, which is } \lim_V 1/V \int_{x_0=0} W_0^2(x) d\vec{x} = c^2|_{c=0} = 0, \\ &\text{ according to (2.8).} \end{aligned}$$

$$W_0^2(x) - W_0^2(-x) = iD(x), \text{ according to Eq. (2.1').}$$

Therefore, we have ³²⁾

$$W_0^2(x) = \mathcal{D}^+(x) = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2|\vec{p}|} e^{-i(|\vec{p}|x_0 - \vec{p} \cdot \vec{x})} \quad (2.21)$$

which is the conventional two-point function of a massless free field. Indeed, if we insert in (2.19) the usual n point function of a massless free field

$$\begin{aligned} W_0^{2n}(x_1, \dots, x_n) &= \sum_{\substack{\text{partitions in pairs} \\ (i_1, i_2) \text{ natural order}}} \mathcal{D}^+(x_{i_1} - x_{i_2}) \dots \mathcal{D}^+(x_{i_{2n-1}} - x_{i_{2n}}) \\ W_0^{2n+1}(x_1, \dots, x_n) &= 0 \end{aligned} \quad (2.22)$$

we get the Wightman functions of a covariant free field. It is one of the remaining problems to show that this describes the only solution to our problem.

2.4. - The appropriate Hilbert space of the solutions of $\square A(x) = 0$

The well-known treatment of free fields ^{7),33)} is based on the construction of a Fock space from a Hilbert space of solutions of the linear field equations. It is crucial for the solution of our problem to use in this approach the right Hilbert space of the solutions of $\square A(x) = 0$. This follows from the discussion of vectors $\psi \in \mathcal{H}_0 \oplus \mathcal{H}_1$ of the type

$$\underline{\Psi} = \iint dx dc F(x, c) A(x) |c\rangle + \int dc G(c) |c\rangle \quad (2.23)$$

The norm $\|\psi\|^2$ is determined by the one- and two-point functions Eqs. (2.20), (2.21), and it is expressed simply with help of the Fourier transform $\tilde{F}(p, c)$, Eq. (2.2) :

$$\begin{aligned} \|\underline{\Psi}\|^2 = & \iint dc dp \left\{ |\tilde{F}(p, c)|^2 \Theta(p_0) \delta(p^2) + \right. \\ & \left. + |(2\pi)^{3/2} \tilde{F}(p, c) c + G(c)|^2 \delta^4(p) \right\} \end{aligned} \quad (2.24)$$

The first term is the conventional expression, the second one, which introduces a discrete part of the measure in momentum space at $p=0$, is a consequence of the c^2 term of $W_c^2(x_1, x_2)$, Eq. (2.20), and is typical for a covariant two-point function. We see from this expression for the norm, that the $\psi \in \mathcal{H}_0 \oplus \mathcal{H}_1$ are determined by $G(c)$ and the restriction of $\tilde{F}(p, c)$ to the forward momentum cone $\overline{\mathcal{P}}^+ = \{p | p^2 = 0, p_0 \geq 0\}$. Test functions $F(p, c)$ with support γ outside $\overline{\mathcal{P}}^+ : \gamma \cap \overline{\mathcal{P}}^+ = 0$, represent pure vacuum states $\psi \sim \int dc G(c) |c\rangle \in \mathcal{H}_0$. On the other hand, states $\psi \in \mathcal{H}_1$, orthogonal on the vacuum space $\mathcal{H}_0$, have the physical interpretation of single particle states of mass zero. According to the inner product introduced by Eq. (2.24), they must satisfy the following conditions

$$\psi \in \mathcal{H}_1 : \quad G_\psi(c) = 0, \quad \tilde{F}_\psi(0, c) = 0 \quad (2.25)$$

In order to make the structure of $\mathcal{H}_0 \oplus \mathcal{H}_1$ more transparent, we introduce on the momentum cone $\mathcal{P} = \{p | p^2 = 0\}$ the following measure

$$\begin{aligned} d\mu(p) &= d\mu_+(p) + d\mu_0(p) + d\mu_-(p) = \\ &= \left\{ \Theta(p_0) \delta(p^2) + \delta^4(p) + \Theta(-p_0) \delta(p^2) \right\} dp \end{aligned} \quad (2.26)$$

and construct in the usual manner ³⁴⁾ the Hilbert space of functions on $\mathcal{P}$ which are square integrable with respect to this measure :

$$\begin{aligned} \tilde{\varphi}(p) &\in \mathcal{L}_{\mu}^{(2)} : \\ \|\varphi\|^2 &= \int |\tilde{\varphi}(p)|^2 \delta(p^2) dp + |\tilde{\varphi}(0)|^2 < \infty \end{aligned} \quad (2.27)$$

This $\mathcal{L}_{\mu}^{(2)}$ is naturally decomposed in orthogonal subspaces

$$\mathcal{L}_{\mu}^{(2)} = \mathcal{L}_{\mu_+}^{(2)} \oplus \mathcal{L}_{\mu_0}^{(2)} \oplus \mathcal{L}_{\mu_-}^{(2)}$$

with

$$\tilde{\varphi} \in \mathcal{L}_{\mu_0}^{(2)} : \quad \tilde{\varphi} = \alpha e_0 \quad e_0 = \begin{cases} 1 & \text{for } p=0 \\ 0 & \text{for } p \neq 0 \end{cases}$$

$$\begin{aligned} \tilde{\varphi} \in \mathcal{L}_{\mu_+}^{(2)} : \quad (\tilde{\varphi}, e_0) &= 0 \\ \text{support } \tilde{\varphi} &\subset \{p \mid p^2 = 0, p_0 \geq 0\} \end{aligned}$$

$$\begin{aligned} \tilde{\varphi} \in \mathcal{L}_{\mu_-}^{(2)} : \quad (\tilde{\varphi}, e_0) &= 0 \\ \text{support } \tilde{\varphi} &\subset \{p \mid p^2 = 0, p_0 \leq 0\} \end{aligned} \quad (2.28)$$

The functions on $\mathcal{P}$ are Fourier transforms of solutions of $\square \varphi(x) = 0$. Parametrizing $p \in \mathcal{P}$ by $\vec{p}$ and the energy sign, the Fourier transformation of $\tilde{\varphi} \in \mathcal{L}_{\mu}^{(2)}$ has the form

$$\begin{aligned} \varphi(\vec{x}, x_0) &= (2\pi)^{-3/2} \int \frac{d\vec{p}}{2|\vec{p}|} \left(\tilde{\varphi}^-(\vec{p}) e^{i(|\vec{p}|x_0 - \vec{p} \cdot \vec{x})} + \right. \\ &\quad \left. + \tilde{\varphi}^+(\vec{p}) e^{-i(|\vec{p}|x_0 - \vec{p} \cdot \vec{x})} \right) + \tilde{\varphi}(0) \end{aligned} \quad (2.29)$$

and the inversion

$$\tilde{\varphi}(0) = \lim_V \frac{1}{V} \int_V \varphi(\vec{x}, x_0) d\vec{x} \quad (2.30')$$

$$\tilde{\varphi}^{\pm}(\vec{p}) = \pm i (2\pi)^{-3/2} \int e^{\pm i \vec{p} \cdot \vec{x}} \overleftrightarrow{\partial}_0 (\varphi(\vec{x}, x_0) - \tilde{\varphi}(0)) d\vec{x} \quad (2.30'')$$

with

$$p = (|\vec{p}|, \vec{p})$$

$$\varphi \overleftrightarrow{\partial}_0 \chi = \varphi(x) \frac{\partial}{\partial x_0} \chi(x) - \left(\frac{\partial}{\partial x_0} \varphi(x) \right) \chi(x)$$

As a consequence of $\square \varphi(x) = 0$, $\varphi^\pm(\vec{p})$ is independent of x_0 . The inner product related to the norm Eq. (2.26) reads in configuration space

$$\begin{aligned} (\varphi, \chi) &= \pm i \int (\varphi^*(\vec{x}, x_0) - \tilde{\varphi}(0)) \overleftrightarrow{\partial}_0 (\chi(\vec{x}, x_0) - \tilde{\chi}(0)) d\vec{x} + \\ &+ \tilde{\varphi}^*(0) \tilde{\chi}(0) \\ \tilde{\varphi}, \tilde{\chi} &\in \mathcal{L}_{\mu_\pm} \oplus \mathcal{L}_{\mu_0} \end{aligned} \quad (2.31)$$

This Hilbert space of solutions of the wave equation, in which the constant solution is normalizable, allows "second quantization" of $\square A(x) = 0$. The "c components" of single particle states $\psi = \iint dx dc F(x, c) A(x) |c\rangle$ with $\tilde{F}(0, c) = 0$ [Eq. (2.25)] can be naturally identified with elements of $\mathcal{L}_{\mu_+}^{(2)}$, whereas the elements of $\mathcal{L}_{\mu_0}$ are related to the vacuum components. The Fourier decomposition of $A(x)$, according to (2.30), leads to the introduction of creation and annihilation operators, which allows a complete characterization of the free covariant operator fields. We shall formulate this in the following section.

2.5. - The main theorems on free fields

Let $A(x)$ be a free, massless field covariant under scalar gauge transformations, as defined by Eq. (2.1), which is represented by operators in a separable Hilbert space $\mathcal{H}$ in such a way that the general postulates of GFT (Section 2.1, I-III) are satisfied and that the system of operators $I\mathcal{A}$, generated by $A(f)$ and $U(c) = e^{-iQc}$, is irreducible. Then we consider the Fourier transformation of $A(x)$ as defined for solutions of the wave equation in Eqs. (2.29), (2.30) :

$$\begin{aligned} A(x) &= (2\pi)^{-3/2} \int d^4p \tilde{A}(p) e^{-ipx} = \\ &= (2\pi)^{-3/2} \int \frac{d\vec{p}}{2|\vec{p}|} \left(a(\vec{p}) e^{-i(|\vec{p}|x_0 - \vec{p} \cdot \vec{x})} + a^\dagger(\vec{p}) e^{i(|\vec{p}|x_0 - \vec{p} \cdot \vec{x})} \right) + \\ &+ A_0 \end{aligned} \quad (2.32)$$

or

$$A_0 = \lim_V \frac{1}{V} \int_V d\vec{x} A(\vec{x}, x_0) \quad (2.33')$$

$$a(\vec{p}) = i (2\pi)^{-3/2} \int d\vec{x} e^{i\vec{p}\vec{x}} \overleftrightarrow{\partial}_0 (A(\vec{x}, x_0) - A_0)$$

$$a^+(\vec{p}) = -i (2\pi)^{-3/2} \int d\vec{x} e^{-i\vec{p}\vec{x}} \overleftrightarrow{\partial}_0 (A(\vec{x}, x_0) - A_0) \quad (2.33'')$$

From Eq. (2.1), the commutation relations

$$[a(\vec{p}), a^+(\vec{p}')] = 2|\vec{p}| \delta(\vec{p} - \vec{p}'),$$

$$[a(\vec{p}), a(\vec{p}')] = [a^+(\vec{p}), a^+(\vec{p}')] = 0 \quad (2.34')$$

$$i [A_0, Q] = -1,$$

$$[a(\vec{p}), Q] = [a^+(\vec{p}), Q] = [a(\vec{p}), A_0] = [a^+(\vec{p}), A_0] = 0 \quad (2.34'')$$

follow by formal calculations. Since $A(x)$, or $a(\vec{p})$, $a^+(\vec{p})$, A_0 , respectively, are operator valued distributions, we have to explain the meaning of the expressions Eq. (2.33). We discussed already the precise definition of Eq. (2.33') in Section 2.2 [Eqs. (2.7), (2.8), (2.10)]. Since the assumption $\hat{\mathcal{H}}_0 = \mathcal{H}_0$ [Eq. (2.6)] follows from the irreducibility of $\mathcal{IQ}$, we conclude

$$A_0 = C \quad (2.35)$$

where C is the unbounded operator defined by Eqs. (2.8), (2.10) with the properties (2.11,i,iii) corresponding to Eq. (2.34''). On the other hand, the Fourier transformation, Eqs. (2.29) or (2.33''), defines the creation and annihilation operators $a(\vec{p})$, $a^+(\vec{p})$ with respect to the improper momentum base in $\mathcal{L}_{\mu^+}^{(2)}$, $\mathcal{L}_{\mu^-}^{(2)}$, respectively. Following the usual procedure we introduce in $\mathcal{L}_{\mu^+}^{(2)}$ a complete orthonormal system $\{\tilde{f}_\alpha(\vec{p})\}$ of sufficiently smooth functions

$$\begin{aligned} \int \frac{d\vec{p}}{2|\vec{p}|} \tilde{f}_\alpha^*(\vec{p}) \tilde{f}_\beta(\vec{p}) &= \delta_{\alpha\beta} \\ \sum_\alpha \tilde{f}_\alpha(\vec{p}) \tilde{f}_\alpha^*(\vec{p}') &= 2|\vec{p}| \delta(\vec{p} - \vec{p}') \end{aligned} \quad (2.36)$$

satisfying the condition

$$(\tilde{f}_\alpha, e_0) = \tilde{f}_\alpha(\vec{0}) = 0 \quad (2.37)$$

characteristic for $\tilde{f} \in \mathcal{L}_+^{(2)}$. Equations (2.34') therefore should be considered as equivalent to

$$a(f_\alpha) = \int \frac{d\vec{p}}{2|\vec{p}|} a(\vec{p}) \tilde{f}_\alpha(\vec{p})$$

$$a^+(f_\alpha) = \int \frac{d\vec{p}}{2|\vec{p}|} a^+(\vec{p}) \tilde{f}_\alpha^*(\vec{p})$$

$$[a(f_\alpha), a^+(f_\beta)] = \delta_{\alpha\beta} \quad (2.38)$$

Now we sum up our main results in the form of theorems.

THEOREM 1 - A free massless relativistic field $A(x)$ covariant under scalar gauge transformations, satisfying our general assumptions, is equivalent to the uniquely determined irreducible representations of the canonical commutation relations Eq. (2.34) with the vacuum subspace $\mathcal{H}_0$:

$$a(f) \mathcal{H}_0 = 0 \quad Q \mathcal{H}_0 \subset \mathcal{H}_0 \quad C \mathcal{H}_0 \subset \mathcal{H}_0 \quad (2.39)$$

Proof

An irreducible representation of the commutation relations Eq. (2.34) which in addition satisfy the condition Eq. (2.39), is up to equivalence uniquely defined : in a C representation of $\mathcal{H}_0$ according to Eq. (2.13), we have $a(\vec{p})|c\rangle = 0$. Thus, for each c the $a(\vec{p})$, $a^+(\vec{p})$ generate a unique Fock representation ⁷⁾ with vacuum $|c\rangle$.

On the other hand, given a covariant field $A(x)$ satisfying our assumptions, we define C and Q according to (2.7) and $U(c) = e^{-iQc}$ and $a(\vec{p})$, $a^+(\vec{p})$ according to (2.33'') or (2.38) which satisfy the commutation relations Eq. (2.34). We already have shown $C\mathcal{H}_0 \subset \mathcal{H}_0$ in Eq. (2.11,iv)

and $Q\mathcal{H}_0 \subset \mathcal{H}_0$ in Eq. (2.15). The remaining condition $a(f)\mathcal{H}_0 = 0$ follows from $\|a(f_\alpha)\Omega\|^2 = (2\pi)^3 \int |F(c)|^2 c^2 dc$. $|\tilde{f}_\alpha(\vec{0})|^2 = 0$ because of Eq. (2.37) calculated with help of Eq. (2.24), $\Omega = \int F(c)|c > dc$. This theorem settles also the problem of the Wightman functions of covariant free fields. The direct calculation, following the usual line ⁷⁾ leads to the statement :

COROLLARY - The Wightman functions of the covariant free fields are uniquely given by Eqs. (2.19), (2.22).

The second theorem summarizes the general structure of the fields $\{A(f)\}$ defined in Section 2.1. It is also a direct consequence of Theorem 1 and the definitions given above.

THEOREM 2 - The irreducible representation of the algebra generated by $\{A(x), U(c)\}$ restricted to the fields $A(f)$ allows a factor decomposition with respect to the centre $\{C\}$:

$$\mathcal{H} = \int \mathcal{H}^c dc, \quad C \mathcal{H}^c = \mathcal{H}^c$$

where $\mathcal{H}^c$ is the Fock space over $\mathcal{L}_{\mu^+}^2$. The action of $A(f)$ in $\mathcal{H}^c$ looks like

$$A(f) = a(\tilde{f}_+) + a^+(\tilde{f}_-) + (2\pi)^{3/2} c \tilde{f}(0) \quad (2.40)$$

$\tilde{f}_\pm$ being the restriction of the test functions to

$$\mathcal{P}^\pm = \{p \mid p^2 = 0, p_0 \gtrless 0\}$$

The gauge group acts transitively on the family of Hilbert spaces

$$e^{-iQc'} \mathcal{H}^c = \mathcal{H}^{c+c'}$$

We want to elaborate a little bit further on the properties and the physical interpretation of the vectors of the Hilbert space $\mathcal{H}$. First we remark that in spite of the reducible representation of the field, there are cyclic vectors, in particular cyclic vacua. Because of Eq. (2.38), the cyclic vacua have the form $\int G(c) |c\rangle dc$ with $G(c) \neq 0$ and decreasing faster than any power of c .

The covariant description of SSB implies some modifications of the particle interpretation of the quantized field obeying the linear field equation. As already mentioned in Section 2.4, we have to define the one-particle states as normalizable eigenstates of the mass operator orthogonal on $\mathcal{H}_0$. The one-particle states have mass zero and look like

$$a^+(f_\alpha) |\Omega\rangle = A(f_\alpha^*) |\Omega\rangle = \int dc \frac{d\vec{p}}{2|\vec{p}|} G(c) \tilde{f}_\alpha^*(\vec{p}) |\vec{p}, c\rangle$$

$$|\Omega\rangle = \int G(c) |c\rangle dc \in \mathcal{H}_0$$

$$f_\alpha \in L_{\mu_+}^{(2)}$$

with the improper base being normalized as

$$\langle \vec{p}, c | \vec{p}', c' \rangle = 2|\vec{p}| \delta(\vec{p} - \vec{p}') \delta(c - c')$$

The degeneracy of the vacuum carries over to the one-particle states. It is related to a kind of superselection rule¹⁶⁾ because all observables generated from $A(f)$ commute with C and therefore do not mix the subspaces $\mathcal{H}^C$. Furthermore, the S matrix commutes with Q too, which implies that the description of the scattering processes of these particles is identical in each factor space $\mathcal{H}^C$. This will be shown in Section 3.

With help of the creation operators $a^+(f_\alpha)$ we can construct as usual a complete orthonormal base in $\mathcal{H}$:

$$\begin{aligned} |f_{\alpha_1}, \dots, f_{\alpha_n}; c\rangle &= N(\alpha_i) a^+(f_{\alpha_1}) \dots a^+(f_{\alpha_n}) |c\rangle = \\ &= N \cdot \int \dots \int \frac{d\vec{p}_1}{2|\vec{p}_1|} \dots \frac{d\vec{p}_n}{2|\vec{p}_n|} \tilde{f}_{\alpha_1}(\vec{p}_1) \dots \tilde{f}_{\alpha_n}(\vec{p}_n) |\vec{p}_1, \dots, \vec{p}_n; c\rangle \end{aligned} \quad (2.41)$$

$$n = 0, 1, 2, \dots$$

and with

$$\begin{aligned}
 \langle f_{\alpha_1}, \dots, f_{\alpha_n}; c \mid f_{\alpha'_1}, \dots, f_{\alpha'_m}; c' \rangle &= \\
 &= \delta_{nm} \delta(c-c') N^2(\alpha_i) \sum_{\pi} \delta_{\alpha_1 \alpha'_{i_1}} \dots \delta_{\alpha_n \alpha'_{i_n}} \\
 \langle \vec{p}_1, \dots, \vec{p}_n; c \mid \vec{p}'_1, \dots, \vec{p}'_m; c' \rangle &= \\
 &= \delta_{nm} \delta(c-c') 2|\vec{p}'_1| \dots 2|\vec{p}'_n| \sum_{\pi} \delta^3(\vec{p}_1 - \vec{p}'_{i_1}) \dots \delta^3(\vec{p}_n - \vec{p}'_{i_n})
 \end{aligned}
 \tag{2.42}$$

$$N(\alpha_1, \dots, \alpha_n) = n_1! \dots n_r! \quad \text{for } n_i \text{ equal indices } \alpha_k.$$

In spite of the degeneracy in c , these states can be interpreted as n particle states, since the degeneracy plays the same rôle as discussed above. From the considerations of the multi-particle states, it is evident that the infinite degeneracy of the one-particle states does not describe multiplets of mass zero particle states, because the composition of multiplets would require the tensor product also with respect to the label c of the multiplet members.

2.6. - Wick ordering. Haag's expansion

Particle properties and reaction amplitudes are described by matrix elements of the scattering operator, currents, etc., in the many-particle base corresponding to Eq. (2.41). The well-known techniques like Wick ordering ³⁵⁾ and Haag's expansion ³⁶⁾ - developed for the evaluation of such matrix elements - need some slight modifications in our formalism.

In the definition of Wick ordering, we have to take into consideration the term proportional to C in

$$\tilde{A}(p) = \tilde{A}_{\perp}(p) + (2\pi)^{3/2} \delta^4(p) C = \bar{A}(p) \delta(p^2) + (2\pi)^{3/2} \delta^4(p) C$$

with

$$\bar{A}(p) = \Theta(p_0) a(\vec{p}) + \Theta(-p_0) a^*(-\vec{p})$$

(2.43)

according to Eq. (2.32). Therefore, we define

$$\begin{aligned}
 & : A(f_1) \cdots A(f_n) : = \\
 & = \sum_{\substack{\nu=0 \\ \text{partitions of} \\ i_\nu = 1, 2, \dots, n}}^n (2\pi)^{3\nu/2} \tilde{f}_{i_1}(0) \cdots \tilde{f}_{i_\nu}(0) \mathcal{C} : \tilde{A}_\perp(\tilde{f}_{i_{\nu+1}}) \cdots \tilde{A}_\perp(\tilde{f}_{i_n}) : \quad (2.44)
 \end{aligned}$$

with the usual convention of Wick ordering of the $\tilde{A}_\perp(\tilde{f})$ on the right-hand side.

Now we consider Haag's expansion of "observables", i.e., linear operators L in $\mathcal{A}$ with $[\bar{L}, \bar{C}] = 0$

$$L = \sum_{n=0}^{\infty} \frac{1}{n!} \iint dp_1 \cdots dp_n F_n(p_1, \dots, p_n) : \tilde{A}(p_1) \cdots \tilde{A}(p_n) : \quad (2.45)$$

As usual this infinite sum is meant as weak limit of matrix elements with respect to the multiparticle states. Whereas under conventional conditions only a finite number of terms contribute to matrix elements between states with finite number of particles, in our case the situation is more involved. Because of the $\mathcal{C}$ terms hidden in $: \tilde{A}(p_1) \cdots \tilde{A}(p_n) :$, F_n with arbitrary n contribute to such matrix elements. Equation (2.44) leads to

$$\begin{aligned}
 & \langle p_1, \dots, p_m; c | L | p_{m+1}, \dots, p_n; c' \rangle = \\
 & = \sum_{\nu=0}^n \frac{1}{\nu!} \iint dq_1 \cdots dq_\nu G_\nu(q_1, \dots, q_\nu) \cdot \\
 & \quad \cdot \langle p_1, \dots, p_m; c | : \tilde{A}_\perp(q_1) \cdots \tilde{A}_\perp(q_\nu) : | p_{m+1}, \dots, p_n; c' \rangle
 \end{aligned} \quad (2.46')$$

with

$$\begin{aligned}
 & G_\nu(q_1, \dots, q_\nu) = \\
 & = \sum_{k=0}^{\infty} \frac{1}{k!} (2\pi)^{3k/2} F_{\nu+k}(q_1, \dots, q_\nu, 0, \dots, 0) c^k \quad (2.46'')
 \end{aligned}$$

The convergence of these sums is crucial for the existence of Haag's expansion. In the applications the convergence follows from the transformation properties of L with respect to the gauge group.

It is known that the matrix elements of L determine the coefficients $G_v(q_1, \dots, q_v)$ of the conventional Haag expansion (2.46') uniquely for $q_i^2 = 0$, $q_i \neq 0$ (as usual up to points of measure zero). We have to show that also the coefficients $F_n(p_1, \dots, p_n)$ of the generalized Haag expansion (2.45) are uniquely determined for points $p_i^2 = 0$, including the special points with non-vanishing measure $p_i = 0$. But this follows from the uniqueness of G , because the F 's are coefficients of a power series expansion of G in c according to Eq. (2.46"). Since we used an improper vacuum base, the derivation is somewhat formal. A more rigorous derivation can be given.

Now we consider gauge invariant observables L , i.e., those with $[L, Q] = 0$. Their expansion coefficients have the characteristic property

$$F_n(p_1, \dots, p_i = 0, \dots, p_n) = 0, \quad p_j^2 = 0 \quad (2.47)$$

as a consequence of the uniqueness of Haag's expansion and

$$[L, Q] = (2\pi)^{3/2} \sum_{n=0}^{\infty} \frac{1}{n!} \int \dots \int dp_1 \dots dp_n F_{n+1}(p_1, \dots, p_n, 0) : \tilde{A}(p_1) \dots \tilde{A}(p_n) : = 0$$

This implies for the infinite sum in Eq. (2.46") that only the first term is different from zero. Hence, for invariant operators there is no convergence problem. These zeros of F_n have physical consequences when the F_n have analytic properties which often follow from locality.

It is worth while to mention that for $p_i^2 = 0$, $p_i \neq 0$, the well-known expression for the coefficients of Haag's expansion remains valid :

$$F_n(p_1, \dots, p_n) = \varepsilon(p_1) \dots \varepsilon(p_n) \langle \Omega | [\dots [L, \bar{A}(-p_1)], \dots, \bar{A}(-p_n)] | \Omega \rangle \quad (2.48)$$

with $\bar{A}$ defined in Eq. (2.43). This follows from the calculation of the matrix elements of L ³⁷⁾ in the multiparticle base and the fact that matrix elements of invariant operators are independent of c :

$$\begin{aligned} \langle p_i; c | L | p_k^i; c' \rangle &= \langle p_i; c | U(-d) L U(d) | p_k^i; c' \rangle = \\ &= \langle p_i; c+d | L | p_k^i; c'+d \rangle \end{aligned}$$

Covariant operators with the transformation property $[L, Q] = i$ may be treated along the same line by considering the invariant operator $L-A(x)$.

2.7. - Relation between the generator Q of the gauge symmetry and the conserved current

In the case of a quantum mechanical symmetry, the space integral of the "charge density" $j_0(x)$ of a conserved current $j_\lambda(x)$, $\partial_\lambda j_\lambda(x) = 0$, converges weakly to the generator of the symmetry, i.e., the "global charge" Q ¹¹⁾:

$$\lim_{V \rightarrow \mathbb{R}_3} \int_V d\vec{x} j_0(\vec{x}, t) = Q \quad (2.49)$$

The scalar gauge symmetry is related by the Noether theorem to the conserved current

$$j_\mu(x) = \partial_\mu A(x) \quad (2.50)$$

In the following we want to derive a relation similar to Eq. (2.49) between the charge integral of this current and the generator Q of the scalar gauge group as defined for our covariant free massless field. Since for the usual irreducible fields such a symmetry generator Q does not exist ³⁸⁾, the impossibility to give a meaning to the charge integral is often considered as characteristic for a spontaneously broken symmetry. For this reason the discussion of the charge integral is illuminating for the comparison of our treatment of SSB based on covariant fields to the conventional one based on irreducible fields.

In order to give the expression Eq. (2.49) a precise meaning we follow the usual procedure ¹¹⁾ and introduce functions $g_r(x) = v_r(|\vec{x}|)\tau(x_0)$ with the properties

$$\begin{aligned}
 (a) \quad & \psi_r(|\vec{x}|) = 1 \quad \text{for } |\vec{x}| \leq r, \quad \psi_r(|\vec{x}|) = 0 \quad \text{for } |\vec{x}| \geq r+d \\
 (b) \quad & \psi_r(y) = \mathcal{L}^\infty \quad \text{for } y \geq 0 \\
 (c) \quad & \int \tau(x_0) dx_0 = 1, \quad \tau(x_0) \geq 0 \\
 (d) \quad & \text{support}(\tau(x_0)) : \text{compact}, \quad \tau(x_0) : \mathcal{L}^\infty
 \end{aligned} \tag{2.51}$$

Then we consider as definition of the charge integral the limit of the smeared field operator Q_r , $r \rightarrow \infty$:

$$Q_r = \int j_0(x) g_r(x) dx = A(-\partial_0 g_r) \tag{2.52}$$

Because of (2.51c) and $\partial_\mu j_\mu(x) = 0$, the time averaging in Eq. (2.51) is formally irrelevant.

We continue with the known remark ^{10),11)} on the existence of $\lim_{r \rightarrow \infty} [Q_r, A(f)]$ for $f \in \mathcal{F}$. In our special example of spontaneously broken symmetry, we get from Eqs. (2.1'), (2.3) :

$$[Q_r, A(f)] = -i(\partial_0 g_r, f) \xrightarrow{r \rightarrow \infty} -i(2\pi)^{3/2} \tilde{f}(0) \tag{2.53}$$

This agrees with $[Q, A(f)]$ according to Eqs. (1.5) or (2.1''), therefore we have

$$\lim_{r \rightarrow \infty} [Q_r, A(f)] = [Q, A(f)] \tag{2.54}$$

Now we consider the limit of matrix elements of Q_r between states $A\Omega = A(f_1) \dots A(f_n)\Omega$, $\Omega = \int dc F(c) |c\rangle \in \mathcal{H}_0$, $F(c) \in \mathcal{F}$, and compare it with matrix elements of Q . From Eqs. (2.53), (2.54) it follows directly that

$$\begin{aligned}
 & \lim_{r \rightarrow \infty} \langle \Omega | A^\dagger Q_r A | \Omega \rangle - \langle \Omega | A^\dagger Q A | \Omega \rangle = \\
 & = \lim_{r \rightarrow \infty} \langle \Omega | A^\dagger A' Q_r | \Omega \rangle - \langle \Omega | A^\dagger A' Q | \Omega \rangle
 \end{aligned} \tag{2.55}$$

Therefore we have to calculate only matrix elements $\langle \Omega | \mathcal{A} Q | \Omega \rangle$, and $\lim_r \langle \Omega | \mathcal{A} Q_r | \Omega \rangle$. These follow immediately from Eq. (2.15), (2.17) :

$$\begin{aligned} \langle \Omega | A(f_n^*) \cdots A(f_1^*) Q | \Omega \rangle &= \\ &= -i \int dc \, W_c^n(f_n^*, \dots, f_1^*) F^*(c) \frac{d}{dc} F(c) \end{aligned} \quad (2.56)$$

and from Eqs. (2.19), (2.22), together with

$$\lim_{r \rightarrow \infty} \iint dx \, dx' \, f^*(x) D^+(x-x') \partial_0 g_r(x') = \frac{1}{2} (2\pi)^{3/2} \tilde{f}(0) :$$

$$\begin{aligned} \lim_{r \rightarrow \infty} \langle \Omega | A(f_n^*) \cdots A(f_1^*) Q_r | \Omega \rangle &= \\ &= \lim_{r \rightarrow \infty} \langle \Omega | A(f_n^*) \cdots A(f_1^*) A(-\partial_0 g_r) | \Omega \rangle = \\ &= \frac{i}{2} \int dc \left(\frac{d}{dc} W_c^n(f_n^*, \dots, f_1^*) \right) F^*(c) F(c) \end{aligned} \quad (2.57)$$

For real $F(c) : F^*(c) = F(c)$, the right-hand sides of Eqs. (2.56) and (2.57) are equal. This allows us to state the following.

THEOREM 3 - Let Ω be a cyclic vacuum vector : $\Omega = \int dc F(c) |c\rangle$, $F(c) \in \mathcal{Y}$, $F(c) \neq 0$, with the conjugation property $F^*(c) = F(c)$ relative to the phase convention of the vacuum base $e^{-iQd} |c\rangle = |c+d\rangle$ and let $\mathcal{D}_j \subset \mathcal{H}$ be the dense set generated from Ω by the polynomials $\text{POL}[A(f_i)]\Omega$ of the quasi-local field operators $A(f)$, $f \in \mathcal{Y}$. Then the charge integral converges weakly on $\mathcal{D}_j$ towards the symmetry generator Q :

$$\begin{aligned} \lim_{r \rightarrow \infty} (\phi', Q_r \phi) &= (\phi', Q \phi) \\ \phi, \phi' &\in \mathcal{D}_j \end{aligned}$$

3. - REMARKS ON INTERACTING FIELDS AND SCATTERING THEORY

3.1. - Interacting fields with covariant scalar gauge symmetry

Now we study the covariant description of spontaneously broken symmetry for fields with a non-trivial S matrix. For that we follow the general line of GFT. We assume the existence of a field operator $A(x)$ satisfying the postulates (I)-(III) of Section 2.1. Instead of the specific commutation relations (2.1') of free fields we postulate

$$(IV) \text{ Locality : } [A(x), A(x')] = 0 \quad \text{for} \quad (x-x')^2 < 0$$

We assume no special field equation, but the field algebra generated by $\{A(f)\}$ should allow the same spontaneously broken symmetry group as the free fields. Therefore in our covariant description we postulate

$$U^{-1}(c) A(f) U(c) = A(f) + (2\pi)^{3/2} \tilde{f}(0) \\ [U(c), U(\lambda, a)] = 0 \quad (3.1)$$

Having in mind the possibility of dynamical rearrangement of symmetry, this means an essential simplification. Nevertheless, already this simple model reveals some features typical for S matrix theories with SSB. The scalar gauge symmetry may be related to a local conserved current $j_\mu(x)$:

$$U(c) j_\mu(x) U^{-1}(c) = j_\mu(x) \\ \text{and} \quad \partial_\mu j_\mu(x) = 0 \quad (3.1')$$

The charge integral formally generates the gauge symmetry. The precise definition of this charge integral raises questions similar to those already discussed in Section 2.7 for the free field.

Since the interacting and the free fields are subject to the same postulates of GFT, some general features of $A(f)$ can be derived as for the free fields in Section 2.

- i) There is an infinite dimensional vacuum subspace $\mathcal{H}_0$.
- ii) The unbounded operator

$$C = \lim_{V \rightarrow \mathbb{R}_3} \frac{1}{V} \int_V d\vec{x} A(\vec{x}, x_0) \quad (3.2)$$

can be defined according to the procedure following Eq. (2.7), and it has the properties (2.11). Whereas in the case of the free field we could make use of the wave equation in order to show that the states $\emptyset = A(f_1), \dots, A(f_n)\Omega$ are dense in $\mathcal{H}$, even if all $f_i(x_0)$ have compact support with a common bound, we have to assume this property for interacting fields ("Zeitschicht axiom" ³⁹⁾).

iii) There is an improper base in $\mathcal{H}_0$ related to C :

$$\Omega = \int G(c) |c\rangle dc \in \mathcal{H}_0$$

$$C |c\rangle = c |c\rangle, \quad U(c') |c\rangle = |c+c'\rangle \quad (3.3)$$

as in Eqs. (2.13)-(2.15). Related to this base in $\mathcal{H}_0$ there is a factor decomposition of the Hilbert space $\mathcal{H}$:

$$\mathcal{H} = \int dc \mathcal{H}^c, \quad A(f_1) \cdots A(f_n) |c\rangle \in \mathcal{H}^c \quad (3.4)$$

iv) From the covariance of the fields follows the symmetry property of the Wightman functions $W_c^n(x_1, \dots, x_n)$, defined according to Eq. (2.17) as :

$$(\Omega_1, A(x_1) \cdots A(x_n) \Omega_2) = \int dc G_1^*(c) G_2(c) W_c^n(x_1, \dots, x_n) \quad (3.5)$$

and symmetric according to Eqs. (2.18), (2.19)

$$W_c^n(f_1, \dots, f_n) =$$

$$= \sum_{v=0}^n (2\pi)^{3v/2} c^v \sum_{\substack{\text{partitions} \\ i_1, \dots, i_n}} \tilde{f}_{i_1}(\theta) \cdots \tilde{f}_{i_n}(\theta) W_0^{n-v}(f_{i_1}, \dots, f_{i_n}) \quad (3.6)$$

v) The clustering property ⁴⁰⁾

$$\langle \Omega, A U(1, \vec{x}) B \Omega \rangle \rightarrow$$

$$\xrightarrow{|\vec{x}| \rightarrow \infty} \langle \Omega | A | \Omega \rangle \langle \Omega | B | \Omega \rangle \quad (3.7)$$

$$A = A(f_1) \cdots A(f_n), \quad B = A(f'_1) \cdots A(f'_m)$$

which plays an important rôle in the Haag-Ruelle scattering theory ⁴¹⁾, is not valid for our fields, because covariant fields are necessarily reducible and it is well known ^{17),31)} that irreducibility and unique clustering are equivalent conditions. However, we have weak clustering for the factors, and therefore

$$\begin{aligned} \lim_{|a| \rightarrow \infty} W_c^{n+m}(x_1+a, \dots, x_n+a, y_1, \dots, y_m) &= \\ &= W_c^n(x_1, \dots, x_n) \cdot W_c^m(y_1, \dots, y_m) \end{aligned} \quad (3.8)$$

This property suggests the introduction of truncated Wightman functions W_T , defined as usual

$$\begin{aligned} W_c^n(x_1, \dots, x_n) &= \\ &= \sum_{\text{partitions}} W_{Tc}^{r_1}(x_{k_1,1}, \dots, x_{k_1,r_1}) \cdots W_{Tc}^{r_s}(x_{k_s,1}, \dots, x_{k_s,r_s}) \end{aligned} \quad (3.9)$$

with

$$\begin{aligned} W_c^0 &= 1 & W_c^1 &= W_{Tc}^1(x_1) = c \\ W_c^2(x_1, x_2) &= W_{Tc}^2(x_1, x_2) + c^2 \end{aligned} \quad (3.10)$$

Inside distinct W_T the order of the space-time points x_i has to be the same as in W on the left-hand side of Eq. (3.9). Of course, in our case the sum of the right-hand side contains factors corresponding to the non-vanishing one-point function, e.g.,

$$\begin{aligned} W_c^3(x_1, x_2, x_3) &= W_{Tc}^3(x_1, x_2, x_3) + \\ &+ c \left(W_{Tc}^2(x_1, x_2) + W_{Tc}^2(x_1, x_3) + W_{Tc}^2(x_2, x_3) \right) + c^3 \end{aligned} \quad (3.11)$$

In our very simple model, the truncated functions W_T are independent of c for $n \neq 1$; the entire c dependence comes from the one-point function. Therefore, the decomposition (3.9) looks similar to Eq. (3.6). Now we proof this.

THEOREM 4 - $W_{Tc}^n(x_1, \dots, x_n)$ is independent of c for $n \neq 1$.

Proof

We assume the validity of the statement for $m = 1, 2, \dots, n-1$.
Differentiation of W_c^n according to (3.9) leads to

$$\begin{aligned} \frac{\partial}{\partial c} W_c^n(x_1, \dots, x_n) &= \frac{\partial}{\partial c} W_{Tc}^n(x_1, \dots, x_n) + \\ &+ \sum_{k=1}^n \sum_{\substack{\text{partitions} \\ i=1,2,\dots,n \\ i \neq k}} W_{Tc}^{r_1}(x_{j_1,1}, \dots, x_{j_1,r_1}) \cdots W_{Tc}^{r_s}(x_{j_s,1}, \dots, x_{j_s,r_s}) \end{aligned}$$

On the other hand, differentiation of (3.6) yields

$$\frac{\partial}{\partial c} W_c^n(x_1, \dots, x_n) = \sum_{k=1}^n W_c^{n-1}(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \quad (3.13)$$

According to the definition (3.9) this sum is equal to the second term on the right-hand side of Eq. (3.13), so that $(\partial/\partial c)W_c^n = 0$.

The unique Wightman functions of the free fields are simply characterized by the truncated functions

$$W_{Tc}^1(x) = c$$

$$W_{Tc}^2(x_1, x_2) = i D^+(x_1 - x_2) \quad (3.14')$$

$$W_{Tc}^n(x_1, \dots, x_n) = 0 \quad \text{for } n > 2 \quad (3.14'')$$

3.2. - The invariant S matrix

The physical interpretation of a general field theory is based on a theorem which assures the existence of a unitary S matrix. One of the essential assumptions of the Haag-Ruelle scattering theory is the existence of one-particle states with masses $m \neq 0$, isolated in the mass spectrum. This assumption contradicts the well-known result, that in a field theory with SSB there exist states with masses arbitrarily close to zero ¹²⁾. Therefore we should be prepared to encounter difficulties in the formulation of the scattering theory. On the other hand, infra-red divergences do not appear in perturbation theory calculations in models showing this type of SSB. It is beyond the scope of this paper to clarify possible relations

between SSB and the existence of the S matrix. We restrict ourselves to the discussion of scattering theory based on the assumption of the LSZ asymptotic condition.

Because of our general postulates on $A(x)$, in particular that the spontaneously broken gauge symmetry is related to a conserved current and that $A(x)$ are operators in a Hilbert space with positive norm, the assumptions of the Goldstone theorem are satisfied ¹²⁾. This theorem assures the existence of massless particle states $|p, c\rangle$ with

$$\langle c | j_\mu(x) | p, c' \rangle = \frac{i F}{(2\pi)^{3/2}} p_\mu e^{-ipx} \delta(c-c') \quad (3.15)$$

We assume that the fields $A(x)$ are interpolating, renormalized, fields of this Goldstone particles, i.e.,

$$\langle c | A(x) | p, c' \rangle = (2\pi)^{-3/2} e^{-ipx} \delta(c-c') \quad (3.16)$$

Now we introduce the local operator ²⁶⁾

$$B_\mu(x) = j_\mu(x) + F \partial_\mu A(x) \quad (3.17)$$

which satisfies

$$\langle c | B_\mu(x) | p, c' \rangle = 0 \quad (3.18)$$

As a consequence of current conservation and definition (3.17), we calculate

$$\square A(x) = F^{-1} \cdot \partial_\mu B_\mu(x) \quad (3.19')$$

$$\square j_\mu(x) = \partial_\nu (\partial_\nu B_\mu - \partial_\mu B_\nu) = i_\mu(x) \quad (3.19'')$$

Now we can formulate the asymptotic condition. We assume that Eqs. (3.19) can be solved by

$$A(x) = A_{in}(x) - \int d\xi D_R(x-\xi) F^{-1} \partial_\mu B_\mu(\xi) \quad (3.20')$$

$$j_\mu(x) = -F \partial_\mu A_{in}(x) - \int d\xi D_R(x-\xi) i_\mu(\xi) \quad (3.20'')$$

with free fields $A_{in}(x)$ satisfying

$$\square A_{in}(x) = 0, \quad [A_{in}(x), A_{in}(x')] = i D(x-x')$$

Similar solutions exist with the advanced Green's function D_A and $A_{out}(x)$, as it follows from CPT. As usual, the integral in (3.20) is understood as weak limit.

Similar to the case of a quantum mechanical symmetry, we will show that the covariance properties of the interacting field $A(x)$ with respect to the spontaneously broken gauge symmetry imply covariance of the asymptotic fields $A_{ex}(x)$ ($ex=in, out$), and invariance of the S matrix.

First we remark that

$$U^{-1}(c) A_{ex} U(c) = A_{ex}(x) + c \quad (3.21)$$

which follows from applying the unitary operator $U(c)$ on (3.20') and using the invariance of $B(x)$ according to Eqs. (3.1'), (3.17). In the same way we get

$$[C, A_{ex}(x)] = 0 \quad (3.22)$$

For reasons of simplicity, we restrict our model further by assuming asymptotic completeness in the following

$$\mathcal{O}_{in} \mathcal{H}_0 = \mathcal{O}_{out} \mathcal{H}_0 = \mathcal{H} \quad (3.23)$$

where $\mathcal{O}_{\text{ex}}$ denotes the algebra generated by the fields $A_{\text{ex}}(f)$. Under these circumstances we have

$$C = \lim_{V \rightarrow \mathbb{R}_3} \frac{1}{V} \int_V A_{\text{ex}}(x) d\vec{x} \quad (3.24)$$

corresponding to the definition Eq. (3.2).

Now we define the S operator in the usual way as unitary equivalence transformation between the free fields A_{in} and A_{out} :

$$A_{\text{in}}(x) = S A_{\text{out}}(x) S^{-1} \quad (3.25')$$

with the phase condition

$$S |c\rangle = |c\rangle \quad (3.25'')$$

This definition is unique and leads to the important properties

$$[C, S] = 0 \quad (3.26')$$

$$[Q, S] = 0, \text{ or } U(c) S U^{-1}(c) = S \quad (3.26'')$$

The first property $[C, S] = 0$ follows rather directly from the definition Eq. (3.25) : since $A_{\text{in}}(x)$ and $A_{\text{out}}(x)$ are irreducibly represented in the factor spaces $\mathcal{H}^{(c)}$, inequivalent for different c , S must leave the factor spaces invariant ; this is equivalent to Eq. (3.26'). This result allows to postulate the phase condition of the S matrix (3.25''), which is necessary for the proof of Eq. (3.26''). For this we consider

$$[Q, A_{\text{in}}] = [Q, S A_{\text{out}} S^{-1}]$$

which implies

$$\begin{aligned} 0 &= S A_{out} [Q, S^{-1}] + [Q, S] A_{out} S^{-1} = \\ &= A_{in} S [Q, S^{-1}] + [Q, S] S^{-1} A_{in} \end{aligned}$$

Because of

$$0 = [Q, S S^{-1}] = S [Q, S^{-1}] + [Q, S] S^{-1}$$

we get

$$[A_{in}, S [Q, S^{-1}]] = 0$$

Therefore $S [Q, S^{-1}]$ has to be an element of the centre :

$$S [Q, S^{-1}] = \alpha \cdot \mathbb{C}$$

The vacuum expectation values $\langle \Omega | S [Q, S^{-1}] | \Omega' \rangle$ ($\Omega, \Omega' \in \mathcal{H}_0$) vanish because of Eq. (3.25"). Therefore we have $\alpha = 0$, which proves $[Q, S] = 0$. Since S transforms the irreducible algebra generated by $\{A_{in}, Q\}$ into that generated by $\{A_{out}, Q\}$, it is uniquely determined up to a phase which is fixed by Eq. (3.25").

According to Eq. (2.42) we introduce n particle bases related to A_{in} or A_{out} , respectively, which describe states with n incoming particles $|p_1, \dots, p_n, c\rangle^{in}$ or m outgoing particles $|p'_1, \dots, p'_m, c\rangle^{out}$. As usual the S matrix elements describe the physical transition amplitudes

$$\begin{aligned} &{}^{out}\langle p_1, \dots, p_n, c | p'_1, \dots, p'_m, c' \rangle^{in} = \\ &= {}^{in}\langle p_1, \dots, p_n | S(c) | p'_1, \dots, p'_m \rangle^{in} \delta(c - c') \end{aligned} \quad (3.27).$$

The two equations (3.26) contain the main statements on the structure of the S matrix in a covariant field theory with a spontaneously broken symmetry. The first one means that the S matrix leaves invariant the irreducible factors $\mathcal{R}^{(c)}$ of the field algebra. It follows directly from the second one that the S matrix elements are identical in different factors, i.e., the matrix elements (3.27) are independent of c . Therefore, in principle, the consideration of the S matrix could have been restricted from the beginning to a single factor. This means that the modifications of quantum mechanics by giving up irreducibility of the field algebra is irrelevant for the physical interpretation. However, the spontaneously broken symmetry has consequences for the S matrix which become transparent by embedding the S operator in a covariant field theory. Under the conditions of Eqs. (3.26), the coefficients in Haag's expansion of the S matrix

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} \int \cdots \int dp_1 \cdots dp_n S_n(p_1, \dots, p_n) : \tilde{A}^{\dagger}(p_1) \cdots \tilde{A}^{\dagger}(p_n) : \quad (3.28)$$

vanish whenever one of the momenta is zero

$$S_n(p_1, \dots, p_i = 0, \dots, p_n) = 0 \quad (3.29)$$

This we have proved in Section 2.6. It is the main advantage of the covariant description of SSB that such zeros of the connected part of the S matrix, which are similar to the Adler zeros^{25),42)} of spontaneously broken $SU(2) \times SU(2)$ in current algebra, appear as self-consistency conditions on the S matrix with the properties (3.26), independent of any specific dynamical model.

It would be possible to derive from the asymptotic condition the reduction formulae for the S matrix^{20),37)}. This would allow to express the coefficients $S_n(p_1, \dots, p_n)$ by retarded or time-ordered functions which would combine the zeros (3.29) with the analytic properties of a local field theory.

3.3. - Concluding remarks

Our discussion of a scalar gauge symmetry exhibits the characteristic problems related to spontaneously broken symmetry : the structure of the degenerate vacuum, the covariance properties of the vacuum expectation values, the connection between conserved currents and symmetry generators, the representation of the free fields and the form of Haag's expansion, the asymptotic condition and the consistency properties of the S matrix. Of course, now it would be desirable to study this general structure in special dynamical models. The simplest model with scalar gauge symmetry is the gradient coupling model ²⁷⁾ defined by the Lagrangian

$$L = \frac{1}{2} (\partial A)^2 - \bar{\psi} (-i \gamma_\mu \partial^\mu + m + i f \gamma_\mu \gamma_5 \partial^\mu A) \psi$$

formally invariant under the transformation $A(x) \rightarrow A(x) + c$. In so far as this model contains a coupling to the matter field ψ we would have to generalize slightly our approach. It is more serious that this model is not renormalizable and also rather unphysical. Extended to the coupling of isovector pions and nucleons it would become more realistic. This would imply a generalization of the scalar gauge group to that of $ISO(3)$, already mentioned in Section 1, which is possible without serious complications. But, of course, the really challenging problem would be that of the treatment of $SU(2) \times SU(2)$, because it is believed to be an approximate symmetry of nature. In addition this would allow a close comparison of our method with the familiar techniques of current algebra - in particular with respect to the "existence" of the charge integral in a covariant field theory. Besides the appearance of dynamical rearrangement of symmetry ¹⁸⁾ a first look made us believe that a covariant description is also possible in this case and leads to the typical problems of SSB discussed here. We shall leave such model considerations in the framework of $SU(2) \times SU(2)$ for discussions in the future.

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