Archives

POSITRONIUM ON NULL PLANE AND MELOSH TRANSFORMATION

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The null plane dynamics of positronium-like systems is studied in a quasipotential approach, in approximations depending on c , the velocity of light, being large. Neglecting terms in the quasipotential of order c^{-3} (relative to particle rest energy) symmetries appear whose generators are analogous to the good null plane charges of relativistic $SU(6)$ hadron classification schemes. These symmetries are broken by terms of order c^{-3} , which can, however, be transformed away by an appropriate approximation to the Melosh transformations. There then emerge better symmetries, broken only at order c^{-4} . When both internal and over-all velocities of a composite system are small of first order, and terms of second and higher order are neglected, these better symmetries coincide with those that appear in a Foldy-Wouthuysen description when spin-orbit coupling is neglected. More generally the better symmetries remain good for such a composite system boosted arbitrarily in the three-direction. For this reason we find the original, boost invariant, "first" Melosh transformation more appropriate than the "second" Melosh transformation.

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In an earlier paper ¹⁾ we presented a null plane description of a spin $\frac{1}{2}$ particle moving about a fixed force centre - i.e., of a hydrogen-like system. We showed how in that context the appropriate approximation to the Melosh transformation arises in describing the approximate spin-orbit degeneracy. Here we make similar observations for two moving spin $\frac{1}{2}$ particles - i.e., for a positronium (or charmonium ?) - like system.

The present considerations will be restricted to the one boson exchange approximation to the potential. That potential will be defined by the null plane analogue of a procedure ²⁾ leading to the familiar Breit equation. The potential is required simply to reproduce in Born approximation the one-boson exchange scattering matrix element.

That scattering matrix element has the form

$$(2\pi)^4 \delta(p'_{0A} + p'_{0B} - p_{0A} - p_{0B}) \delta^3(\vec{p}'_A + \vec{p}'_B - \vec{p}_A - \vec{p}_B) \\ \times \frac{u'_A u'_B \mathcal{M}(\vec{p}'_A, \vec{p}'_B, \vec{p}_A, \vec{p}_B) u_A u_B}{\mu^2 + q^2} \quad (1)$$

for particles A, B with initial and final momenta $(\vec{p}, \vec{p}')_{A,B}$ and initial and final Dirac four-spinors $(U, U')_{A,B}$, where

$$p_{0A} = \sqrt{m_A^2 + \vec{p}_A^2} \quad p_{0B} = \sqrt{m_B^2 + \vec{p}_B^2} \quad (2)$$

$$\vec{q} = \vec{p}'_A - \vec{p}_A = \vec{p}_B - \vec{p}'_B \quad (3)$$

$$q_0 = p'_{0A} - p_{0A} = p_{0B} - p'_{0B} \quad (4)$$

In null plane dynamics each third component of momentum p_3 is thought of as a function of a quantity

$$\eta = p_3 + \sqrt{m^2 + \vec{p}^2}$$

and of the other momenta. The factor

$$\delta(p'_{3A} + p'_{3B} - p_{3A} - p_{3B})$$

can be replaced in (1) by

$$\delta(\eta'_A + \eta'_B - \eta_A - \eta_B)$$

Moreover the Dirac spinors can be written in terms of their "good" components

$$u = \begin{pmatrix} \sqrt{\eta} \chi \\ \frac{m\sigma_3 - \vec{\sigma}_1 \cdot \vec{p}_1}{\sqrt{\eta}} \chi \end{pmatrix} \quad (5)$$

in the representation

$$\alpha_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad \beta = \begin{pmatrix} & \sigma_3 \\ \sigma_3 & \end{pmatrix} \quad \vec{\alpha}_1 = \begin{pmatrix} & -\vec{\sigma}_1 \\ -\vec{\sigma}_1 & \end{pmatrix} \quad (6)$$

We use here the normalization

$$u^\dagger u = 2\sqrt{m^2 + \vec{p}^2} \quad \chi^\dagger \chi = 1$$

So the scattering matrix element can be rewritten

$$(2\pi)^4 \delta(p'_{0A} + p'_{0B} - p_{0A} - p_{0B}) \delta^2(\vec{p}'_{1A} + \vec{p}'_{1B} - \vec{p}_A - \vec{p}_B) \\ \times \delta(\eta'_A + \eta'_B - \eta_A - \eta_B) \frac{\chi'^{\dagger}_A \chi'_B \hat{M} \chi_A \chi_B}{\mu^2 + q^2} \quad (7)$$

with $\hat{M}$ now a 4×4 rather than a 16×16 matrix.

The energy conserving factor

$$2\pi \delta(p'_{0A} + p'_{0B} - p_{0A} - p_{0B})$$

can be thought of as arising from the integration of a time-dependent Schrödinger equation. The remaining factors in (1) can then be considered to define an effective potential, in momentum space. Strictly speaking one could add to any such potential arbitrary terms containing a factor

$$(\vec{p}'_A + \vec{p}'_B - \vec{p}_A - \vec{p}_B) \quad (8)$$

- which would not contribute to the scattering matrix element in Born approximation. Such an ambiguity arises in particular in deciding which of various expressions for q^2 , equivalent when all four components of momentum are conserved, to use in (1) and (7). For definiteness let us choose the form, symmetric between A and B,

$$q^2 = (\vec{p}'_A - \vec{p}_A) \cdot (\vec{p}'_B - \vec{p}_B) - (p'_A - p_A)(p'_B - p_B)$$

We have calculated $\hat{M}$ for exchange of scalar, pseudoscalar, vector, and pseudovector bosons, with the usual non-derivative parity-conserving vertices :

$$\begin{array}{ll} g_A^S \bar{u}'_A u_A & g_B^S \bar{u}'_B u_B \\ g_A^{PS} \bar{u}'_A i\gamma_5 u_A & g_B^{PS} \bar{u}'_B i\gamma_5 u_B \\ g_A^V \bar{u}'_A i\gamma_\mu u_A & g_B^V \bar{u}'_B i\gamma_\mu u_B \\ g_A^{PV} \bar{u}'_A i\gamma_\mu \gamma_5 u_A & g_B^{PV} \bar{u}'_B i\gamma_\mu \gamma_5 u_B \end{array}$$

The first order in small quantities

$$\begin{aligned} \hat{M} (2m_A 2m_B)^{-1} = & (g_A^S g_B^S - g_A^V g_B^V) \left(1 + \left[\frac{\vec{\sigma}_3 \cdot \vec{\sigma}_1 \cdot \vec{p}'_1}{2m} - \frac{\vec{\sigma}_3 \cdot \vec{\sigma}_1 \cdot \vec{p}_1}{2m} \right]_A + \left[\frac{\vec{\sigma}_3 \cdot \vec{\sigma}_1 \cdot \vec{p}'_1}{2m} - \frac{\vec{\sigma}_3 \cdot \vec{\sigma}_1 \cdot \vec{p}_1}{2m} \right]_B \right) \\ & + g_A^{PV} g_B^{PV} \left(\vec{\sigma}_A \cdot \vec{\sigma}_B + \left[\frac{\vec{\sigma}_3 \cdot \vec{\sigma}_1 \cdot \vec{p}'_1}{2m} \right]_A \vec{\sigma}_A \cdot \vec{\sigma}_B - \vec{\sigma}_A \cdot \vec{\sigma}_B \left[\frac{\vec{\sigma}_3 \cdot \vec{\sigma}_1 \cdot \vec{p}_1}{2m} \right]_A \right. \\ & \left. + \left[\frac{\vec{\sigma}_3 \cdot \vec{\sigma}_1 \cdot \vec{p}'_1}{2m} \right]_B \vec{\sigma}_A \cdot \vec{\sigma}_B - \vec{\sigma}_A \cdot \vec{\sigma}_B \left[\frac{\vec{\sigma}_3 \cdot \vec{\sigma}_1 \cdot \vec{p}_1}{2m} \right]_B \right) \end{aligned} \quad (9)$$

Here we treated as small of first order

$$(\vec{p}_\perp / m)_{A,B} \quad , \quad (\vec{p}'_\perp / m)_{A,B} \quad (10)$$

and also differences from unity of each of the ratios

$$\frac{\eta'_A}{\eta_A} \quad , \quad \frac{\eta'_B}{\eta_B} \quad , \quad \left(\frac{\eta_A}{m_A} / \frac{\eta_B}{m_B} \right) \quad (11)$$

We could in fact, in a non-relativistic approximation, consistently have made the stronger assumption

$$1 - \left(\frac{\eta, \eta'}{m} \right)_{A,B} = \text{small of first order}$$

But this was not necessary to obtain (9), as will be interesting in a moment. In fact the invariant matrix element, quite accurately, can depend only on ratios of η 's, for all η 's scale identically under Lorentz boost in the three-direction.

In the particular case of pseudovector exchange, to obtain (9), it is at some points necessary to suppose that the small quantities just listed remain small when multiplied by (m/μ) . Such factors arise from the second term in the pseudovector propagator

$$\frac{\delta_{\mu\nu} + q_\mu q_\nu / \mu^2}{\mu^2 + q^2} \quad (12)$$

This term is equivalent to pseudoscalar exchange with effective coupling

$$(g^S_{A,B})_{\text{eff}} = \left(\frac{2m}{\mu} g^V \right)_{A,B} \quad (13)$$

When the small terms involving $p_\perp$'s in (9) are ignored, the only $\vec{\sigma}$ dependence is in the form $\vec{\sigma}_A \cdot \vec{\sigma}_B$. Thus there is a symmetry, represented by the infinitesimal variation of spinors

$$\delta \chi = -i \vec{\omega} \cdot \vec{\sigma} \chi \quad (14)$$

where $\vec{w}$ is an arbitrary infinitesimal vector, the same for all four wave functions

$$(\chi, \chi')_{A, B}$$

In the absence of pseudovector exchange, and therefore of the zero-order spin dependence $\vec{\sigma}_A \cdot \vec{\sigma}_B$, there is a higher symmetry in which the spinors belonging to particles A and B are transformed separately - a group $SU(2)_A \times SU(2)_B$ instead of simply $SU(2)$.

The symmetry of the last paragraph is broken by the first order $\vec{\sigma}$ dependent terms in (9). However, again, as in the external potential case ¹⁾, there is a better symmetry which can be made evident by a unitary transformation

$$\chi = \left(1 + \frac{\sigma_3 \vec{\sigma}_1 \cdot \vec{p}_1}{2m} + \dots \right) \phi \quad (15)$$

Corresponding to the use of ϕ rather than χ is a transformed matrix

$$\hat{M}_M (2m_A 2m_B)^{-1} = \left(g_A^S g_B^S - g_A^V g_B^V \right) + g_A^{PV} g_B^{PV} \vec{\sigma}_A \cdot \vec{\sigma}_B \quad (16)$$

+ second order terms

The spin-orbit coupling has been removed to a higher order, and we have a better $SU(2)$ symmetry

$$\delta \phi = - i \vec{w} \cdot \vec{\sigma} \phi \quad (17)$$

Again this is enlarged to $SU(2)_A \times SU(2)_B$, in the absence of pseudovector exchange, in which A and B spinors are transformed independently.

The transformation (15) can be considered as the first order approximation to the second Melosh transformation ^{*)}

*) See for example Eq. (70) of Ref. 3) and the good representation of $\vec{v}_1$ in Section 3 of the Appendix to that paper. We could be accused here of terminological inexactitude. Because (18) is not boost invariant, Melosh actually proposed a more complicated many-particle construction referring explicitly to the over-all momentum of a composite system, or (equivalently) recommended the use of (18) only in the centre-of-mass frame. See also Buccella et al. 4).

$$\chi = \frac{m + \sqrt{q^2} + \sigma_3 \vec{\sigma}_1 \cdot \vec{p}_1}{\sqrt{(m + \sqrt{q^2})^2 + \vec{p}_1^2}} \phi \quad (18)$$

so that ϕ can be identified ³⁾ to that order with the Foldy-Wouthuysen spinor. In this picture the simple spin dependence of S matrix elements, and corresponding potentials, illustrated in (16), follows rather simply from rotation and inversion invariance. The pseudovector $\vec{\sigma}_A$ has to be contracted with another pseudovector ; if this latter is not just $\vec{\sigma}_B$, it has to contain two small momenta, e.g. :

$$\vec{p}_1 \times \vec{p}'_1, \quad \vec{\sigma}_B \cdot \vec{p}_1 \vec{p}'_1$$

To first order the only permissible spin dependence is $\vec{\sigma}_A \cdot \vec{\sigma}_B$.

Instead of regarding (15) as a first order approximation to the second Melosh transformation (18), one can instead regard it as a first order approximation to the first Melosh transformation ^{*)}

$$\chi = \frac{m + \sqrt{m^2 + \vec{p}_1^2} + \sigma_3 \vec{\sigma}_1 \cdot \vec{p}_1}{\sqrt{(m + \sqrt{m^2 + \vec{p}_1^2})^2 + \vec{p}_1^2}} \phi \quad (19)$$

In fact (19) reduces to (15) when $(\vec{p}_1/m)^2$ is small, regardless of the magnitude of $((\gamma/m) - 1)$. Moreover ϕ defined by (15) or (19), but not by (18), is invariant under Lorentz boosts in the three-direction. Likewise the symmetry operations (17). So states which transform as an irreducible representation of this group continue to do so when boosted in the three-direction. So as regards motion in the three-direction, the symmetry depends on the slowness of internal motion in a composite system, and the over-all motion can be fast. As confirmation of this one can repeat the calculations leading to (9), and verify that the η 's separately do not have to be $\approx m$ for the validity of the result given ; it is sufficient that the ratios (11), each involving a pair of η 's, are close to unity.

^{*)} See Eq. (48) of Ref. 3).

The point just made tends to rehabilitate the first Melosh transformation at the expense of the second ^{*)}. And this in spite of the greater compatibility of the second with rotational invariance ⁴⁾⁻⁶⁾. Note by the way that we have not here found any reason for preferring the full form (19), of the first Melosh transformation, to the approximate form (15).

The point about boost invariance may be quite important, because in practice ^{**)} the Melosh transformation is applied in calculating (by PCAC) matrix elements for processes like

$$\pi + (56, L=0) \leftarrow (76, L=1)$$

where there is no frame of reference in which both 70 and 56 move slowly.

To relate these symmetries to the algebra of null plane charges, note the relation between spinors χ and one-particle states $|1\rangle$

$$\sqrt{4} \chi = \langle 0 | \hat{\psi}_g(0,0,0,0) | 1 \rangle \quad (20)$$

where $|0\rangle$ is vacuum, and $\hat{\psi}_g$ denotes the good components of the null-plane field operator

$$\hat{\psi}(t, x, y, z) = \psi(t-z, x, y, z)$$

where ψ is the ordinary Heisenberg operator. The variation (14) of spinors then corresponds to

$$\delta |1\rangle = -i \hat{F}(\vec{\omega} \cdot \vec{\sigma}) |1\rangle$$

where $\hat{F}$ is the null plane charge

$$\hat{F}(\vec{\omega} \cdot \vec{\sigma}) = \vec{\omega} \cdot \int d^4x \delta(t) \hat{\psi}_g^+ \vec{\sigma} \hat{\psi}_g$$

^{*)} See, however, the footnote on p. 5.

^{**)} For recent practice see, for example, Hey, Litchfield and Cashmore ⁷⁾, and references therein.

The generators of the $SU(2)_A \times SU(2)_B$ group (14) are then the null plane charges

$$\left(2 \int d^4x \delta(t) \hat{\psi}_g^\dagger \vec{\sigma} \hat{\psi}_g \right)_{A,B} \quad (21)$$

The better symmetry (17) is likewise generated by the corresponding constructions from Melosh transformed field operators.

Note that for identical particles, $A=B$, only a single $SU(2)$ subgroup of the $SU(2)_A \times SU(2)_B$ can be generated in this way by null plane charges. The larger group is specific to the two-particle effective Schrödinger equation description.

A related point has to be made for the case of A and B antiparticles of one another. Again one does not have two independent fields; ψ_A and ψ_B are charge conjugates of one another. Nevertheless a field theoretic $SU(2) \times SU(2)$ algebra can be formed⁵⁾

$$2 \int d^4x \delta(t) \hat{\psi}_g^\dagger \vec{\sigma} \left(\frac{1 \pm \epsilon}{2} \right) \hat{\psi}_g \quad (22)$$

by introducing ϵ , defined in momentum space by

$$\epsilon = \eta / \sqrt{\eta^2} \quad (23)$$

As is well known the $SU(2)_W$ algebra of good null plane charges

$$2 \int d^4x \delta(t) \hat{\psi}_g^\dagger (\sigma_x, \sigma_y, \sigma_z) \hat{\psi}_g$$

acts differently on particles and antiparticles, so it is not this subgroup which survives spin-spin coupling $\vec{\sigma}_A \cdot \vec{\sigma}_B$, but rather the other

$$2 \int d^4x \delta(t) \hat{\psi}_g^\dagger (\epsilon \sigma_x, \epsilon \sigma_y, \sigma_z) \hat{\psi}_g$$

- or, better, the Melosh modified version thereof - which acts similarly on particles and antiparticles.

When A and B are particle and antiparticle there will be, as well as exchange diagrams

annihilation diagrams

For these the propagator factor

$$(\mu^2 + \vec{q}^2 - q_0^2)$$

is replaced by

$$\mu^2 + (\vec{p}_A + \vec{p}_B)^2 - (p_{0A} + p_{0B})^2 \quad (24)$$

When $\mu \ll m$, such terms can be regarded as small of second order relative to the exchange contributions. In any case the general arguments following (18) again apply and show that the spin dependence of the leading terms, after Melosh transformation, will again be of the form (16). Note that for annihilation contributions spin-spin coupling, $\vec{\sigma}_A \cdot \vec{\sigma}_B$, arises even with ordinary vector mesons.

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