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TREATMENT OF THE CLASSICAL RELATIVISTIC STRING
IN ANY ORTHONORMAL GAUGE

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ABSTRACT

It is shown that a certain set of gauge invariant functions are, for an appropriate choice of a parameter on which they depend, equal to the Fourier components of the classical relativistic string in any orthonormal gauge. These variables are natural generalizations of the classical DDF operators recently introduced by Goddard, Hanson and Ponzano. The Poisson algebra of the relativistic string in any orthonormal gauge (including the proper time gauge) is written down. Application to quantization is briefly discussed.

1. INTRODUCTION

Ever since the discovery that the relativistic string is a space-time picture underlying the dual models, a considerable interest has been devoted to this object ¹⁾. A fundamental and intriguing property of the canonically quantized relativistic string is that the Lorentz algebra is only satisfied when the dimension of space-time is 26, and the intercept $\alpha_0 = 1$ of the leading Regge trajectory (implying the presence of a tachyon state) ²⁾. Since up to now a canonical quantization has only been possible in a light-like orthonormal gauge, there have been some speculations that this result is due to an inappropriate gauge choice and that a more physical result would come out in a time-like orthonormal gauge ³⁾. Rohrlich's approach is here the most elaborate one. However, it can be seen (cf., Appendix C) that this approach does not follow from the Nambu action ⁴⁾. In a recent treatment by Goddard, Hanson and Ponzano ⁵⁾, the opposite point of view were advocated, namely that $D=26$ and $\alpha_0 = 1$ is retained in any time-like gauge. Although their treatment follows from the Nambu action, we feel that it is incomplete and that the conclusion might turn out to have been drawn too easily. The present paper, which is inspired by their paper, will exhibit new results on the classical string, and in particular, we shall give explicit expressions of the string in any orthonormal gauge in terms of canonical co-ordinates, formulae which should serve as a starting point for a future complete investigation of the quantum string in any orthonormal gauge.

In Section 2 we present a new gauge mapping theorem, and in Section 3 we briefly discuss its application to the quantization problem. In Appendix A we derive the main formulae used in the text, and in Appendix B we give a separate proof of a particular consequence of the gauge mapping theorem. In Appendix C the Poisson algebra of the string in any gauge is presented.

2. A GAUGE MAPPING THEOREM FOR THE CLASSICAL RELATIVISTIC STRING IN THE ORTHONORMAL GAUGE

The equations of motion for a free relativistic string are obtained by minimizing the action functional ⁴⁾

$$S = \int_{-\infty}^{\infty} d\tau \int_0^{\pi} d\sigma \mathcal{L} \quad (2.1)$$

where

$$\mathcal{L} = -\frac{1}{2\pi\alpha'} \{ (\dot{X}^0)^2 - (\dot{X}^1)^2 \}^{1/2} \quad (2.2)$$

where α' is a constant with dimension length over energy (choosing $c=1$). In the quantum case α' is the slope of the leading Regge trajectory. We are using the notation $\dot{X}^\mu \equiv (\partial/\partial\tau) X^\mu(\tau, \sigma)$ and $X'^\mu \equiv (\partial/\partial\sigma) X^\mu(\tau, \sigma)$, and our metric is time-like.

The action (2.1) is a reparametrization-invariant expression for a surface imbedded in space-time where we have chosen the conventional τ independent end points in the σ parameter. The equations of motion are

$$\frac{\partial}{\partial\tau} \frac{\partial\mathcal{L}}{\partial\dot{X}_\mu} + \frac{\partial}{\partial\sigma} \frac{\partial\mathcal{L}}{\partial X'^\mu} = 0 \quad (2.3)$$

with the boundary conditions

$$\frac{\partial\mathcal{L}}{\partial X'^\mu}(\tau, \sigma=0) = \frac{\partial\mathcal{L}}{\partial X'^\mu}(\tau, \sigma=\pi) = 0 \quad (2.4)$$

(The closed string will not be treated in the present paper.) These equations are very difficult to solve, in general, due to their non-linearity. However, there exist a particular set of parameter choices such that Eqs. (2.3) linearize, namely any orthonormal (ON) parametrization (gauge) which by definition satisfies

$$\dot{X} \cdot X' = 0, \quad \dot{X}^2 + X'^2 = 0 \quad (2.5)$$

The equations of motion then become

$$\begin{aligned} \ddot{X}^\mu - X''^\mu &= 0, \quad 0 \leq \sigma \leq \pi \\ X'^\mu &= 0 \quad \text{at } \sigma = 0, \pi \end{aligned} \quad (2.6)$$

whose solution may be written as

$$\begin{aligned} X^\mu(\tau, \sigma) &= \sqrt{2\alpha'} \left(q^\mu + a_0^\mu \tau + \right. \\ &\quad \left. + i \sum_{m \neq 0} \frac{1}{m} a_m^\mu \cos m\sigma e^{-im\tau} \right) \end{aligned} \quad (2.7)$$

where reality requires $(a_m^\mu)^* = a_{-m}^\mu$.

The Lagrangian (2.2) yields in the ON gauge (2.5) the following expressions for the conserved energy-momentum vector and angular-momentum tensor

$$P^\mu = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \dot{X}^\mu = \frac{a_0^\mu}{\sqrt{2\alpha'}} \quad (2.8)$$

$$\begin{aligned} \mathcal{I}^{\mu\nu} &= \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) = \\ &= q^\mu a_0^\nu - q^\nu a_0^\mu + i \sum_{m \neq 0} \frac{1}{m} a_m^\mu a_{-m}^\nu \end{aligned} \quad (2.9)$$

Now the ON gauge does not specify the parameters τ and σ uniquely in (2.7). There remains a further freedom of reparametrization characterized by conformal transformations ^{1),2)}. A unique parameter (gauge) choice is obtained by adding the following constraints to (2.5)

$$\eta \cdot X = 2\alpha' (\eta \cdot P) \tau \quad (2.10)$$

where η^μ is a vector which has to be time- or light-like for reasons soon to be apparent.

In terms of the normal modes in (2.7), the orthonormality conditions (2.5) become

$$L_n = 0 \quad \text{any integer } n \quad (2.11)$$

where

$$L_n \equiv \frac{1}{2} \sum_{m=-\infty}^{\infty} a_m \cdot a_{n-m} \quad (2.12)$$

which are just classical versions of the Virasoro constraints. Notice that (2.7) is a solution to the equations of motion (2.3), only if it is consistent with these constraints. Equation (2.10) becomes

$$\eta \cdot q = 0 ; \eta \cdot a_n = 0 , \quad n \neq 0 \quad (2.13)$$

In the following we shall denote the normal modes satisfying the constraints (2.11) by q^μ and a_n^μ , and the normal modes satisfying both (2.11) and (2.13) by q_η^μ and $a_{\eta n}^\mu$. The set $q^\mu, \{a_n^\mu\}_{n=-\infty}^{+\infty}$ we shall call the Virasoro space, and $q_\eta^\mu, \{a_{\eta n}^\mu\}_{n=-\infty}^{+\infty}$ the physical space according to their different degrees of freedom.

An important and useful observation is that the string is completely determined by one of its end points in the ON gauge. Consider the variable

$$X^\mu(\tau) \equiv q^\mu + a_0^\mu \tau + i \sum_{n \neq 0} \frac{1}{n} a_n^\mu e^{-in\tau} \quad (2.14)$$

which, apart from the constant factor $\sqrt{2\alpha'}$ whose exclusion will turn out to be convenient later on, is just one end point of the string. The velocity of the end point has the following properties written in terms of $\dot{X}^\mu(\tau)$

$$\dot{X}^\mu(\tau) = \dot{X}^\mu(\tau + 2\pi), \text{ all } \tau \quad (2.15)$$

$$\dot{X}^2(\tau) = 0, \text{ all } \tau \quad (2.16)$$

Notice that (2.15) and (2.16) both follow from and imply (2.14) and (2.11), respectively. The string (2.7) is now given by

$$X^\mu(\tau, \sigma) = \sqrt{\frac{\alpha'}{2}} (X^\mu(\tau + \sigma) + X^\mu(\tau - \sigma)) \quad (2.17)$$

We are then ready for the gauge mapping theorem. Consider the string in different gauges within the ON gauge, denoting by $X_\xi^\mu(\tau_\xi, \sigma_\xi)$ the string in the gauge determined by the vector ξ^μ and by $X(\tau, \sigma)$ in a general ON gauge [only satisfying the constraints (2.5)]. Since the string in the different gauges describes one and the same physical string, we must have the equalities

$$X_\xi^\mu(\tau_\xi, \sigma_\xi) = X_\eta^\mu(\tau_\eta, \sigma_\eta) = X^\mu(\tau, \sigma) \quad (2.18)$$

The consequences of these equalities are now easily derived by making use of the above observation that the string in the ON gauge is completely determined by its end point; i.e., by replacing (2.18) by the following equivalent equations

$$X_{\xi}^{\mu}(\tau_{\xi}) = X_{\eta}^{\mu}(\tau_{\eta}) = X^{\mu}(\tau) \quad (2.19)$$

Taking the scalar product of the last equality with η^{μ} , we arrive at

$$\tau_{\eta} = \lambda_{\eta} \eta \cdot X(\tau) \quad (2.20)$$

where

$$\lambda_{\eta} \equiv (\eta \cdot a_0)^{-1} \quad (2.21)$$

(In the following we shall omit the index η for simplicity.) Inserting (2.20) into (2.19) we get

$$X_{\eta}^{\mu}(\lambda_{\eta} \eta \cdot X(\tau)) = X^{\mu}(\tau) \quad (2.22)$$

In order to determine what this equality implies for the normal modes, we take the derivative

$$\lambda_{\eta} \eta \cdot \dot{X}(\tau) \dot{X}_{\eta}^{\mu}(\lambda_{\eta} \eta \cdot X(\tau)) = \dot{X}^{\mu}(\tau) \quad (2.23)$$

Notice that this equality requires the factor $\lambda_{\eta} \cdot \dot{X}$ to be different from zero, which implies that η^{μ} has to be time- or light-like since $\dot{X}^{\mu}$ is light-like. Multiplying (2.23) by $e^{in\lambda_{\eta} \cdot X(\tau)}$ and integrating over τ from zero to 2π [notice that $\lambda_{\eta} \cdot \dot{X}(\tau + 2\pi) = \lambda_{\eta} \cdot \dot{X}(\tau) + 2\pi$] we get

$$a_{\eta n}^{\mu} = A_n^{\mu}(\eta) \equiv A_n^{\mu}(\eta; a_k^{\nu}, q^{\nu}) \quad (2.24)$$

where

$$A_n^{\mu}(\eta; a_k^{\nu}, q^{\nu}) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\tau \dot{X}^{\mu}(\tau) e^{in\lambda_{\eta} \cdot X(\tau)} \quad (2.25)$$

Then integrating (2.22) over τ from zero to 2π and making use of formula (A.5) in Appendix A, we obtain

$$q_p^\mu = Q^\mu(\eta) \equiv Q^\mu(\eta; a_n^\nu, q^\nu) \quad (2.26)$$

where

$$Q^\mu(\eta; a_n^\nu, q^\nu) \equiv \lambda F^{\mu\nu} \eta_\nu \quad (2.27)$$

Repeating the same calculation, starting from the first equality in (2.19), we obtain

$$a_{\eta n}^\mu = A_n^\mu(\eta; a_5^\nu k, q_5^\nu) \quad (2.28)$$

$$q_\eta^\mu = Q^\mu(\eta; a_5^\nu k, q_5^\nu) \quad (2.29)$$

Thus, the functions $A_n^\mu(\eta)$ and $Q^\mu(\eta)$ are gauge invariant functions which constitute a mapping from any gauge in the physical space as well as from the Virasoro space to the gauge η^μ in the physical space. Equations (2.24) to (2.29) are the main results of the present paper; in the sequel we shall give some checks and consequences of them, as well as discussing their application to the quantization problem.

Notice that $A_n^\mu(k)$ where k^μ is light-like are just classical versions of the DDF operators α_n^μ , and they were, together with $Q^\mu(k)$, first introduced in Ref. 5).

The following relations are easily checked

$$\eta \cdot Q(\eta) = \eta \cdot A_n(\eta) = 0, \quad n \neq 0 \quad (2.30)$$

$$A_0^\mu(\eta) = a_0^\mu = \sqrt{2\alpha'} P^\mu \quad (2.31)$$

$$\begin{aligned}
 Q^\mu(\eta)A_0^\nu - Q^\nu(\eta)A_0^\mu + \lambda i \sum_{n \neq 0} \frac{1}{n} A_n^\mu(\eta) A_{-n}^\nu(\eta) \\
 = J^{\mu\nu}
 \end{aligned} \tag{2.32}$$

The last relation is shown by means of formula (A.5). We have furthermore

$$\begin{aligned}
 \bar{L}_n &\equiv \frac{1}{2} \sum_{m=-\infty}^{\infty} A_m(\eta) \cdot A_{n-m}(\eta) = \\
 &= \frac{1}{4\pi} \int_0^{2\pi} d\tau \frac{\dot{X}^2}{\lambda \eta \cdot \dot{X}} e^{in\lambda \eta \cdot X} = 0, \text{ any integer } n
 \end{aligned} \tag{2.33}$$

due to (2.16) and the requirement $\lambda \eta \cdot \dot{X} \neq 0$. The following special case of (2.28)-(2.29) is also easily checked

$$\begin{aligned}
 a_{\eta n}^\mu &= A_n^\mu(\eta; a_{\eta k}^\nu, q_{\eta}^\nu) \\
 q_{\eta}^\mu &= Q^\mu(\eta; a_{\eta k}^\nu, q_{\eta}^\nu)
 \end{aligned} \tag{2.34}$$

The relations (2.30)-(2.34) were also shown in Ref. 5) for the special case $\eta^\mu = k^\mu$, $k^2 = 0$.

Combining (2.28)-(2.29) with (2.24) and (2.26), one obtains

$$A_n^\mu(\eta) = A_n^\mu(\eta; A_k^\nu(\xi), Q^\nu(\xi)) \tag{2.35}$$

$$Q^\mu(\eta) = Q^\mu(\eta; A_k^\nu(\xi), Q^\nu(\xi)) \tag{2.36}$$

The last equation follows also from the definition of $Q^\mu(\eta)$ and (2.31)-(2.32). Equation (2.35) is explicitly checked in Appendix B.

We have also checked that (2.24)-(2.29) remain true within the Hamiltonian formulation of the string utilizing Dirac's method for constrained systems ⁷⁾⁻⁹⁾. The Poisson algebra of $a_{\eta n}^{\mu}$ and q_{η}^{μ} in any gauge (which is given in Appendix C) has been shown to be identical to the algebra of $A_n^{\mu}(\eta)$ and $Q^{\mu}(\eta)$.

3. DISCUSSION AND APPLICATION TO QUANTIZATION

In this Section we shall give some remarks, and discuss the application of the gauge mapping theorem (2.24)-(2.29) to the quantization of the string in any ON gauge.

The equality (2.24) gives an explanation of the previously observed isomorphic algebra of the DDF operators ⁶⁾ and the normal modes in the light-like gauge ^{2),10)}, a property which also was used in Ref. 5). Equation (2.24) makes us believe that this isomorphism will be true in any gauge. We shall, at least, assume this and only discuss the quantization on the physical space below.

The Dirac bracket (C.2) is the Poisson bracket (PB) on the physical space, and since it has all the properties of a PB ¹¹⁾, it can also be expressed in terms of physical canonical coordinates in the conventional form ¹²⁾. In order to be able to canonically quantize the string in any ON gauge, we have to find such canonical co-ordinates in the conventional string in terms of these, and then impose the canonical commutation relations on the canonical co-ordinates. The number of canonical co-ordinates is determined by the number of independent physical degrees of freedom of the system. Naively, q^{μ} and a_n^{μ} have four degrees of freedom each. The Virasoro constraints (2.11) could be said to reduce the degrees of freedom of a_n^{μ} to three each. The constraint (2.13) then finally reduces the independent degrees of freedom to three for q^{μ} , a_0^{μ} , and to two for a_n^{μ} ($n \neq 0$). (Notice that the "c.m." coordinates q^{μ} and a_0^{μ} have the same degrees of freedom as a free relativistic particle.) For a light-like η^{μ} we can choose the two transverse components of q^{μ} , a_n^{μ} plus one additional component of q^{μ} and a_0^{μ} as canonical coordinates [apparent from the relations (C.3-C.4)]. This was shown in Ref. 2) by solving the constraints explicitly, and expressing the dependent components in terms of the independent ones. It is well known that these constraints are difficult to solve in a time-like gauge. Now, from (C.4) and (C.6), we have that the following PB relation is true for any gauge η^{μ}

$$\{a_{\eta^n}^\mu, a_{\eta^m}^\nu\}^* = in \delta_{n,-m} g^{\mu\nu} - in \lambda \eta^\nu a_{\eta, n+m}^\mu +$$

$$+ im \lambda \eta^\mu a_{\eta, n+m}^\nu + i \lambda^2 \eta^2 nm \sum_{k \neq 0} \frac{1}{k} a_{\eta, n+k}^\mu a_{\eta, m-k}^\nu \quad (3.1)$$

and this relation tells us that no solution of the above type exists for a time-like gauge, since no component of the string then can be a canonical co-ordinate. [The reason why this occurs is the fact that the PB (C.1) of the constraints (2.13) does not vanish weakly ^{7),9)}.] It is here the utility of the gauge mapping theorem enters. Since canonical co-ordinates for one gauge are canonical co-ordinates for all gauges, we can express the string in any gauge in terms of canonical coordinates by means of (2.28)-(2-29), where ξ^μ is chosen to be light-like and where this light-like gauge is expressed in terms of its canonical coordinates as in Ref. 2).

The canonical quantization is now straightforward but far from trivial due to severe ordering problems. The ordering must be so performed that the Poincaré algebra is satisfied and that the string transforms as a four vector, i.e., corresponding relations to (C.3) and (C.5) must be satisfied. If we, furthermore, want to imbed the theory in a Fock space in the usual way, we have to require that $J^{\mu\nu}$, P^μ and P^2 are normal ordered.

There are three possibilities for what might occur for a time-like gauge:

- i) there is no solution which satisfies the above requirements;
- ii) there is a solution which is unitarily equivalent to the usual one ²⁾
($D=26$, $\alpha_0=1$);
- iii) there is a solution which is not unitarily equivalent to the usual one
($D=26$, $\alpha_0=1$ no longer necessarily the case).

Due to the anomalous properties of the quantum string, the naive expectation (ii) is far from obvious. Of course, the possibility (iii) is the most interesting one, but we think that (i) is the most likely. A fact which is not so well known is that even the usual quantization ²⁾ does not quite

satisfy the above requirements (the dependent component does only transform correctly for $D=8$ and $\alpha_0 = 1/4!$), which might mean that these requirements could be weakened.

We hope to come back to these questions elsewhere.

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THE ESSENTIAL FORMULAE USED IN THE TEXT
AND THEIR DERIVATIONS

Formula 1:

$$i \sum_{k \neq 0} \frac{1}{k} A_{n+k}^{\mu} A_{m-k}^{\nu} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\tau d\tau' \times$$

$$\dot{X}^{\mu}(\tau) \dot{X}^{\nu}(\tau') e^{in\lambda \eta \cdot X} e^{im\lambda \eta \cdot X'} \times$$

$$\left(i \sum_{k \neq 0} \frac{1}{k} e^{ik(\tau-\tau')} + \lambda (\eta \cdot r(\tau) - \eta \cdot r(\tau')) \right) \quad (A.1)$$

where we have introduced the relative motion $r^{\mu}(\tau)$ of the end point given by

$$r^{\mu}(\tau) = i \sum_{n \neq 0} \frac{1}{n} a_n^{\mu} e^{-in\tau} \quad (A.2)$$

Proof of formula 1:

$$i \sum_{k \neq 0} \frac{1}{k} A_{n+k}^{\mu} A_{m-k}^{\nu} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\tau d\tau' \dot{X}^{\mu}(\tau) \dot{X}^{\nu}(\tau') \times$$

$$e^{in\lambda \eta \cdot X} e^{im\lambda \eta \cdot X'} i \sum_{k \neq 0} \frac{1}{k} e^{ik\lambda (\eta \cdot X - \eta \cdot X')}$$

and

$$i \sum_{k \neq 0} \frac{1}{k} e^{ik\lambda(\eta \cdot X - \eta \cdot X')} = i \sum_{k \neq 0} \frac{1}{k} (e^{ik(\tau - \tau')} + e^{ik(\tau - \tau')} (e^{ik\lambda(\eta \cdot r - \eta \cdot r')} - 1))$$

Furthermore,

$$i \sum_{k \neq 0} \frac{1}{k} e^{ik(\tau - \tau')} (e^{ik\lambda(\eta \cdot r - \eta \cdot r')} - 1) =$$

$$= \lambda(\eta \cdot r - \eta \cdot r') - 2\pi(\eta \cdot r - \eta \cdot r') \int_0^\lambda d\lambda' \times$$

$$\delta(\tau - \tau' + \lambda'(\eta \cdot r - \eta \cdot r')) =$$

$$= \lambda(\eta \cdot r - \eta \cdot r') - 2\pi(\theta(\lambda(\eta \cdot x - \eta \cdot x')) -$$

$$- \theta(\tau - \tau')) = \lambda(\eta \cdot r - \eta \cdot r')$$

(A.3)

since we have here that $\tau - \tau' > 0$ implies $\lambda(\eta \cdot X - \eta \cdot X') > 0$, which is due to the fact that $\lambda \eta \cdot \dot{X} > 0$ [$\eta \cdot \dot{X} \neq 0$ and $\text{sign}(\eta \cdot \dot{X}) = \text{sign}(\eta \cdot a_0)$ (consider $\eta \cdot A_0$)]. Formula 1 is thereby proved.

The following formulae follow from formula 1:

Formula 2:

$$i \sum_{m \neq 0} \frac{1}{m} A_m^\mu A_{n-m}^\nu = \frac{1}{2\pi} \int_0^{2\pi} d\tau \dot{X}^\nu e^{in\lambda \eta \cdot X} (r^\mu - a_0^\mu \lambda \eta \cdot r) + i \sum_{m \neq 0} \frac{\lambda}{m} a_{-m}^\mu \eta \cdot a_m A_n^\nu \quad (\text{A.4})$$

Formula 3:

$$i \sum_{m \neq 0} \frac{1}{m} A_m^\mu A_{-m}^\nu = i \sum_{m \neq 0} \frac{1}{m} a_m^\mu a_{-m}^\nu + \lambda i \sum_{m \neq 0} \frac{\eta \cdot a_m}{m} (a_{-m}^\mu a_0^\nu - a_{-m}^\nu a_0^\mu) \quad (\text{A.5})$$

PROOF OF THE RELATION (2.35)

$$\begin{aligned}
 A_n^\mu(\eta; Q^\nu(\xi), A_m^\nu(\xi)) &= \\
 &= \frac{1}{2\pi} \int_0^{2\pi} d\tau \sum_k A_k^\mu(\xi) e^{-ik\tau} \exp\{in\lambda(\eta \cdot Q(\xi) + \\
 &\quad + (\eta \cdot A_0)\tau + i \sum_{s \neq 0} \frac{1}{s} \eta \cdot A_s(\xi) e^{-is\tau})\} = \\
 &= |\bar{\lambda} \equiv (\xi \cdot a_0)^{-1}| = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} d\tau d\tau' \dot{X}^\mu(\tau') \delta(\bar{\lambda} \xi \cdot X(\tau') - \\
 &\quad - \tau) e^{in\lambda(\dots)} = \frac{1}{2\pi} \int_0^{2\pi} d\tau \dot{X}^\mu(\tau) \exp\{in(\lambda \eta \cdot Q(\xi) + \\
 &\quad + \bar{\lambda} \xi \cdot X(\tau) + i \lambda \sum_{s \neq 0} \frac{1}{s} \eta \cdot A_s(\xi) e^{-is\bar{\lambda} \xi \cdot X(\tau)})\}
 \end{aligned}
 \tag{B.1}$$

$$\begin{aligned}
 i \sum_{s \neq 0} \frac{1}{s} \eta \cdot A_s(\xi) e^{-is\bar{\lambda} \xi \cdot X(\tau)} &= \\
 &= \frac{i}{2\pi} \int_0^{2\pi} d\tau' \eta \cdot \dot{X}(\tau') \sum_{s \neq 0} \frac{1}{s} e^{is\bar{\lambda} (\xi \cdot X(\tau') - \xi \cdot X(\tau))} = \\
 &= \frac{1}{2\pi} \int_0^{2\pi} d\tau' \eta \cdot \dot{X}(\tau') \left(i \sum_{s \neq 0} \frac{1}{s} e^{is(\tau' - \tau)} + \bar{\lambda} \xi \cdot r(\tau') - \right. \\
 &\quad \left. - \bar{\lambda} \xi \cdot r(\tau) \right) = \eta \cdot r(\tau) - \lambda^{-1} \bar{\lambda} \xi \cdot r(\tau) + \\
 &\quad + \bar{\lambda} i \sum_{m \neq 0} \frac{1}{m} \eta \cdot a_{-m} \xi \cdot a_m
 \end{aligned}
 \tag{B.2}$$

where the second equality follows from (A.3).

$$\begin{aligned}
 & \lambda \eta \cdot Q(\xi) + \bar{\lambda} \xi \cdot X(\tau) = \\
 & = \lambda \eta \cdot q + \tau + \bar{\lambda} \xi \cdot r(\tau) + \\
 & + \bar{\lambda} \lambda i \sum_{m \neq 0} \frac{1}{m} \eta \cdot a_m \xi \cdot a_{-m}
 \end{aligned}
 \tag{B.3}$$

The relation (2.35) is now proved by inserting (B.2) and (B.3) into (B.1) and to compare with (2.25).

THE POISSON ALGEBRA OF THE STRING IN ANY ON-GAUGE

Since the relativistic string is a system subjected to constraints (a gauge theory), it is convenient to use Dirac's method for constrained Hamiltonian systems ⁷⁾. The reader unfamiliar with this method is also referred to Ref. 8) where a rather extensive treatment of the string is given.

The conventional definition of canonical momenta leads to the following Poisson bracket (PB) for the normal modes in (2.7):

$$\begin{aligned} \{a_0^\mu, q^\nu\} &= g^{\mu\nu} \\ \{a_n^\mu, a_m^\nu\} &= i n \delta_{n,-m} g^{\mu\nu} \end{aligned} \quad (C.1)$$

This PB is, however, inconsistent with the constraints (2.11) and (2.13), the reason being that the phase space has too many degrees of freedom. On the other hand, the unique correct PB can be expressed in terms of this PB by means of the so-called Dirac bracket ^{7),8)}. Following Ref. 5), we obtain the general formula

$$\begin{aligned} \{\alpha|_\eta, \beta|_\eta\}^* &= \{\alpha, \beta\}|_\eta + \\ &+ i\lambda \sum_{n \neq 0} \frac{1}{n} (\{\alpha, L_n\} \lambda \eta^2 \{L_{-n}, \beta\} + \\ &+ \{\alpha, L_{-n}\} \{\eta \cdot a_n, \beta\} + \{\alpha, \eta \cdot a_{-n}\} \{L_n, \beta\})|_\eta + \\ &+ \lambda (\{\alpha, L_0\} \{\eta \cdot q, \beta\} - \{\alpha, \eta \cdot q\} \{L_0, \beta\})|_\eta \end{aligned} \quad (C.2)$$

where α and β are two dynamical variables defined on the conventional phase space (C.1), and where $|_{\eta}$ means evaluated on the hypersurface determined by the constraints (2.11) and (2.13). The condition $\lambda \eta \cdot \dot{X}(\tau) \neq 0$ requires η^μ to be a fixed time- or light-like vector, or $\eta^\mu = a_0^\mu = \sqrt{2\alpha'} P^\mu$ which is a dynamical gauge choice which we shall call the proper time gauge.

In a fixed time- or light-like gauge, we obtain the following PB algebra from (C.2)

$$\{a_0^\mu, q_\eta^\nu\}^* = g^{\mu\nu} - \lambda a_0^\nu \eta^\mu$$

$$\{a_0^\mu, a_{\eta n}^\nu\}^* = i n \lambda \eta^\mu a_{\eta n}^\nu$$

$$\{F^{\mu\nu}, q_\eta^\lambda\}^* = q_\eta^\mu g^{\nu\lambda} - q_\eta^\nu g^{\mu\lambda} +$$

$$+ \lambda a_0^\lambda (q_\eta^\nu \eta^\mu - q_\eta^\mu \eta^\nu) +$$

$$+ i \lambda \sum_{n \neq 0} \frac{1}{n} a_{\eta n}^\lambda (a_{\eta, -n}^\mu \eta^\nu - a_{\eta, -n}^\nu \eta^\mu)$$

$$\{F^{\mu\nu}, a_{\eta n}^\lambda\}^* = a_{\eta n}^\mu g^{\nu\lambda} - a_{\eta n}^\nu g^{\mu\lambda} +$$

$$+ i n \lambda a_{\eta n}^\lambda (q_\eta^\mu \eta^\nu - q_\eta^\nu \eta^\mu) +$$

$$+ n \lambda \sum_{m \neq 0} \frac{1}{m} a_{\eta, n-m}^\lambda (a_{\eta m}^\nu \eta^\mu - a_{\eta m}^\mu \eta^\nu)$$

(C.3)

which expresses the transformation properties of the string that are non-covariant due to the non-covariant choice of parameters σ and τ (2.10); this in turn induces a change of these along with any Poincaré transformation^{2),10)}. We have, furthermore,

$$\begin{aligned}
 \{q_\eta^\mu, q_\eta^\nu\}^* &= i\lambda^2 \eta^2 \sum_{m \neq 0} \frac{1}{m} a_{\eta, -m}^\mu a_{\eta, m}^\nu \\
 \{q_\eta^\mu, a_{\eta n}^\nu\}^* &= \lambda \eta^\nu a_{\eta n}^\mu - n\lambda^2 \eta^2 \sum_{m \neq 0} \frac{1}{m} a_{\eta m}^\mu a_{\eta, n-m}^\nu, n \neq 0 \\
 \{a_{\eta n}^\mu, a_{\eta m}^\nu\}^* &= i n \delta_{n, -m} g^{\mu\nu} - i n \lambda \eta^\nu a_{\eta, n+m}^\mu \\
 &\quad + i m \lambda \eta^\mu a_{\eta, n+m}^\nu + i \lambda^2 \eta^2 n m \sum_{k \neq 0} \frac{1}{k} a_{\eta, n+k}^\mu a_{\eta, m-k}^\nu
 \end{aligned}
 \tag{C.4}$$

In the proper time gauge we have

$$\begin{aligned}
 \{a_0^\mu, q_P^\nu\}^* &= g^{\mu\nu} - \lambda a_0^\mu a_0^\nu \\
 \{a_0^\mu, a_{Pn}^\nu\}^* &= i n \lambda a_0^\mu a_{Pn}^\nu \\
 \{\mathcal{F}^{\mu\nu}, q_P^\lambda\}^* &= q_P^\mu g^{\nu\lambda} - q_P^\nu g^{\mu\lambda} \\
 \{\mathcal{F}^{\mu\nu}, a_{Pn}^\lambda\}^* &= a_{Pn}^\mu g^{\nu\lambda} - a_{Pn}^\nu g^{\mu\lambda}
 \end{aligned}$$

(C.5)

and

$$\{q_p^\mu, q_p^\nu\}^* = \lambda F^{\mu\nu}$$

$$\{q_p^\mu, a_{pn}^\nu\}^* = \lambda a_0^\nu a_{pn}^\mu - i n \lambda q_p^\mu a_{pn}^\nu$$

$$\{a_{pn}^\mu, a_{pm}^\nu\}^* = i n \delta_{n,-m} g^{\mu\nu} -$$

$$- i n \lambda a_0^\nu a_{p,n+m}^\mu + i m \lambda a_0^\mu a_{p,n+m}^\nu +$$

$$+ \lambda m n i \sum_{k \neq 0} \frac{1}{k} a_{p,n+k}^\mu a_{p,m-k}^\nu$$

(C.6)

The string here is transforming covariantly under homogeneous Lorentz transformations, but not under translations, since we have fixed the zero point of the proper time τ [see (2.10)].

Notice that the relations (C.3) and (C.4) for the appropriate light-like η^μ reduce to the PB relations of Ref. 2) and that they are equivalent to the PB relations of Ref. 8) for any η^μ . Notice also that the above PB relations become singular for any light-like gauge when $P^2 = 0$ and when P^μ and η^μ are parallel, and for the proper time gauge when $P^2 = 0$, all due to the definition of λ (2.21).

For gauge invariant functions γ_i , we may define a consistent PB in the following fashion ⁹⁾

$$\{\gamma_i, \gamma_j\} / \nu \quad (C.7)$$

i.e., by evaluating the conventional PB (C.1) on the Virasoro surface. This is a consistent definition, since

$$\{L_n, L_m\}|_V = \{\gamma_i, L_n\}|_V = 0$$

any integers n and m (C.8)

When one calculates the PB algebra of the gauge invariant functions $J^{\mu\nu}$, $A_n^\mu(\eta)$ and $Q^\mu(\eta)$, in terms of the PB (C.7), one obtains exactly (C.3)-(C.6) when replacing $A_n^\mu(\eta) \rightleftharpoons a_{\eta n}^\mu$ and $Q^\mu(\eta) \rightleftharpoons q_n^\mu$, which is consistent with (2.24) and (2.26). Furthermore, due to (C.8) and the definition (C.2) we have

$$\{\gamma_i, \gamma_j\}|_V = \{\gamma_i, \gamma_j\}|_\eta = \{\gamma_i|_\eta, \gamma_j|_\eta\}^* \quad (C.9)$$

where $\gamma_i = J^{\mu\nu}$, $A_n^\mu(\eta)$, or $Q^\mu(\eta)$. This, in turn, is consistent with (2.28) and (2.29). Equation (C.9) tells us that the Poincaré algebra is always satisfied.

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