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ON SELF-INTERACTING VECTOR FIELDS

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1. Let us consider the following Lagrangian density [Velo's Lagrangian¹⁾]:

$$\mathcal{L} = -\frac{1}{4} R_a^{\mu\nu} R_{a\mu\nu} + \frac{m_a^2}{2} V_{a\mu} V^{a\mu} \quad (1)$$

In Eq. (1) the functions $V_{a\mu}$ are components of n four-vectors ($a = 1, 2, \dots, n$) ($\mu, \nu = 0, 1, 2, 3$); m_a are n constants (masses); $R_a^{\mu\nu}$ are given by

$$R_a^{\mu\nu} = F_a^{\mu\nu} - f_{abc} V_b^\mu V_c^\nu, \quad (2)$$

where f_{abc} are coupling constants having the property

$$f_{abc} = -f_{acb} \quad (3)$$

and

$$F_a^{\mu\nu} = \partial^\mu V_a^\nu - \partial^\nu V_a^\mu \quad (2')$$

$$(\underline{V}^2 = V_\mu V^\mu = V_0^2 - (V^k)^2 = V_0^2 - \vec{V}^2; \quad \partial^\mu = \frac{\partial}{\partial x_\mu} \text{ and so on};$$

$$k = 1, 2, 3$$

From the Lagrangian (1) the Lagrange equations of motion follow:

$$E_a^\mu = \partial_\rho R_a^{\rho\mu} + f_{bca} V_{c\rho} R_b^{\rho\mu} + m_a^2 V_a^\mu = 0. \quad (4)$$

In the particular case in which $n = 2$, $m_1 = m_2$, and

$$\left. \begin{aligned} f_{1,1,2} &= -f_{1,2,1} = f \\ f_{2bc} &= 0 \end{aligned} \right\}, \quad (5)$$

the equations of motion (4) admit the following static solution²⁾:

$$\left. \begin{aligned} V_{10} &= \Phi(r) \\ V_{1k} &= 0 \end{aligned} \right\} \quad \left. \begin{aligned} V_{20} &= 0 \\ V_{2k} &= \Psi(r) \frac{x_k}{r} \end{aligned} \right\} \quad (6)$$

where

$$r^2 = -x_k x^k$$

and

$$\left. \begin{aligned} \Phi(2) &= \frac{1}{f} \frac{1}{r} + \psi(r, m) \\ \Psi(2) &= -\frac{1}{f} \frac{1}{r} + \chi(r, m) \end{aligned} \right\} \quad (6')$$

where $\psi(r,m)$ and $\chi(r,m)$ are regular (indeed zero) for $r = 0$ and are going to zero for any value of r for m going to zero. We had remarked in paper I that for the limiting case $m = 0$ the strength of the field is inversely proportional to the coupling constant f .

It is perhaps important to point out that this holds even for $m \neq 0$. For this purpose it is sufficient to perform the substitution

$$\begin{aligned}\Phi(r) &= \frac{u(r)}{f} \\ \Psi(r) &= \frac{w(r)}{f}\end{aligned}\tag{7}$$

in the equation of motion, and one obtains equations for the function $u(r)$ and $w(r)$ which no longer depend on f . The procedure can be generalized to any Velo's Lagrangian equations³⁾ by putting

$$V_{a\mu} = \frac{u_{a\mu}}{f}\tag{7'}$$

where, for instance,

$$f = \sqrt{\frac{1}{2}f_{abc} f_{abc}}.\tag{7''}$$

The general fact, that the strength of the fields are, for Velo equations, inversely proportional to the strength of coupling, might acquire a considerable physical importance when considering Velo fields, coupled with other fields (for instance, gravitation for measuring the energy) and "soliton type" solutions of the equation of motion.

It is interesting to point out that this result is only due to the imposition of causality (see Ref. 1).

2. In a private discussion with T.D. Lee and G.C. Wick, it was pointed out that, in order to be correct, a singular solution of the type (6') might probably necessitate the introduction of a point external source in the Lagrangian.

This is actually not true, but I have to admit that the thing was far from being clear on the basis of paper I.

I therefore think that here it is necessary to clarify the point.

Taking into account the equations (6) (putting $f = 1$, $f_{1,2,1} = -1$), one easily obtains that in our case the equations of motion [see Eq. (4)] are

$$\begin{aligned} E_1^0 &= \partial_k R_1^{k0} - V_{2k} R_1^{k0} + m^2 V_1^0 = 0 \\ E_2^k &= V_{1,0} R_1^{0k} + m^2 V_2^k = 0 \end{aligned} \quad (8)$$

and the other equations turn out to be identities.

The tricky term for our question is

$$\partial_k R_1^{k0}.$$

Let us make the convention

$$\vec{r} \rightarrow (x^k)_k = 1, 2, 3. \quad (9)$$

Then

$$\left. \begin{aligned} (\partial_k) &\rightarrow \vec{\nabla} \\ (\partial^k) &\rightarrow -\vec{\nabla} \end{aligned} \right\} \quad (9')$$

Taking into account the equations (6) and (6'), we have

$$\partial_k R_1^{k0} = \vec{\nabla} \left[-\vec{\nabla} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\vec{r}}{r^2} \right] \quad (10)$$

plus terms which cannot give rise to trouble if $\psi(r,m)$ and $\chi(r,m)$ are regular functions near $r = 0$.

Now by applying standard procedures of the theory of distribution it is easily seen that

$$\vec{\nabla} \left[\vec{\nabla} \left(\frac{1}{r} \right) + \frac{1}{r} \frac{\vec{r}}{r^2} \right]$$

is zero even for $r = 0$. Therefore, there are not $\partial(\vec{r})$ terms.

All of course depends on the fact that $\psi(r,m)$, $\chi(r,m)$ are regular near the origin. This can be proved. The proof is, however, far from being obvious because the equation which $\psi(r,m)$ satisfied is very complicated and highly singular near the origin. The result can be achieved with a semi-standard method, but the procedure is quite long and I cannot give it here for reasons of space.

I shall merely say that after proving the existence theorem for $\psi(r,m)$ (regular near the origin), it is a trivial matter to prove that $\psi(r,m)$ is going to zero for any value of r when m is going to zero.

It is a pleasure to thank T.D. Lee, G. Velo and G.C. Wick for fruitful discussions, and the Theoretical Studies Division of CERN for its kind hospitality which has made these discussions possible.

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REFERENCES AND FOOTNOTES

- 1) G. Velo, Nuclear Phys. B65, 427 (1973).
- 2) B. Ferretti and G. Velo, Nuovo Cimento Letters 1D, 451 (1974), quoted in the following paper I.
- 3) I owe to Velo the remark that the transformation (7) could immediately be generalized to the transformation (7') because of the structure of his equations.