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SOME COMMENTS ON THE SPHERICITY OF MULTIPARTICLE
STATES IN e^+e^- COLLISIONS

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ABSTRACT

The deviations from sphericity in e^+e^- hadrons collisions, due to various effects, are calculated analytically. The effect of statistical fluctuations on a previous definition of sphericity is especially strong, and a modified definition is given to take this into account. After this is done, jets of a size typical to hadronic reactions should be clearly visible and cannot be confused with trivial other effects.

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1. INTRODUCTION

Most conventional theories on high-energy e^+e^- annihilation into hadrons can be classified into two categories ¹⁾. In the first, particles are produced in jets resembling the jets observed in normal hadronic interactions, with $\langle p_\perp \rangle \approx 350$ MeV and logarithmically rising multiplicity, but with the jet axis oriented randomly ²⁾. The latter follows from the one-photon intermediate state, which allows -- for unpolarized electrons -- only an angular dependence $\propto 1 + a \cos^2 \theta$.

In contrast to this, the second group of models predicts roughly spherical events with momenta distributed according to phase space or similar to transverse momenta in hadronic scattering, and with multiplicities rising proportional to E_{cm} ³⁾.

Present experimental data ^{4),5)} strongly favour jet models, as indicated in particular by three facts:

- i) the mean charged multiplicity rises much slower than the available energy;
- ii) there is an excess of fast particles as compared to phase space models;
- iii) a quantity called "sphericity", defined in Ref. 4), deviates significantly from Monte-Carlo estimates based on isotropic phase space, but agrees with calculations based on a jet model ⁴⁾.

In spite of this evidence, we consider it worth while to reconsider the problem for several reasons. Firstly, several facts indicate that a substantial fraction of final states in e^+e^- are not normal hadronic states and after removing them, the "normal" states might behave quite differently. Secondly, there is also a slight indication for large momenta perpendicular to the jets ⁴⁾. Finally -- and this is our main motivation -- the three experimental findings mentioned above are not independent. For the first two, this is obvious. As is less clear but will be shown later, the definition of sphericity used in Ref. 4) is also directly sensitive to an excess of fast particles, even if they do not originate from jets. (We should stress that there is no logical necessity for fast particles to be produced in jets, although most models do assume this.) It is the aim of the present paper to give a definition of (a-) sphericity which is less sensitive to the particle spectrum (and thus more sensitive to genuine jets), and to estimate various other effects -- as for instance clustering -- that can produce deviations from sphericity.

To obtain reliable numerical estimates, one has of course to do Monte Carlo calculations, taking also into account detection efficiency, resolution, etc. This is not what we shall do. Rather, to get a feeling for the importance of the different effects, we shall perform analytic calculations in those limits where they are feasible.

In Section 2, we define our observables measuring the sphericity. The main difference to the observable defined in Ref. 4) is that the contribution of statistical fluctuations to the asymmetry of individual events is eliminated. The influence of various effects on these observables is then studied in the following sections: energy-momentum conservation in Section 3, resonance production in Section 4, anisotropy of the single-particle distribution (and detection efficiency) in Section 5, and finally jet structure in Section 6.

2. DEFINITIONS AND STATISTICAL FLUCTUATIONS

In Ref. 4), for each event a 3×3 tensor

$$T_{\alpha\beta} = \sum_{i=1}^n \left(\vec{p}_i^2 \delta_{\alpha\beta} - p_{i\alpha} p_{i\beta} \right) \quad (1)$$

is constructed out of the momenta of the observed particles. The eigenvalues of this tensor are called λ_i . For spherical events, $\lambda_1 = \lambda_2 = \lambda_3$. Deviations from this are measured by the sphericity

$$S_p = \frac{3 \lambda_{\min}}{\lambda_1 + \lambda_2 + \lambda_3} \quad (2)$$

Clearly, $0 \leq S_p \leq 1$, with $S_p = 0$ corresponding to a perfect jet, and $S_p = 1$ to a perfect sphere.

Following Bjorken and Brodsky ¹⁾, let us use a slightly different traceless tensor

$$\tilde{T}_{\alpha\beta} = \sum_{i=1}^n \left(p_{i\alpha} p_{i\beta} - \frac{1}{3} \vec{p}_i^2 \delta_{\alpha\beta} \right) \quad (3)$$

Its eigenvalues can be calculated from the invariants

$$\text{tr}(\tilde{T}^2) = \sum_{i,k} [(\vec{p}_i \cdot \vec{p}_k)^2 - \frac{1}{3} \vec{p}_i^2 \vec{p}_k^2] \quad (4)$$

and $\text{tr}(\tilde{T}^3)$, which can be similarly expressed as a triple sum. The effect of statistical fluctuations is particularly clearly seen in Eq. (4): in the absence of correlations ^{*)} and for a spherically symmetric single-particle density, one has

$$\langle (\vec{p}_i \cdot \vec{p}_k)^2 \rangle = \frac{1}{3} \langle \vec{p}_i^2 \cdot \vec{p}_k^2 \rangle \quad \text{for } i \neq k \quad (5)$$

so that the expectation value of $\text{tr}(\tilde{T}^2)$ is reduced to the partial sum with $i = k$,

$$\langle \text{tr}(\tilde{T}^2) \rangle = \frac{2}{3} \langle \sum_i (\vec{p}_i^2)^2 \rangle \neq 0 \quad (6)$$

Similarly, one finds in this case for the trace of $\tilde{T}^3$

$$\langle \text{tr}(\tilde{T}^3) \rangle = \frac{2}{9} \langle \sum_i (\vec{p}_i^2)^3 \rangle \quad (7)$$

Notice that for spherical events, both these expectation values would be zero. Their non-zero values just say that spherical distributions correspond, via statistical fluctuations, to non-spherical events.

With the help of Eqs. (6) and (7), one can make an asymptotic estimate of the mean values of the eigenvalues λ_i entering into Eq. (2), provided one has a model for the single-particle distribution. Let us assume that it is just according to phase space. For a fixed number of n particles, we calculated the average sphericity

$$(S_\rho)_{av} = \frac{3 \langle \lambda_{min} \rangle}{\langle \lambda_1 + \lambda_2 + \lambda_3 \rangle} \quad (8)$$

*) even of kinematical ones!

in the non-relativistic resp. extremely relativistic limit. The way of calculating is sketched in the Appendix, and the result is shown in Fig. 1. One sees that the effect is quite appreciable, and does not vanish rapidly when $n \rightarrow \infty$. Indeed, for large n , it is proportional to $1/\sqrt{n}$, reminiscent of its statistical origin:

$$(S_p)_{av} \approx 1 - \frac{1.07}{\sqrt{n}} \quad (\text{non-rel.}) \quad (9a)$$

$$(S_p)_{av} \approx 1 - \frac{1.52}{\sqrt{n}} \quad (\text{extremely rel.}) \quad (9b)$$

At the highest SPEAR energy (corresponding to $n \approx 8$), the measured sphericity was ~ 0.3 , while the estimate based on phase space was ~ 0.4 . Thus the above is an essential contribution, and a wrong model for the single-particle spectrum (which a priori need not be related to jets) might have induced a considerable error. This is particularly important since the observed excess of fast particles is heavily weighted in Eqs. (6) and (7).

Therefore, we propose to simply drop the partial sum with $i=k$ in Eq. (4), and similarly all partial sums with at least two equal indices in $\text{tr}(\tilde{T}^3)$. The meaning of this becomes particularly transparent if we write the tensor $\tilde{T}$ in "second quantized" form, using the particle density operator $n(\vec{p}) = a^\dagger(\vec{p})a(\vec{p})$ (we use covariant normalization and $d^3p = d^3p/2p_0$):

$$\tilde{T}_{\alpha\beta} = \int d^3p \, n(\vec{p}) (p_\alpha p_\beta - \frac{1}{3} \vec{p}^2 \delta_{\alpha\beta}) \quad (3')$$

Our proposal corresponds in this language simply to replacing $\tilde{T}^2$ and $\tilde{T}^3$ by the normal-ordered operators $:\tilde{T}^2:$ and $:\tilde{T}^3:$.

Since inclusive densities are just expectation values of normal-ordered powers of the particle density,

$$\rho(\vec{p}, \vec{k}) = \langle : n(\vec{p}) n(\vec{k}) : \rangle \quad (10)$$

and

$$\rho(\vec{p}, \vec{k}, \vec{q}) = \langle : n(\vec{p}) n(\vec{k}) n(\vec{q}) : \rangle \quad (11)$$

this leads immediately to the connection with inclusive quantities. One finds easily

$$\langle : t_2(\tilde{T}^2) : \rangle = \int d^3p d^3k \rho(\vec{p}, \vec{k}) [(\vec{p}, \vec{k})^2 - \frac{1}{3} \vec{p}^2 \vec{k}^2] \quad (12)$$

and

$$\begin{aligned} \langle : t_2(\tilde{T}^3) : \rangle &= \int d^3p d^3k d^3\vec{q} \rho(\vec{p}, \vec{k}, \vec{q}) * \\ &* [(\vec{p}, \vec{k})(\vec{k}, \vec{q})(\vec{p}, \vec{q}) - (\vec{p}, \vec{k})\vec{q}^2 + \frac{2}{9} \vec{p}^2 \vec{k}^2 \vec{q}^2] \end{aligned} \quad (13)$$

Except for the normalization, these are already our wanted observables. To normalize them in a suitably way, we define an operator

$$I = \int d^3p n(\vec{p}) \vec{p}^2 \quad (14)$$

The asphericity of the final state distribution is then defined as

$$\begin{aligned} A &= \frac{3 \langle : t_2(\tilde{T}^2) : \rangle}{2 \langle : I^2 : \rangle} = \\ &= \frac{\int d^3p d^3k [3(\vec{p}, \vec{k})^2 - \vec{p}^2 \vec{k}^2] \rho(\vec{p}, \vec{k})}{2 \int d^3p d^3k \vec{p}^2 \vec{k}^2 \rho(\vec{p}, \vec{k})} \end{aligned} \quad (15)$$

and the prolateness as

$$P = \frac{9 \langle : t_2(\tilde{T}^3) : \rangle}{2 \langle : I^3 : \rangle} \quad (16)$$

Notice that we measure in this way only the shape of the distributions, not of single events. In our opinion this is reasonable, as the shape of single events will be strongly influenced by statistical fluctuations which have nothing directly to do with jets.

The names asphericity and prolateness become clear by looking at special examples: a spherical distribution has $A=P=0$; a perfect jet ^{*)} has $A=P=1$, while a perfect disc has $A=1/4$, $P=-1/8$. Quite generally, a prolate distribution has $P>0$, while an oblate one has $P<0$. Notice that the three-particle distribution is necessary to distinguish a jet-like from a disc-like structure (unless A is close to 1), while two-body correlations alone are sufficient to measure deviations from sphericity.

Finally, we should stress that these two observables are the most important ones only if there is no strong anisotropy, as would be the case, e.g., in hadronic collisions. In that case, if a certain direction $\vec{e}$ is distinguished (the beam direction, say), observables like $(e, \tilde{T}e)$ or $:(e, \tilde{T}^2 e):$ contain as important information as A and P .

In the following, we shall consider various contributions to A and P . Surprisingly enough, it will turn out that most effects (except genuine jets) should lead to rather small values for both of them.

3. KINEMATICAL CORRELATIONS

Energy momentum conservation implies that particles cannot be emitted completely uncorrelated. The effect of this on asphericity and prolateness can be estimated assuming pure phase space distributions. Relativistic phase space can, in general, not be handled analytically, and therefore we shall restrict ourselves to the non-relativistic and extremely relativistic limits considered also above. Realistic estimates will fall between, and can be done by Monte-Carlo calculations taking also into account experimental acceptance etc. For fixed particle number n , the calculation sketched in the Appendix gives

$$A = \frac{5}{3(n-1)^2 + 2} \quad (\text{non-rel.}) \quad (17a)$$

$$A = \frac{20}{3n(n+1) + 2} \quad (\text{extremely rel.}) \quad (17b)$$

*) By which we mean a jet with all momenta strictly antiparallel.

and

$$P = - \frac{35}{9n(n^2 - 3n + 5) - 35} \quad (\text{non-rel}) \quad (18a)$$

$$P = - \frac{280}{9n(n^2 + 6n + 13) - 80} \quad (\text{ext. rel.}) \quad (18b)$$

This is also shown in Fig. 2. Two interesting remarks are in order: firstly for large n , $A \propto 1/n^2$ and $P \propto 1/n^3$, corresponding to a contribution $\propto 1/n$ to the sphericity as defined in Eq. (2). Remember that statistical fluctuations gave to the latter a contribution $\propto 1/\sqrt{n}$. If we had not done the normal ordering in Eq. (15), these fluctuations would have led to the much bigger "event" asphericities also shown in Fig. 2. The second remark is that P is negative, showing that energy momentum conservation tends to produce oblate distributions, certainly not jets.

4. CLUSTERING AND RESONANCE PRODUCTION

Since clusters are very vaguely defined objects, clustering clearly could lead to rather odd shapes. As an extreme, production of exactly two clusters might not at all be distinguishable (even in principle!) from jet production. Thus reliable estimates are not very easy. Qualitatively, one can distinguish several extreme cases:

- i) production of light clusters, produced with large momenta but decaying with small Q values. This leads to several "jets" of nearly parallel particles (a similar effect is expected from Bose statistics), and therefore tends to make A positive;
- ii) in the other extreme, when very massive clusters with small momenta are produced, one also expects positive A , since in this case one gets kinematic correlations corresponding not to the observed particle number but to the number of particles per cluster.

In between these two extremes, when the cluster momentum is comparable to the relativistic momenta of its decay products, one should however expect a negative contribution to A . This is easily seen for a two-particle decay, looking at Eq. (15): for fixed cluster mass and momentum $\vec{p} + \vec{k}$, $\vec{p}^2 \cdot \vec{k}^2$ will be maximal when the relative momentum $\vec{p} - \vec{k}$ is orthogonal to $\vec{p} + \vec{k}$, in which case $\vec{p} \cdot \vec{k}$ is however small.

To check this quantitatively, we have studied the simple model of a two-body resonance with fixed mass M and energy distribution $\rho(E) \propto e^{-ET}$ (T is an effective temperature, and should be $T^{-1} \sim 150$ MeV), decaying isotropically. The asymmetry of the two pions produced in this way is shown in Fig. 3, as a function of T and for two typical resonance masses. The expected behaviour is found, and moreover one sees that for typical hadronic temperatures the asymmetry is very small. To show that this effect is not specific for two-body clusters (although it is most pronounced there), let us consider the asphericity of a moving cluster in the non-relativistic limit. Let us assume that the decay products of the cluster, which move with a fixed c.m. momentum $\vec{P}$, have isotropic distributions $\rho_{cl}(p^2)$ and $\rho_{cl}(\vec{p}, \vec{k})$ in the cluster rest frame. Then the numerator in the definition Eq. (15) of the asymmetry becomes

$$\begin{aligned} & \int d^3p d^3k \rho_{cl}(\vec{p} + \vec{P}, \vec{k} + \vec{P}) [\vec{p} \cdot \vec{k} - \vec{P} \cdot \vec{k}] = \\ & \approx \int d^3p d^3k \rho_{cl}(p, k) [\vec{p} \cdot \vec{k} - \vec{P} \cdot \vec{k}] - \frac{20}{3} \int d^3p \vec{p}^2 \rho_{cl}(p) \vec{P}^2 \\ & + 2 \langle n(n-1) \rangle_{cl} \vec{P}^4 \end{aligned}$$

From this we see that, if we increase the cluster momentum $\vec{P}$, keeping ρ_{cl} fixed, the asymmetry goes through a minimum and may well become negative. It should be pointed out that in Fig. 3 the asymmetry of two particles known to arise from a common resonance is shown. The over-all effect is of course much smaller, being proportional to the chance $(\langle n_{cl} \rangle - 1)/(n-1)$ that two arbitrary pions stem from the same resonance ($\langle n_{cl} \rangle$ is the mean number of particles into which the resonance decays). This latter remark is even more relevant for the prolateness, where the clustering effect is proportional to the chance that a triplet of pions entirely stems from one cluster.

In conclusion, we estimate the contribution to A from clustering to be very small (smaller than ~ 0.1), for clusters of typical size ($M \sim 1-2$ GeV, 3-4 particles/clusters), and the contribution to P to be even much smaller.

5. ANISOTROPY OF SINGLE-PARTICLE SPECTRA

All previous estimates [as well as the Monte Carlo calculations in Ref. 4)] were done under the assumption that single-particle spectra are isotropic. At the highest SPEAR energies, this is, however, not true^{4),5)}.

Denoting the angle between the produced particle and the beam by θ , and its azimuthal angle (with respect to the horizontal plane) by φ , the most general density compatible with a one-photon intermediate state is ⁴⁾

$$\rho(\vec{p}) = a(p^2) + b(p^2) (\cos^2 \theta + P^2 \sin^2 \theta \cos 2\varphi) \quad (19)$$

Here P is the beam polarization (which rises sharply at the highest SPEAR energy), and $a(p^2)$ and $b(p^2)$ are arbitrary functions constrained only by $|b(p^2)| \leq a(p^2)$.

In quark parton models, a non-vanishing $b(p^2)$ is predicted together with jets. Experimentally, it is indeed positive, which one might therefore take also as a hint for jets. On the other hand, even if this is not true (and a priori this is quite possible), an anisotropy of the single-particle distribution clearly implies that the average shape of events will not be spherical. To estimate its effect on A and P in the absence of genuine jets, we assume

$$\rho(\vec{p}, \vec{k}) = \rho(\vec{p}) \rho(\vec{k}) \quad (20)$$

and similarly for the three-body density. Defining

$$\alpha = \frac{\int d^3 p \, b(p^2) p^4}{\int d^3 p \, a(p^2) p^2} \quad (21)$$

we get after a straightforward calculation

$$A = \left[\frac{2\alpha}{5(3+\alpha)} \right]^2 (1 + 3P^2) \quad (22)$$

and

$$P = \left[\frac{2\alpha}{5(3+\alpha)} \right]^3 (1 - 9P^2) \quad (23)$$

Since $|\alpha| \leq 1$, and experimentally $\alpha > 0$, we get

$$\alpha \leq \frac{1+3P^2}{100} \quad |P|^2 \leq \left| \frac{1-9P^2}{1000} \right| \quad (24)$$

which represents a surprisingly small effect. Furthermore, while A increases with the $e^+ - e^-$ polarization (and thus with energy around $E \sim 7$ GeV), the prolateness decreases, making the shape disc- rather than jet-like [which might of course already have been guessed from Eq. (19)].

A similar anisotropy of the single-particle distribution arises from the fact that the detector does not, in general, cover the full solid angle. In the LBL-SPEAR detector, for instance, particles with $|\cos \theta| < a$, $a = \cos \theta_0 \approx 1/\sqrt{2}$, are not observed. The effective single-particle density thus becomes

$$\tilde{\rho}(\vec{p}) = \rho(\vec{p}) \Theta(a^2 - \cos^2 \theta) \quad (25)$$

A simple calculation then gives, in the case of an isotropic $\rho(\vec{p})$,

$$A = \left(\frac{\sin \theta_0}{\sqrt{2}} \right)^4 \approx \frac{1}{16} \quad (26)$$

and

$$P = - \left(\frac{\sin \theta_0}{\sqrt{2}} \right)^6 \approx - \frac{1}{64} \quad (27)$$

Again, this is a small effect, producing furthermore also a disc- rather than a jet-like structure. The situation can, however, change quite remarkably if this shape of the detector interferes with a marked anisotropy and a strong beam polarization. Qualitatively, this is easy to see while the detector cuts out a "disc" perpendicular to the beams, Eq. (17) describes for $P \approx 1$ a "disc" in a plane including the beam axis. The net result will then resemble a jet in the intersection of these two discs. A somewhat lengthy calculation indeed shows

$$A = \frac{1}{4(1 + \frac{\cos^2 \theta_0}{3} \alpha)} \left[\left(\sin^2 \theta_0 - \frac{\cos^2 \theta_0 (9 \cos^2 \theta_0 - 5) \alpha}{15} \right)^2 + \frac{3}{4} \alpha^2 P^4 \left(1 - \frac{2}{3} \cos^2 \theta_0 + \frac{1}{5} \cos^4 \theta_0 \right) \right] \approx$$

$$\approx \frac{1}{16(1 + \alpha/6)^2} \left[\left(1 + \frac{\alpha}{30} \right)^2 + \frac{1849}{1200} \alpha^2 P^4 \right]$$

(28)

The expression for the prolateness is too lengthy to be given here, but already from Eq. (28) one sees that the effect can, in particular circumstances (when $a \approx P \approx 1$), become quite appreciable. When doing realistic Monte-Carlo calculations, one should therefore take it into account by using a phase space weighted by $a + b \cos^2 \theta$.

6. THE ASPHERICITY AND PROLATENESS IN THE CASE OF A JET STRUCTURE

In case the individual events are of a jet-like nature, the jet axis will be distributed according to Eq. (17), but the asymmetry A and the prolateness P will of course be independent of this distribution. With the help of Ref. 6) one can calculate these parameters analytically and find for the

$$A = (1-x)^2 + \varepsilon x^2$$

where

$$x = \frac{3 \langle k_{\perp}^2 \rangle \langle n \rangle (d+1)}{\alpha S}$$

with

$$\alpha = \frac{1}{2} \left(1 + \frac{d(d+1)}{(d+2)(d+3)} \right) \frac{\langle n \rangle}{\langle n \rangle - 1 - \frac{d}{d+1}}$$

$$\varepsilon = \alpha \frac{\langle n \rangle^2 - 2\lambda^2 \psi'(\lambda)}{\langle n \rangle (\langle n \rangle - 1 - \frac{\lambda}{\lambda+1})} - 1$$

$\psi'(\lambda)$ is the trigamma funct.: $\frac{d^2}{d\lambda^2} \ln \Gamma(\lambda)$

and where λ has to be determined from the average multiplicity $\langle n \rangle$

$$\langle n \rangle = \lambda \ln \frac{s}{\langle k_{\perp}^2 \rangle} - 2\lambda \psi'(\lambda) - \lambda - 1$$

$\lambda = .5772\dots$

Taking $\lambda = 3$ and $\langle k_{\perp}^2 \rangle = 0.156$, one then obtains:

$$\text{at } s = 50 \text{ GeV}^2 \rightarrow \langle n \rangle = 8.55 \text{ and } A = 0.40$$

$$\text{at } s = 100 \text{ GeV}^2 \rightarrow \langle n \rangle = 10.65 \text{ and } A = 0.57.$$

For the prolateness one can write down a similar formula, but with rather more complicated results:

$$P = 0.17 \text{ at } s = 50 \text{ GeV}^2 \text{ and } P = 0.33 \text{ at } s = 100 \text{ GeV}^2.$$

7. CONCLUSIONS

We have defined two observables, the asphericity A and the prolateness P , in terms of two- and three-particle densities. They are closely related to the traces of the square resp. third power of a tensor used in previous analyses. The main advantage of the present definition is that it eliminates the contribution of statistical fluctuations, which would otherwise contribute substantially. Other trivial contributions are estimated analytically, and found to be rather small: energy momentum conservation via a distribution uniform in phase space yields a contribution $\propto 1/n^2$ to A and $\propto 1/n^3$ to P , both of which are less than 10% at highest SPEAR energies. The effects of resonance production or clustering and of anisotropy of phase space should each contribute less than 5% to A . Adding all these up (which should be allowed as long as they represent small deviations from sphericity), we get $A \lesssim 20\%$ and $|P| \lesssim 5-10\%$. Only if a large beam polarization and a big $\cos^2 \theta$ term in the single particle distribution come together, they could

interact with the particular shape of the SPEAR detector to produce a significant effect. Although it is presumably small on an absolute scale, it could explain part of the observed decrease of the sphericity when going from 6.4 GeV to 7.8 GeV ⁴⁾.

If, on the other hand, the individual events are of a jet-like structure with transverse momenta typical for hadronic reactions, one expects $A \approx 0.4$ at the highest SPEAR energy. Therefore it seems that one should be able to distinguish clearly between these two cases.

APPENDIX

If the produced particles are distributed according to phase space, the semi-inclusive densities are given by

$$\rho_n(p_1, \dots, p_k) = \text{const.} * R_{n-k} \left[\left(P - \sum_{i=1}^k p_i \right)^2 \right], \quad (\text{A.1})$$

where $R_n(P^2)$ is the invariant n particle phase space, given by

$$R_n(P^2) = \text{const.} * \int d^4x e^{i P \cdot x} [v(x^2)]^n \quad (\text{A.2})$$

with

$$v(x^2) = \int \frac{d^3k}{2k_0} e^{-i k \cdot x} \quad (\text{A.3})$$

In the general case, this cannot be integrated analytically. Exceptions are the non-relativistic and the extreme relativistic limits, in which cases ⁷⁾

$$R_n(P^2) = \begin{cases} \text{const.} * \left(P^2 - \left(\sum_{i=1}^n m_i \right)^2 \right)^{\frac{3n-5}{2}} & (\text{non-rel.}) \\ \text{const.} * (P^2)^{n-2} & (\text{extremely rel.}) \end{cases} \quad (\text{A.4a})$$

(A.4b)

All estimates for these two limits mentioned in the paper can be calculated straightforwardly by using Eqs. (A.4) and (A.1). A somewhat more elegant method consists, however, in using Eq. (A.2). Inserting it into the definitions, Eqs. (13) and (14), we can convert all momentum powers into differentiations, and get

$$A = \frac{\int d^4x e^{iP \cdot x} v^{n-2} [\vec{r}^2 (\frac{\partial}{\partial x^2})^2 v]^2}{\int d^4x e^{iP \cdot x} v^{n-2} [\vec{r}^2 (\frac{\partial}{\partial x^2})^2 v + \frac{3}{2} \frac{\partial}{\partial x^2} v]^2} \quad (A.5)$$

and

$$P = \frac{\int d^4x e^{iP \cdot x} v^{n-3} [\vec{r}^2 (\frac{\partial}{\partial x^2})^2 v]^3}{\int d^4x e^{iP \cdot x} v^{n-3} [\vec{r}^2 (\frac{\partial}{\partial x^2})^2 v + \frac{3}{2} \frac{\partial}{\partial x^2} v]^3} \quad (A.6)$$

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FIGURE CAPTIONS

Fig. 1 : The average sphericity given by Eq. (2), and averaged according to Eq. (8), in the case of no correlations and isotropic single particle distributions.

Fig. 2 : Average asphericity A and prolateness P , as defined in Eqs. (15) and (16). Solid lines: non-relativistic limit; broken lines: extremely relativistic limit. The dotted and doubly dotted lines give the average event asymmetry, defined as in Eq. (15) but without normal ordering, in these two limits.

Fig. 3 : Average asphericity of a pion pair originating from an isotropic two-body resonance, with fixed mass M and energy distribution $\propto e^{-ET}$.

FIG. 1

FIG. 2

FIG. 3