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CALCULABLE OBSERVABLES IN QUANTUM CHROMODYNAMICS. IG. Tiktopoulos^{*)}

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ABSTRACT

A simple heuristic proof of infra-red finiteness to all orders of perturbation theory is given for a class of observables in massless field theories including Quantum Chromodynamics. This class includes the average energy $E(\Omega)$ carried into a given solid angular region Ω by final state particles in e^+e^- annihilation and its positive integral powers; it also includes energy correlations between different angular regions: $E(\Omega_1) E(\Omega_2) \dots E(\Omega_N)$. It is argued that appropriately defined inclusive jet cross sections are infra-red finite even for vanishing fractional energy resolutions. Extensions to lepton-hadron and hadron-hadron collisions are described.

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1. INTRODUCTION

It is reasonable to expect that within the framework of asymptotically free Quantum Chromodynamics (QCD) certain observable quantities associated with large momentum transfer processes are calculable by perturbation theory. This belief is supported by the recent finding that for processes like deep inelastic lepton production, heavy μ pair and high p_T hadron production in hadronic collisions infra-red singularities arising in the QCD calculations of the relevant "hard" subprocesses are, as far as they have been checked, absorbable into appropriate quark and gluon distribution and decay functions¹⁾.

For e^+e^- annihilation into hadrons it appears that certain observables are altogether free of infra-red singularities and therefore perturbation theory provides for them a calculable asymptotic expansion. A quantity of this kind other than the total cross section was proposed by Sterman and Weinberg²⁾. They defined a jet cross section as the cross section for those final states in e^+e^- annihilation in which all but at most a fraction ϵ of the total available energy is emitted within two oppositely directed cones of half-angle δ and found it to be finite to lowest non-trivial order in QCD in the limit of zero quark and gluon masses. One understands intuitively why perturbation theory works for this cross section: it does not distinguish between events which differ in the number and individual momenta of any set of particles with collinear momenta as long as the total momentum of the set is the same. Based on the same idea, a number of such IR finite (at least to lowest non-trivial order) quantities have been suggested since by other authors. Such variables as sphericity³⁾ and thrust or maximum directed momentum⁴⁾ also measure the extent to which a configuration of hadronic momenta is jet-like. Another, simpler, quantity is the angular distribution of hadronic energy⁵⁾ (antenna pattern).

Quite apart from the basic assumption made in connection with the above observables that calculations in QCD of transition probabilities involving quarks and gluons in the final state give a reliable asymptotic expansion of the corresponding quantities in terms of hadronic states, one should check whether the expansion is indeed infra-red finite to all orders of perturbation theory. Work of this kind has been presented by Sterman⁶⁾ who, on the basis of a detailed classification of the singularities of cut vacuum polarization graphs, concludes that cross sections for e^+e^- transitions into certain "multi-jet" ensembles of final states are indeed IR singularity-free in perturbation theory.

Let $E(\Omega)$ be the energy carried by final state particles into a given angular region Ω in the e^+e^- centre-of-mass frame. Sterman's N jet ensemble is defined by the conditions

$$\begin{aligned} E_i - \delta E_i &\leq E(\Omega_i) \leq E_i + \delta E_i \quad ; \quad i = 1, 2, \dots, N \\ E_{TOT} - \sum_{i=1}^N E(\Omega_i) &\leq \delta E_0 \end{aligned} \quad (1)$$

where $\{\Omega_i\}$ are N given non-overlapping regions on the unit sphere, E_i and δE_i are the corresponding jet energy central values and resolutions and E_{TOT} the total available c.m. energy. According to Sterman such N jet cross sections are IR finite in perturbation theory provided the fractional energy resolutions $\delta E_i/E_i$ and the solid angles Ω_i are kept at non-zero values. In the limit of small Ω_i and small $\delta E_i/E_i$, these jet cross sections develop logarithmic singularities in these variables²⁾. One may call these cross sections exclusive N jet cross sections since, were the energy resolutions to vanish, the N jets would carry the whole of the available c.m. energy.

In this paper I present an investigation of the IR finiteness of a class of observables in QCD which is based on a rather simple heuristic all-orders proof of the IR finiteness of $E(\Omega)$. The proof is far from rigorous but seems compelling enough to stimulate more detailed work along the same lines. Quite apart from the question of rigour, however, the approach seems different in some respects from the methods applied by Sterman to the study of exclusive jet cross sections and may lead to new insights. Indeed, I am led in Section 3 below to argue that inclusive jet cross sections in which a number of jets are specified which do not contain all the available energy (see precise definition below) should be IR finite even in the limit of vanishing fractional energy resolutions.

In Section 4, I briefly discuss applications of similar ideas to processes other than e^+e^- in which hadrons are produced at large transverse momenta: lepton-hadron and hadron-hadron collisions. Similar methods applied in the presence of heavy quarks will be described in a forthcoming paper,

2. ADDITIVE OBSERVABLES IN e^+e^- ANNIHILATION

Consider the class of observables of the form

$$\sum_j \psi(\vec{p}_j) \quad (2)$$

for e^+e^- annihilation. Here $\vec{p}_j$ is the momentum of the j -th particle in the final state and j is summed over all final particles. Our aim is to find out which "additive" quantities of this type are IR finite in massless perturbation theory. We begin by noting that e^+e^- cross sections are directly related to cut vacuum polarization graphs. More generally, consider any graph G for a 2-point function in a massless field theory. Let $T(q)$ be the associated Feynman integral where q is the external four-momentum taken timelike: $-q^2 > 0$ ^{*)}. Denote by D_α the operation of taking the discontinuity ^{**) in q^2 associated with a given Cutkosky cut α :}

$$D_\alpha T \equiv \int T_{\alpha,L}^* [dq]_\alpha T_{\alpha,R} \quad (3)$$

where $T_{\alpha,L}$ and $T_{\alpha,R}$ are the amplitudes to the left and to the right of the cut α and $[dq]_\alpha$ stands for the appropriate multiparticle differential phase-space element. (Spin indices and summations over them are not explicitly shown.)

Individual discontinuities $D_\alpha T$ will in general be IR divergent, so that some IR cut-off method must be employed in intermediate stages. In QCD it will be convenient to just give a mass μ to the gluon keeping the quarks massless. Quantities remaining finite in the limit $\mu \rightarrow 0$ will be said to be IR finite.

The total discontinuity of T across the real positive $-q^2$ axis $DT \equiv \sum_\alpha D_\alpha T$ is a contribution to the total e^+e^- cross section and is IR finite.

Given a function $\psi(\vec{q})$ of a single momentum $\vec{q}$ we define the operations D_α^ψ and D^ψ as follows

$$D_\alpha^\psi T \equiv \int T_{\alpha,L}^* \left\{ \sum_{j \in \alpha} \psi(q_j) \right\} [dq]_\alpha T_{\alpha,R} \quad (4)$$

*) The Lorentz notation $q^2 = q_1^2 + q_2^2 + q_3^2 - q_0^2$ is used throughout this paper.

**) All discontinuities are assumed to be divided by $2i$.

$$\mathcal{D}^\Psi T = \sum_{\alpha} \mathcal{D}_{\alpha}^\Psi T \quad (5)$$

The sum in Eq. (4) runs over all intermediate particles i.e. lines cut by α . The e^+e^- final state average of $\Sigma\psi$ is

$$\langle \Sigma\psi \rangle = \sum_{\text{all graphs}} \mathcal{D}^\Psi T$$

Our task is to choose ψ so that $\langle \Sigma\psi \rangle$ is IR finite.

The special merit of variables of the form (2) is that $\mathcal{D}^\Psi T$ for a given graph can be written as

$$\mathcal{D}^\Psi T = \sum_j \int \frac{d^4 k}{(2\pi)^3} \psi(\vec{k}) \delta_+(k^2) \mathcal{D}M^{(j)}(q; k), \quad (6)$$

where j ranges over all the lines of the graph (spinor and vector polarization wave functions and summations over discrete indices are omitted) and $\mathcal{D}M^{(j)}$ is the discontinuity in the variable $(q-k)^2$ of the amplitude $M^{(j)}$ associated with the graph obtained by removing the j -th line (Fig. 1).

If terms in the sum of Eq. (6) in which j ranges over the lines of the same propagator subgraph are grouped and summed first we have

$$\mathcal{D}^\Psi T = \sum_{\sigma} \int \frac{d^4 k}{(2\pi)^3} \mathcal{D}^\Psi \Delta^{(\sigma)}(k) \mathcal{D}T^{(\sigma)}(q; k) \quad (7)$$

where σ runs over all (maximal) propagator subgraphs of G their amplitudes being $\Delta^{(\sigma)}$ and $\mathcal{D}T^{(\sigma)}$ is the "total" discontinuity (i.e. the sum over all relevant Cutkosky discontinuities) in the variable $(q-k_{\sigma})^2$ of the amplitude $T(q; k)$ obtained by removing from G the $\Delta^{(\sigma)}$ subgraph and thereby creating two external lines carrying momenta k and $-k$ (see Fig. 2).

The integration in Eq. (7) is limited by the constraints (arising from the $\mathcal{D}^\Psi$ operation on $\Delta^{(\sigma)}$):

$$-k^2 \geq 0, \quad k_0 \geq 0 \quad (8)$$

For $-k^2 > 0$ we can take $D^\psi \Delta^{(\sigma)}$ to be IR finite if we proceed by induction on the order of perturbation theory, because $\Delta^{(\sigma)}$ is a two-point amplitude or a product of two-point amplitudes (times bare propagators) of lower order than G itself; and the lowest order $\Delta^{(\sigma)}$ is the bare propagator itself. Also $T^{(\sigma)}(q,k)$ [and therefore $DT^{(\sigma)}(q,k)$] are IR finite for $-k^2 > 0$, since all external lines are off-shell.

It remains to discuss possible singularities and their integrability at $k^2 = 0$. An important point is that $T^{(\sigma)}(q,k)$ [and therefore $DT^{(\sigma)}(q,k)$] is IR finite for $k^2 = 0$ (but $k_0 \neq 0$) because by construction it corresponds to a two-particle irreducible (2PI) graph i.e. one cannot separate a piece to which the external lines k and $-k$ are attached by cutting across less than three internal lines. The IR finiteness of such 2PI graphs is always true for non-vector theories with dimensionless coupling constants⁷⁾ (i.e. for $\bar{\psi}\psi\phi$, $\bar{\psi}\gamma_5\psi\phi$, ϕ^4 but not for ϕ^3 interactions) and also for gauge theories in axial gauges. It is remarkable that the IR finiteness of 2PI graphs in axial gauges also simplifies the demonstration that mass singularities factorize in the moments of forward quark and gluon amplitudes^{*)}.

One way to control the behaviour of $D^\psi \Delta^{(\sigma)}(k)$ at $k^2 = 0$ is to pick ψ to be "collinearly additive" i.e. for any two momenta $\vec{p}_1$ and $\vec{p}_2$ that are collinear to have

$$\psi(\vec{p}_1 + \vec{p}_2) = \psi(\vec{p}_1) + \psi(\vec{p}_2) \quad ; \quad \vec{p}_1 \parallel \vec{p}_2 \quad (9)$$

From Eq. (9) one can trivially derive the collinear additivity of ψ for any number of parallel momenta: $\psi(\sum p_j) = \sum \psi(p_j)$. Thus if $|\vec{k}|$ is kept fixed at a non-zero value, as $k^2 \rightarrow 0$ each of the intermediate mass-shell momenta $\vec{p}_j$ associated with any Cutkosky discontinuity of $\Delta^{(\sigma)}$ either becomes parallel to $\vec{k}$ or vanishes, so that $\sum \psi(p_j)$ becomes equal to $\psi(\vec{k})$. One may thus write

$$D^\psi \Delta^{(\sigma)}(k) \xrightarrow{k^2 \rightarrow 0} \psi(\vec{k}) D \Delta^{(\sigma)}(k) \quad (10)$$

*) See the second and third paper of Ref. 1).

For fixed $\vec{k}$, $\Delta^{(\sigma)}(k)$ is analytic in the k_0 plane except for cuts along the real axis from $-\infty$ to $-|\vec{k}|$ and from $|\vec{k}|$ to ∞ (these cuts reflect the cut in k^2 from 0 to ∞). In exploring the integrability at $k^2 = 0$ we limit the integration to a small neighbourhood of $k_0 = |\vec{k}|$ and write:

$$\int d^4k \mathcal{D}^\psi \Delta^{(\sigma)}(k) \mathcal{D}T^{(\sigma)}(q; k) \simeq \left. \int d^3k \mathcal{D}T^{(\sigma)}(q; k) \right|_{k_0 = |\vec{k}|} \cdot \psi(E) \int_C dk_0 \Delta^{(\sigma)}(k) \quad (11)$$

where the contour C is around the right hand cut in the k_0 plane (Fig. 3). As long as $|\vec{k}| \neq 0$ the contour C can be kept away from the singularity of $\Delta^{(\sigma)}(k)$ at $k_0 = |\vec{k}|$.

For theories with dimensionless coupling constants (i.e. Yukawa, ϕ^4 and gauge vector interactions but not ϕ^3) the singularity of $\Delta^{(\sigma)}(k)$ at $k^2 = 0$ is worse than a pole by only some logarithmic power:

$$\Delta^{(\sigma)}(k) \propto \frac{(\ln k^2)^N}{k^2} \quad (12)$$

where N is a positive integer.

The nature of the singularity of $\Delta^{(\sigma)}$ at $k^2 = 0$ given by Eq. (12) is related to the integrability near $\vec{k} = 0$. Indeed as $|\vec{k}| \rightarrow 0$ the two branch points at $k_0 = \pm|\vec{k}|$ pinch the contour C and the integral can be evaluated asymptotically on the basis of Eq. (12).

$$\int_C dk_0 \Delta^{(\sigma)}(k) \underset{|\vec{k}| \rightarrow 0}{\propto} \frac{[\ln |\vec{k}|]^N}{|\vec{k}|} \quad (13)$$

To proceed one needs an estimate of the $|\vec{k}| \rightarrow 0$ behaviour of $\mathcal{D}T(q, k)|_{k^2=0}$. It is straightforward to verify that at the tree-graph level it behaves like $|\vec{k}|^{-2}$; and it is reasonable to expect that higher order contributions are worse

only by powers of $\ln|\vec{k}|$. This plausible assumption together with Eq. (13) implies that the right hand side of Eq. (11) converges provided $\psi(\vec{k})$ satisfies:

$$\int \frac{d^3k}{|\vec{k}|^3} \ln^N |\vec{k}| \psi(\vec{k}) < \infty \quad \text{for all } N \quad (14)$$

Thus we find that if ψ satisfies the conditions (9) and (14) then $D^\psi T$ is IR finite. Actually, it is not difficult to show that functions satisfying condition (9) must necessarily be of the form

$$\psi(\vec{k}) = |\vec{k}| f(\hat{k}) \quad (15)$$

where $\hat{k}$ is the unit vector in the direction of $\vec{k}$. Thus condition (14) turns out to be a consequence of (9). We conclude that variables of the form

$$V_f = \sum_j |\vec{k}_j| f(\hat{k}_j) \quad (16)$$

are calculable (i.e. IR finite) in perturbation theory. An important case is the energy flow $E(\Omega)$ into a solid angular region Ω :

$$E(\Omega) = \sum_j |\vec{k}_j| \chi_\Omega(\hat{k}_j) \quad (17)$$

where $\chi_\Omega(\hat{k}_j) = 1$ for $\hat{k}$ in Ω and $= 0$ otherwise. Taking an infinitesimal Ω in the direction of some unit vector $\hat{n}$ we define the antenna pattern⁵⁾:

$$\left. \frac{dE(\Omega)}{d\Omega} \right|_{\hat{n}} = \frac{1}{2\pi} \sum_j |\vec{k}_j| \delta(1 - \hat{n} \cdot \hat{k}_j)$$

Note that for any given f the quantity V_f of Eq. (16) can be obtained by integrating the antenna pattern over $\hat{n}$ with f as a weight function. Thus one may regard the IR finiteness of the antenna pattern as the more basic result.

It is important to realize that the argument would not have worked for conserved additive observables like the electric charge, the third component of isospin, strangeness or a colour component all of which commute with energy-momentum: a relation analogous to Eq. (10) would hold, of course, for them but none of these quantities vanishes at zero four momentum.

The above argument may be generalized to show IR finiteness for any positive power of V_f [where V_f is of the form (16)]. This may be adequately illustrated by considering V_f^2 . Let D^{ψ^2} denote an operation defined as in Eqs (4) and (5) but with the quantity in curly brackets squared. Then, by summing first over propagator subgraphs as before, one obtains in place of Eq. (7) the relation (see Figure 4)

$$\begin{aligned} D^{\psi^2} T &= \sum_{\sigma} \int \frac{d^4 k}{(2\pi)^3} D^{\psi^2} \Delta^{(\sigma)}(k) D T^{(\sigma)}(q; k) \\ &+ \sum_{\sigma \neq \tau} \int \frac{d^4 k_1}{(2\pi)^3} \frac{d^4 k_2}{(2\pi)^3} D^{\psi^2} \Delta^{(\sigma)}(k_1) D^{\psi^2} \Delta^{(\tau)}(k_2) D T^{(\sigma, \tau)}(q; k_1, k_2) \end{aligned} \quad (18)$$

where $T^{(\sigma, \tau)}(q; k_1, k_2)$ is the 2PI amplitude obtained by removing the $\Delta^{(\sigma)}$ and $\Delta^{(\tau)}$ propagator subgraphs and thereby creating two external lines with momenta k_1 and $-k_1$ and two external lines with momenta k_2 and $-k_2$.

As before we note that $T^{(\sigma)}(k^2 = 0)$ is IR finite (in an axial gauge) and behaves like $[\ln|\vec{k}|]^N / |\vec{k}|$ and $T^{(\sigma, \tau)}$ has similar properties with respect to k_1 and k_2 separately. The integrability with respect to k and k_1, k_2 is then established as above.

Finally, one may go one step further, by essentially the same argument as above, by considering several functions $f_1, f_2, \dots, f_N$. Thus one can show that an arbitrary monomial like

$$\left(V_{f_1}\right)^{n_1} \left(V_{f_2}\right)^{n_2} \dots \left(V_{f_N}\right)^{n_N} \quad (19)$$

is IR finite ⁸⁾.

3. INCLUSIVE JET CROSS SECTIONS

In this Section I pursue the consequences of the assertion that all positive integral powers of $E(\Omega)$ have IR finite e^+e^- final state averages. In particular, it is interesting to investigate the possibility that the average of an arbitrary function of $E(\Omega)$ might be IR finite.

One might argue as follows. Consider the observable ($\langle \dots \rangle$ denotes the final state average normalized so that $\langle 1 \rangle = \sigma_{TOT}$)

$$\left\langle \delta \left(\frac{E(\Omega)}{q_0} - \eta \right) \right\rangle ; \quad 1 \geq \eta \geq 0.$$

Obviously, if this delta function is IR finite any reasonable function of $E(\Omega)$ is IR finite. For a given vacuum polarization graph consider the sum-over-cuts quantity

$$F(\eta; \mu) = \sum_{\alpha} \int T_{\alpha, L}^* \delta \left(\frac{E(\Omega)}{q_0} - \eta \right) [dq]_{\alpha} T_{\alpha, R}$$

with the notation of Eq. (4). An infra-red cut-off mass μ is introduced to allow for the possibility that this quantity is IR singular. Suppose now that as the IR cut-off μ approaches zero $F(\eta; \mu)$ develops IR singularities which appear in the form of a polynomial in $\ln \mu$:

$$F(\eta, \mu) = \sum_{j=1}^n A_j(\eta) [\ln \mu]^j + F_{reg}(\eta) \quad (20)$$

We know, however, that $E^n(\Omega)$ is IR finite. Therefore,

$$\lim_{\mu \rightarrow 0} \int_0^1 d\eta \eta^N F(\eta; \mu) < \infty \quad \text{for all } N \quad (21)$$

Were we allowed to interchange the order of the integration with the taking of the $\mu \rightarrow 0$ limit we would substitute Eq. (20) into Eq. (21) to obtain:

$$\int_0^1 d\eta \eta^N A_j(\eta) = 0 \quad \text{for all } N \text{ and } j$$

which implies that $A_j(\eta) \equiv 0$ so that $F(\eta, \mu)$ is finite for $\mu \rightarrow 0$. We would then conclude that $\delta_j[E(\Omega)/q_0 - \eta]$ and thus any reasonable function of $E(\Omega)$ is IR finite.

Actually, the argument just outlined fails near $\eta = 1/2$ which is the limiting value of η below which the total energy and total momentum contained in Ω cannot be kinematically balanced by a single massless particle outside Ω . One can see this clearly in an explicit calculation to lowest non-trivial order, done in the Feynman gauge for simplicity.

Take Ω such that it is wholly contained in some hemisphere. Then the one-loop contribution (Fig. 5a) to (the total cross section times the average of) $\delta[E(\Omega)/q_0 - \eta]$ is given by

$$F_0(\eta) = \frac{\Omega}{2\pi} \sigma_{TOT}^{(0)} \delta\left(\frac{1}{2} - \eta\right) \quad (22)$$

where $\sigma_{TOT}^{(0)}$ is the one-loop (i.e. the zeroth order in g) total e^+e^- cross section. Consider now the two-loop contributions. From the graph 5b we have the following contribution from the cut α_1 with $\vec{k}$ in Ω :

$$F_{b1}(\eta, \mu) = -\frac{\Omega}{4\pi} \sigma_{TOT}^{(0)} \Sigma_2(k^2) \Big|_{k^2=0} \delta\left(\frac{1}{2} - \eta\right) \quad (23)$$

Here Σ_2 is the quark self-energy part of which we need keep only the IR singular part. We have

$$\Sigma_2(k^2) \Big|_{k^2=0} = C_F \frac{g^2}{16\pi^2} \ln\left(\frac{M^2}{\mu^2}\right) + (\text{IR - finite}) \quad (24)$$

where M is a renormalization point mass and C_F the Casimir eigenvalue for the fermions.

From the cut α_2 of graph 5b we have, for $\eta \rightarrow 1/2$:

$$F_{b2}(\eta; \mu) = \frac{\Omega}{4\pi} \sigma_{TOT}^{(0)} \frac{2g_0^2}{\pi} \left[\frac{1}{-k^2} \Sigma_2^{\Omega}(k) \right]_{-k^2 = (2\eta-1)g_0^2} \quad (25)$$

where

$$\Sigma_2^{\Omega}(k) = \frac{C_F g^2}{8\pi^2} \int_{\Omega} d^4p \delta_+(p^2 + \mu^2) \delta_+((k-p)^2) \quad (26)$$

The integral is only over values of $\vec{p}$ such that both $\vec{p}$ and $\vec{k} - \vec{p}$ are in Ω . Other contributions are ignored because they vanish at $\eta = 1/2$.

For $-k^2 \ll \delta^2 k^2$ where δ is the angular radius of Ω we have

$$\Sigma_2^{\Omega} \simeq \frac{C_F g^2}{16\pi} \theta\left(\eta - \frac{1}{2} - \frac{\mu^2}{2g_0^2}\right) + \dots \quad (27)$$

The dots stand for contributions which vanish at $\eta = 1/2$. Adding the contributions from cuts α_1 and α_2 we have :

$$F_{b1} + F_{b2} = \frac{\Omega}{4\pi} \sigma_{TOT} \frac{C_F g^2}{16\pi} \left\{ -\ln \frac{M^2}{\mu^2} \delta\left(\eta - \frac{1}{2}\right) + \frac{\theta(\eta - \frac{1}{2} - \mu^2/(2g_0^2))}{\eta - \frac{1}{2}} + \dots \right\} \quad (28)$$

We see now how the general argument based on Eq. (20) fails: there is a non-integrable singularity in $F_{reg}(\eta)$ [i.e. the second term in curly brackets in Eq. (28)] located at a distance of order μ^2/q_0^2 from the end point value of η , which upon integration over any smooth weight function of η [e.g. η^N in Equation (21)] gives rise to $\ln \mu$ terms which cancel the $\ln \mu$ already present in $F(\eta; \mu)$.

Making use of the relation

$$\lim_{\epsilon \rightarrow 0} \left\{ -(\ln \epsilon) \delta(y) - \frac{\theta(y-\epsilon)}{y} \right\} = \ln y \frac{d}{dy} \quad (29)$$

the quantity in curly brackets of Eq. (28) may be replaced by

$$- \ln(\eta - \frac{1}{2}) \frac{d}{d\eta} + (\text{regular at } \eta = \frac{1}{2}) \quad (30)$$

The derivative is meant to operate on some function of η to be used as a weight function for smearing $\delta[E(\Omega)/q_0 - \eta]$ over a range of η values about $\eta = 1/2$.

Similarly the cut α_2 of the graph of Fig. 5c gives a contribution proportional to

$$\frac{\theta(-k^2 - \mu^2)}{k^2} \ln(-k^2/q_0^2)$$

which upon integration over k^2 generates a $(\ln \mu^2)^2$ IR divergence which cancels a corresponding $(\ln \mu^2)^2$ singularity from cut α_1 of the same graph.

Consider now a jet cross section of the Stermann-Weinberg type defined in terms of the ensemble of final states in which the energy in Ω equals $q_0/2$ within a fractional energy resolution ϵ :

$$(\frac{1}{2} + \epsilon) q_0 > E(\Omega) > (\frac{1}{2} - \epsilon) q_0 \quad (31)$$

In calculating this cross section the expression of Eq. (30) is applied to the weight function $\theta(\frac{1}{2} + \epsilon - \eta) \cdot \theta(\eta - \frac{1}{2} + \epsilon)$ and then η is integrated over resulting in $\ln \epsilon$ terms. This means that ϵ cannot be taken too small if the perturbation expansion is to be meaningful. Thus the smoothness of the weight function near $\eta = \frac{1}{2}$ seems to determine how fast, if at all, perturbation

theory converges asymptotically for a given observable.

The significance of the value $\eta = \frac{1}{2}$ derives from the fact that it is the value at which exactly two oppositely directed jets, each of zero invariant mass, are kinematically possible (one jet is in Ω and the other in the angular region diametrically opposite to Ω ; recall that Ω was taken to be small enough to be contained in a hemisphere).

Based on the foregoing evidence I suggest that, except near $\eta = \frac{1}{2}$ (and near $\eta = 0$, see below), the quantity

$$\frac{d\sigma}{d\eta}(\Omega, \eta) = \left\langle \delta\left(\frac{E(\Omega)}{q_0} - \eta\right) \right\rangle \quad (32)$$

is IR finite and therefore calculable in perturbation theory at zero quark masses. It is the cross section per unit fractional energy for those events in which a fraction η of the total available energy is deposited in the angular region Ω .

If Ω is wholly contained in some hemisphere, then it is easy to see that there is a kinematical upper bound for η which is less than one. In particular, if Ω is the region within a cone of half-angle δ then $\max \eta = 1/(1 + \cos \delta)$. This is always above the critical value of $1/2$ but very near it for small δ . Thus, if one is interested in describing the final hadronic states in terms of narrow jets, the quantity Eq. (32) will be useful for values of η below $1/2$ and at some distance so as to avoid large terms like $\alpha_s/(1/2 - \eta)$. It can then be termed as the inclusive cross section for the production of two or more jets in addition to the one specified by Ω and η .

One may proceed to generalize to inclusive jet cross sections in which N jets are specified by $\{\Omega_j, \eta_j\}$, $j = 1, 2, \dots, N$:

$$\frac{d\sigma}{d\eta_1 \dots d\eta_N}(\Omega_1, \eta_1; \dots; \Omega_N, \eta_N) \equiv \left\langle \prod_{j=1}^N \delta\left(\frac{E(\Omega_j)}{q_0} - \eta_j\right) \right\rangle \quad (33)$$

These cross sections should be IR finite provided the jet data $\{\Omega_j, \eta_j\}$ stay away from "massless-jet threshold" values. This requires that for all choices of N unit vectors $\hat{e}_1, \dots, \hat{e}_N$ such that $\hat{e}_j$ points into Ω_j we have

$$\left(1 - \sum_{j=1}^N \eta_j\right)^2 - \left(\sum \hat{e}_j \eta_j\right)^2 > 0$$

A brief comment on the jet angular widths Ω_j is in order here. The total invariant mass of a jet of energy E and angular radius $\delta < \frac{\pi}{2}$ may range from zero to a kinematical maximum of $E \sin \delta$. Thus, as $\delta \rightarrow 0$, the jet cross section receives no adequate "smearing" over its invariant mass M near the singularities at $M = 0$ and, as a result, large logarithms of δ appear. Thus, in general jet cross sections must be defined with angular regions of a finite size. [Note the contrast with the antenna pattern $dE(\Omega)/d\Omega$ which is IR finite, even though an infinitesimal angle is involved ^{*)}]. Adequate smearing near $M = 0$ also requires that the η_j 's stay away from 0:

$$\eta_j > 0$$

One feels that the set of all multiple jet angle and energy correlation functions of the type of Eq. (33) [perhaps those of Eq. (19) form an equivalent set] contain all the information that perturbation theory can reveal to us at the level of approximation where quantities of order m_q/q_0 are neglected. Granted that variables like sphericity³⁾ and thrust⁴⁾ seem to be of a still different type; however, they too can be approximated by sequences of polynomials in N variables $E(\Omega_1), E(\Omega_2), \dots, E(\Omega_N)$ where $\Omega_1, \Omega_2, \dots, \Omega_N$ form a sufficiently fine partition of the unit sphere into angular cells.

4. LEPTON-HADRON AND HADRON-HADRON COLLISIONS

The arguments presented in connection with e^+e^- annihilation also apply with some modifications to processes like lepton-hadron and hadron-hadron collisions where large momentum transfers are involved.

The heuristic argument given in Section 2 for $E(\Omega)$ may be applied to lepton production, the cut vacuum polarization graphs being replaced by cut quark (or gluon) current scattering graphs. Figure 2 is thus replaced by Fig. 6. Note, however, that T is 2PI with respect to the $(k, -k)$ channel but not with respect to the $(p, -p)$ channel. But this means that T contains large logarithms of p^2/Q^2 which, just as in total cross section calculations, should be absorbable

^{*)} For $e^+e^- \rightarrow$ hadrons at any rate. See remarks on leptonproduction in Section 4.

into appropriate quark (or gluon) distribution functions. Also, since T is not 2PI with respect to the (k, p) channels k must not be allowed to become collinear to p . This means that Ω must not contain the target fragmentation direction, a most reasonable limitation.

A new feature in lepton-hadron collisions is that the antenna pattern $dE(\Omega)/d\Omega$ in the current-target centre-of-mass frame becomes singular in the direction of the current. This can be readily seen already at the zeroth order level at which the antenna pattern has a delta function in the direction of the current. To order α_s the pattern behaves like $1/\theta^2$ as θ , the angle with the direction of the current approaches zero. (This is quite similar to the $\eta \rightarrow 1/2$ limit of the inclusive jet cross section in e^+e^- annihilation.) This happens because the validity of Eq. (10) requires a certain smoothness of $\psi(\vec{k})$ in this case. To get a meaningful result one must smear in angle; if, e.g. one computes $E(\Omega)$ with Ω a neighbourhood of $\theta = 0$ of angular radius δ , one obtains terms proportional to $\ln \delta$.

The foregoing discussion indicates that one may define and calculate IR finite jet cross sections in leptonproduction just as in e^+e^- annihilation provided one avoids the target fragmentation direction. Also, a jet in the direction of the current would require a finite fractional energy resolution.

Similar ideas apply to hadron-hadron collisions in which the production of hadrons at large angles in the centre of mass frame is assumed to take place predominantly via a single subprocess of hard collision between hadron constituents. Accordingly, one may adapt the argument of Section 2 to quark (gluon) - quark (gluon) scattering graphs. Figure 2 is then replaced by Fig. 7 where p_1 and p_2 are the momenta of the colliding constituents and the variable $s = -(p_1 + p_2)^2$ is large (i.e. $\alpha_s(s) \ll 1$). Terms proportional to powers of $\ln(p_1^2/s)$ and $\ln(p_2^2/s)$ are present in T but they should be absorbable into in principle measurable distribution functions. Note also that k should not be allowed to become collinear to either p_1 or p_2 because T is not 2PI in the (k, p_1) or (k, p_2) channels. Thus $E(\Omega)$ is calculable in terms of the quark and gluon distribution functions provided the "forward" directions i.e. the directions of $\vec{p}_1$ and $\vec{p}_2$ are not contained in Ω . Similarly inclusive jet cross-sections are IR finite if $\vec{p}_1$ and $\vec{p}_2$ lie outside of all jet angular regions Ω_i .

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FIGURE CAPTIONS

- Figure 1 : A graphical representation of the integral in Eq. (6).
- Figure 2 : A graphical representation of the integral in Eq. (7).
- Figure 3 : The contour C in the k_0 plane [see Eq. (11)].
- Figure 4 : A graphical representation of the second integral of Eq. (18).
- Figure 5 : Graphs considered in the evaluation of $\langle \delta(E(\Omega)/q_0 - \eta) \rangle$.
- Figure 6 : Current-quark (or gluon) scattering graph (analogous to that of Figure 2) for leptonproduction.
- Figure 7 : Hadron constituent scattering graph (analogous to that of Figure 2) for hadron-hadron collisions.

FIG.1

FIG.2

FIG. 3

FIG. 4

FIG.5

FIG.6

FIG.7