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EXPLICIT SOLUTION OF THE INTEGRAL EQUATIONS FOR FIREBALL DECAY

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A B S T R A C T

The exact integral equations for the single particle distribution and multiplicity distribution of a decaying fireball are explicitly solved. The asymptotic behaviour of the solutions for large fireball mass is shown to be the same as in the simplified model with linear decay chain. The possibility of obtaining the many-particle distributions is also discussed.

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Diffractive excitation models for particle production ^{1),2),3)} recently enjoyed a considerable interest. In these models the excitation of "diffractively excited objects" (clusters, novas, fireballs,...) is described by Regge-pole exchange. The produced objects (we shall call them "fireballs") then decay statistically in a cascading process into the stable (or unstable) hadrons the different distributions of which are the experimentally accessible quantities.

In a previous paper ⁴⁾ integral equations were written down for the statistical decay process determining the single particle distribution and multiplicity distribution emerging from a fireball. The solutions of the integral equations were investigated in the thermodynamic bootstrap model ⁵⁾ hence our approach is closest to that of Ref. 3) where essentially the empirical function $F(\lambda)$ of Ref. 6) was determined assuming two produced fireballs and Regge-pole exchange. In I the recoil and "cooling down" of the fireballs along the decay chain were exactly taken into account. The solutions were given for $M \rightarrow \infty$ (large fireball mass) and the deviations from the asymptotic solutions for finite M were numerically investigated. It was assumed, however, that only two-body decays occur during the decay chain producing always a pion and another fireball ("linear decay chain"). This is possible to take as a first approximation but is certainly not the exact situation in reality. The decay chain has several branch points corresponding to three and more-body decays and to two-body decays into two fireballs. In the present paper the branching of the decay chain is also exactly taken into account and the explicit solutions of the equations are given using the method recently discovered by Yellin ⁷⁾.

Let us denote by $\tau(M^2)$ the density of hadronic states (fireballs) at mass M . Frautschi's strong bootstrap equation ⁸⁾ states that

$$\tau(M^2) = D \delta(M^2 - m^2) + \sum_{l=2}^{\infty} \frac{AB^l}{l!} \int dM_1^2 \dots dM_l^2 \tau(M_1^2) \dots \tau(M_l^2) \varrho^{(l)}(M^2; M_1^2, \dots, M_l^2) \quad (1)$$

where $\varrho^{(l)}$ denotes the l body phase space

$$\varrho^{(l)}(r^2; M_1^2, \dots, M_l^2) = \prod_{j=1}^l \left\{ \int d^4 r_j \delta(r_j^2 - M_j^2) \theta(r_{j0}) \right\} \delta^4(r - r_1 - \dots - r_l) \quad (2)$$

and m is the mass of the ground state hadron. Equation (1) is in fact slightly more general than the one considered in Ref. 7) due to the presence of the additional parameters A , B in it ^{*}). It is a highly non-linear Volterra-type integral equation for $\tau(M^2)$. It is, in fact, "better than the Volterra-type" as the upper bound of the integration region is smaller by a finite amount than the argument of the unknown function [here $\tau(M^2)$]. The iterative solution is therefore exact and for a given M^2 it has always a finite number of non-vanishing terms. The solution given by Yellin is essentially the iterative one written in the form

$$\tau(M^2) = \sum_{\ell=1}^{\infty} g_{\ell} \theta(M^2 - \ell^2 m^2) \varphi^{(\ell)}(M^2, m^2, \dots, m^2) \quad (3)$$

$$\varphi^{(1)}(M^2, m^2) \equiv \delta(M^2 - m^2).$$

The coefficients g_{ℓ} are determined by a function $G(t)$ satisfying ^{**})

$$\sum_{\ell=1}^{\infty} g_{\ell} t^{\ell} \equiv G(t) = Dt + A \left(e^{BG(t)} - 1 - BG(t) \right). \quad (4)$$

It was shown in Ref. 7) that the radius of convergence of the power series defining $G(t)$ is

$$R = D^{-1} \left[(A + B^{-1}) \log(1 + A^{-1} B^{-1}) - B^{-1} \right]. \quad (5)$$

The function $G(t)$ has a square-root branch point at $t=R$ where it behaves as

^{*}) D is, like in Ref. 7), the degeneration parameter of the ground state. Equation (2.1) in Ref. 7) corresponds to the choice $A = m^2 (4\pi)^{-1}$ and $B = 2m^{-2}$. The parameters A , B represent somehow the volume of the "hadron box" in the non-covariant formulation of the bootstrap equation ⁸⁾. As in an ℓ -body bound state there are $(\ell-1)$ relative co-ordinates and the finiteness of the hadron volume restricts only the relative co-ordinates, one is tempted to put in Eq. (1) $AB^{\ell} = B^{\ell-1}$.

^{**}) Such an equation was first considered by Nahm ⁹⁾.

$$Q(t) \sim B^{-1} \log(1 + A^{-1} B^{-1}) - \sqrt{(R-t) 2DB^{-1}(1+AB)^{-1}}. \quad (6)$$

In the following we assume that the state density $\tau(M^2)$ is known and we consider the statistical decay process of the fireball. In the statistical decay the weights of the different final states are determined by the available phase space depending on the density of states at the masses of the decay products. If r is the four-momentum of the fireball then the normalized probability distribution $P_\ell(r, r_1, \dots, r_\ell)$ of the decay into ℓ other fireballs with four momenta $r_1, \dots, r_\ell$ is given by

$$P_\ell(r; r_1, \dots, r_\ell) d^4r_1 \dots d^4r_\ell = \frac{AB^\ell}{\ell!} \frac{\tau(r_1^2) \dots \tau(r_\ell^2)}{\tau(r^2)} \delta^4(r - r_1 - \dots - r_\ell) d^4r_1 \dots d^4r_\ell \quad (7)$$

Let us first consider the integral equation for the mean multiplicity $n(M)$ of the decay products of a mass M fireball ^{*}). The solution of the integral equations for the single particle distribution or multiplicity distribution is very similar to the case of $n(M)$ (only a bit more complicated), therefore, it is enough to treat only $n(M)$ in detail.

The average multiplicity of the decaying fireballs is the sum of average multiplicities of the fireballs which it decays into, therefore, owing to the symmetry of P_ℓ in $r_1, \dots, r_\ell$:

$$\begin{aligned} \tau(M^2) n(M) &= \delta(M^2 - m^2) + \sum_{\ell=2}^{\infty} \frac{AB^\ell}{(\ell-1)!} \int dM_1^2 \dots dM_\ell^2 \tau(M_1^2) \dots \tau(M_\ell^2) \times \\ &\quad \times Q^{(\ell)}(M^2; M_1^2, \dots, M_\ell^2) n(M_1^2). \end{aligned} \quad (8)$$

Here in front of the inhomogeneous term there is no factor D because, if the ground state is D fold degenerate then picking up a definite particle out of the D degenerate ones reduces the multiplicity by a factor of $1/D$. This compensates the factor D in the density of states.

^{*}) In what follows we generally use the same notations as in I.

In general, if the density of states $\tau(M^2)$ is given then Eq. (8) is a linear integral equation for $\tau(M^2)n(M)$. In the strong thermodynamic bootstrap τ is given by Eq. (3) and the solution of Eq. (8) goes as follows: define the functions Δ_H and γ_H by

$$\begin{aligned}\Delta_H(x^2) &= (2\pi)^3 \int_{m^2}^{\infty} dM^2 \tau(M^2) \Delta_F(x^2; M^2) \\ \gamma_H(x^2) &= (2\pi)^3 \int_{m^2}^{\infty} dM^2 \tau(M^2) n(M) \Delta_F(x^2; M^2)\end{aligned}\quad (9)$$

with $\Delta_F(x^2; M^2)$ the Feynman propagator of mass M . Using the relation

$$\prod_{j=1}^l (2\pi)^3 \Delta_F(x^2; M_j^2) = \int_{(M_1+\dots+M_l)^2}^{\infty} dM^2 \rho^{(l)}(M^2; M_1^2, \dots, M_l^2) (2\pi)^3 \Delta_F(x^2; M^2) \quad (10)$$

one obtains from Eqs. (1) and (8)

$$\begin{aligned}\Delta_H(x^2) &= D(2\pi)^3 \Delta_F(x^2; m^2) + A \left(e^{B\Delta_H(x^2)} - 1 - B\Delta_H(x^2) \right), \\ \gamma_H(x^2) &= (2\pi)^3 \Delta_F(x^2; m^2) + AB\gamma_H(x^2) \left(e^{B\Delta_H(x^2)} - 1 \right).\end{aligned}\quad (11)$$

For brevity, let us denote $(2\pi)^3 \Delta_F(x^2; m^2)$ by t and $\Delta_H(x^2)$, $\gamma_H(x^2)$ by $G(t)$ and $N(t)$, respectively. Then the first line in Eq. (11) goes over into Eq. (4) whereas the second one into

$$N(t) = t + ABN(t) \left(e^{BG(t)} - 1 \right). \quad (12)$$

The solution for $N(t)$ is clearly $D^{-1} t dG(t)/dt$, therefore the coefficients in the Taylor series of $N(t)$ are given by $D^{-1} l g_l$. Comparing Eqs. (11) and (9) and using once more the relation (10) the solution of Eq. (8) is obtained as

$$\tau(M^2)n(M) = \sum_{l=1}^{\infty} D^{-1} l g_l \Theta(M-lm) \rho^{(l)}(M^2; m^2, \dots, m^2). \quad (13)$$

The asymptotic behaviour of the series in Eqs. (3) and (13) can be determined from the singularities of $G(t)$ and $N(t)$, respectively, in the point $t=R$. The propagator function takes on real values in the spacelike region, hence, the singularity of $\Delta_H(x^2)$, $\mathcal{J}_H(x^2)$ corresponding to $t=R$ is at some spacelike distance x_R . According to the well-known explicit form of the propagator at spacelike distances the singularity is situated at x_R given by

$$R = 2\pi m^2 \frac{K_1(m\sqrt{-x_R^2})}{m\sqrt{-x_R^2}}. \quad (14)$$

In the integral representations of $\Delta_H(x^2)$ and $\mathcal{J}_H(x^2)$ the singularity is reproduced by the $M \rightarrow \infty$ behaviour of the integrand where the asymptotic form of the propagator can be used

$$(2\pi)^3 \Delta_F(x^2; M^2) \sim \frac{1}{2} \sqrt{M} \left(\frac{2\pi}{\sqrt{-x^2}} \right)^{3/2} e^{-M\sqrt{-x^2}}. \quad (15)$$

This approximation holds for spacelike x , if $M\sqrt{-x^2}$ tends to infinity. It is easy to show that the singularity of $G(t)$ in Eq. (6) is reproduced if at high M the function $\tau(M^2)$ behaves as

$$\tau(M^2) \sim M^{-3} e^{\frac{M}{T_0}} (2\pi)^{-3/2} \frac{m}{T_0} \sqrt{\frac{DK_2(\frac{m}{T_0})}{B(1+AB)}}; \quad T_0 \equiv \frac{1}{\sqrt{-x_R^2}}. \quad (16)$$

Here T_0 denotes Hagedorn's limiting temperature. Its dependence on the model parameters A, B, D can be read off from Eqs. (14) and (5). The expression of T_0 given in Eq. (16) shows the intimate connection between T_0 and the position x_R of the singularity of $\Delta_H(x^2)$ ^{*}). The singularity of the function $N(t)$ can be obtained simply by differentiation from Eq. (6) and then the asymptotic behaviour of $\tau(M^2)n(M)$ can be determined in the same way as for $\tau(M^2)$. The result is that for large M

^{*}) The asymptotic behaviour in Ref. 7) was not determined from the singularity structure of $\Delta_H(x^2)$ but from approximate expressions for the phase space. Our expression is, therefore, more general because it does not involve any assumption about the model parameters. Equation (3.8) of Ref. 7) corresponds, for instance to putting $m=0$ in Eq. (16).

$$n(M) \sim n_{\infty}(M) = \frac{M}{DE_{\infty}} ; \quad \bar{E}_{\infty} = \frac{m K_2(\frac{m}{T_0})}{K_1(\frac{m}{T_0})} . \quad (17)$$

For non-degenerate ground state ($D=1$) this coincides exactly with the result derived in I for the linear decay chain.

The integral equation for the invariant single particle distribution $D(r^2, r \cdot k)$ is rather similar to the mean multiplicity equation (8). It is the following:

$$\begin{aligned} \tau(r^2) D(r^2, r \cdot k) = & \frac{1}{2} S^4(r \cdot k) + \sum_{l=2}^{\infty} \frac{A B^l}{(l-1)!} \int d^4 r_1 \dots d^4 r_l \tau(r_1^2) \dots \tau(r_l^2) \times \\ & \times S^4(r - r_1 - \dots - r_l) D(r_1^2, r_1 \cdot k) . \end{aligned} \quad (18)$$

Here k denotes the four-momentum of the observed (stable) particle. If E is its energy in the rest frame of the fireball then $D(r^2, r \cdot k) = D(M^2, ME) = d(M, E)$. The solution of the equation and the asymptotic behaviour for $M \rightarrow \infty$ (E fixed) is

$$\begin{aligned} d(M, E) = & \frac{1}{2\tau(M^2)} \left\{ \sum_{l=2}^{\infty} \frac{l q_l}{D} \Theta(M - ml) \varrho^{(l-1)}(M^2 - 2ME + m^2; m^2, \dots, m^2) + S^4(r \cdot k) \right\} \simeq \\ & \simeq d_{\infty}(M, E) = n_{\infty}(M) e^{-\frac{E}{T_0}} \left[4\pi m T_0 K_1\left(\frac{m}{T_0}\right) \right]^{-1} . \end{aligned} \quad (19)$$

This is again exactly the same as $d_{\infty}(M, E)$ for the linear decay chain. A consequence is that the asymptotic spectrum in the thermodynamic model is strictly exponential, the recoils, more body decays, etc., do not cause a deviation from that ^{*)}.

The generating function of the multiplicity distribution is defined ⁴⁾ by

^{*)} Moreover, a more careful analysis shows that for large E the exponential is approached from below instead of a deviation above the exponential which could be concluded from a naive reasoning.

$$\psi(M, h) = \sum_{\ell=0}^{\infty} V_{\ell}(M) (1+h)^{\ell} \quad (20)$$

Here $V_{\ell}(M)$ is the probability that the fireball decays into a number ℓ of ground state hadrons. If in the first decay step the fireball decays into a certain number of fireballs then the generating function of its multiplicity distribution is the product of the generating functions for the individual decay product fireballs. The equation for $\psi(M, h)$ is therefore:

$$\tau(M^2) \psi(M, h) = (D+h) \delta(M^2 - m^2) + \sum_{\ell=2}^{\infty} \frac{AB^{\ell}}{\ell!} \int dM_1^2 \dots dM_{\ell}^2 \varrho^{(0)}(M^2; M_1^2, \dots, M_{\ell}^2) \times$$

$$\times \tau(M_1^2) \dots \tau(M_{\ell}^2) \psi(M_1, h) \dots \psi(M_{\ell}, h). \quad (21)$$

This is rather similar to Eq. (1) for the state density. The solution is

$$\psi(M, h) = \frac{1}{\tau(M^2)} \sum_{\ell=1}^{\infty} g_{\ell} \left(1 + \frac{h}{D}\right)^{\ell} \varrho^{(0)}(M^2; m^2, \dots, m^2) \Theta(M - \ell m). \quad (22)$$

The asymptotic behaviour for large M is also here identical to $\psi_{\infty}(M, h)$ given in Section 5 of I. It is a real surprise that the rather different (and realistic) decay chain considered in the present paper reproduces the asymptotic forms corresponding to the chain decay of I. This indicates that the power of M in front of the exponential in $\tau(M^2)$ is not really important. In the asymptotic region the solution possesses properties dictated by the exponential increase of the state density ^{*)}.

The solution in Eq. (22) for the multiplicity distribution has a very simple physical interpretation. Consider, for simplicity, from now on only the case of non-degenerate ground state ($D=1$) and call the only stable particle of mass m a "pion" ^{**)} . Then, comparing Eqs. (22) and (20) it follows that

^{*)} The assumptions of I correspond to a certain approximate linearization of the thermodynamic bootstrap, which, however, asymptotically behaves in the same way as the non-linear bootstrap.

^{**)} The quantum numbers as charge, etc., are disregarded here in the same way as in I. Hence our "pion" is neutral.

$$V_l(M) = \frac{g_l}{\tau(M^2)} \rho^{(l)}(M^2; m^2, \dots, m^2). \quad (23)$$

Therefore, the fireball can be thought as if it were built up virtually from the pions it decays into, $V_l(M)$ giving the probability that the number of its "virtual constituent" pions is l ^{*}). This picture can be extended even further. Introducing the virtual pion distribution inside the fireball of four-momentum r as

$$V_l(r; k_1, \dots, k_l) d^4k_1 \dots d^4k_l = \frac{g_l \theta(r^2 - l^2 m^2)}{\tau(r^2)} \delta^4(r - k_1 - \dots - k_l) \times \\ \times \delta(k_1^2 - m^2) \theta(k_{10}) \dots \delta(k_l^2 - m^2) \theta(k_{l0}) d^4k_1 \dots d^4k_l \quad (24)$$

the single particle distribution $d(M, E)$ given in Eq. (19) is the mean number of pions at energy E (if the fireball of mass M is at rest). Hence the distribution in Eq. (24) is something like the l particle exclusive cross-section. It is therefore straightforward to calculate from it not only the multiplicity distribution or the single particle distribution we considered up to now but also the many-particle distributions. It is the property of the statistical decay that the solution of the statistical bootstrap has such a simple physical interpretation in terms of the decay products of the fireballs.

Equation (24) has clearly a number of applications. To give an example, let us imagine that our fireball moves fast along the z axis and we are interested in the pion distribution in the longitudinal variables $x_j = (k_{j0} + k_{jz}) / (r_0 + r_z)$. After some algebra it follows from Eq. (24) that the distribution in x_j is

$$V_l(M^2; x_1, \dots, x_l) dx_1 \dots dx_l = \frac{g_l \theta(M^2 - l^2 m^2)}{\Gamma(l-1) \tau(M^2)} \frac{\pi}{2} \left(\frac{\pi M^2}{2} \right)^{l-2} \left[1 - \frac{m^2}{M^2} \left(\frac{1}{x_1} + \dots + \frac{1}{x_l} \right) \right]^{l-2} \times \\ \times \theta\left(\frac{M^2}{m^2} - \frac{1}{x_1} - \dots - \frac{1}{x_l}\right) \delta(1 - x_1 - \dots - x_l) dx_1 \dots dx_l. \quad (25)$$

^{*}) In reality the fireballs are bound states of other fireballs according to the thermodynamic bootstrap. The "virtual pion constituents" are materialized only during the statistical decay process.

For $M^2 \gg m^2$ and for not very small x (i.e., $1/x \ll M^2/m^2$) this expression simplifies considerably as the Θ function and the expression in the square brackets can be omitted. The mean number of pions at x turns out then to be

$$V(M; x) \cong (1-x)^{-1/2} \left(\frac{M}{2T_0} \right)^2 e^{-\frac{M}{T_0}(1-\sqrt{1-x})} \quad (26)$$

This has a consequence that if in high energy collisions only one or two fireballs are produced like in the models of Refs. 1)-3) and the probability of producing the fireball of mass M is $\sim \text{const. } M^{-2}$ then the single particle distribution near $x \sim 1$ at high enough energy behaves as $\text{const. } (1-x)^{-1/2}$. This looks like a "leading particle effect" as in our simple model there is only a single kind of stable hadrons. It would be interesting to check this result in a model with quantum numbers but this is already out of the scope of the present paper ^{*)}.

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*) A more detailed paper on this subject will be published elsewhere.

R E F E R E N C E S

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