CM-P00057933

Ref.TH.1495-CERN

Archive

CURRENT ALGEBRA SUM RULES FOR REGGEONS

R. Carlitz ⁺)
CERN - Geneva

A B S T R A C T

The interplay between the constraints of chiral $SU(2) \times SU(2)$ symmetry and Regge asymptotic behaviour is investigated. We review the derivation of various current algebra sum rules in a study of the reaction $\pi + \alpha \rightarrow \pi + \beta$. These sum rules imply that particles may be classified in multiplets of $SU(2) \times SU(2)$ and that each of these multiplets may contain linear combinations of an infinite number of physical states. Extending our study to the reaction $\pi + \alpha \rightarrow \pi + \pi + \beta$, we derive new sum rules involving commutators of the axial charge with the Reggeon coupling matrices of the ρ and f Regge trajectories. Some applications of these new sum rules are noted, and the general utility of these and related sum rules is discussed.

⁺) Research supported by a National Science Foundation Postdoctoral Fellowship.

1. INTRODUCTION

This paper explores some of the constraints that chiral $SU(2) \times SU(2)$ symmetry imposes on narrow resonance amplitudes. As is well known, the low energy theorems of chiral symmetry ¹⁾ fix the threshold values of amplitudes involving one or more massless pions. By means of dispersion relations these threshold constraints may be written in the form of sum rules ²⁾.

A systematic study of these current algebra sum rules for pion-hadron scattering amplitudes has been made by Gilman and Harari ³⁾. Their work was generalized somewhat by Weinberg ⁴⁾ who emphasized that the sum rules may be cast in an elegant algebraic form. The axial vector coupling matrix $\underline{X}$ and the isospin $\underline{I}$ were shown to generate the algebra $SU(2) \times SU(2)$:

$$[I_a, I_b] = i \epsilon_{abc} I_c \quad (1.1)$$

$$[I_a, X_b] = i \epsilon_{abc} X_c \quad (1.2)$$

$$[X_a, X_b] = i \epsilon_{abc} I_c \quad (1.3)$$

Furthermore, the mass matrix M was found to satisfy the commutation relation

$$[X_a, [X_b, M^2]] = \delta_{ab} \mathcal{F}, \quad (1.4)$$

where $\mathcal{F}$ is some unspecified matrix.

Relations (1.3) and (1.4) follow from the existence of unsubtracted dispersion relations for the $I=1$ and $I=2$ exchange amplitudes, respectively, of the process $\pi_a + \alpha \rightarrow \pi_b + \beta$. This, in turn, depends on the assumptions

$$\alpha_{I=1}(0) < 1 \quad (1.5)$$

and

$$\alpha_{I=2}(0) < 0, \quad (1.6)$$

where $\alpha_{I=1}$ and $\alpha_{I=2}$ denote the highest lying Regge trajectories with isospin 1 and 2 respectively.

The relations (1.1)-(1.3) indicate that the physical states on which $\tilde{X}$ and $\tilde{I}$ are defined form a representation of the algebra $SU(2) \times SU(2)$. Each irreducible representation will consist, in general, of linear combinations of a number of physical states.

Thus, for example, Gilman and Harari ³⁾ and Weinberg ⁴⁾ have proposed grouping the helicity zero states of the particles π , σ , A and ρ into multiplets as follows:

$$\pi_a \sin \psi + A_a \cos \psi, \quad \sigma \quad \in \quad \left(\frac{1}{2}, \frac{1}{2}\right) \quad (1.7)$$

$$\pi_a \cos \psi - A_a \sin \psi, \quad \rho_a \quad \in \quad (1,0) \oplus (0,1). \quad (1.8)$$

The relation (1.4) implies that the matrix M^2 transforms as the sum of a chiral scalar, $(0,0)$, and the fourth component of a chiral four-vector, $(\frac{1}{2}, \frac{1}{2})$. In the saturation scheme of Gilman and Harari and Weinberg, this implies relations among the π , σ , A and ρ masses:

$$m_A^2 = m_\rho^2 + m_\sigma^2 \quad (1.9)$$

$$m_\rho^2 / m_\sigma^2 = \tan^2 \psi \quad (1.10)$$

The choice $\psi = \pi/4$ gives a reasonable value for the $\rho \rightarrow \pi\pi$ decay amplitude determined from Eq. (1.8) and provides the attractive looking results $m_\rho^2 = m_\sigma^2$ and $m_A^2 = 2m_\rho^2$.

The apparent success of this approach has inspired a considerable amount of work at guessing saturation schemes ⁵⁾ including a larger number of particles. There is, however, a severe difficulty in the formulation of any realistic saturation scheme. As we shall prove in the following section, each multiplet may contain linear combinations of an infinite number of physical states. The reason is that the matrix elements of $\tilde{F}$ in Eq. (1.4) are all actually either infinite or zero. Hence the commutation $[\tilde{X}_a, [\tilde{X}_b, M^2]]$ may be divergent, which is possible only if the matrix $\tilde{X}$ connects any given state with an infinite number of other states.

That the chiral multiplets should contain an infinite number of physical states is very plausible from the viewpoint of Regge pole theory. In the absence of $I=2$ states the pion may be written as

$$\pi_a = v_a \sin \varphi + t_a \cos \varphi, \quad (1.11)$$

where v_a and t_a denote components of vector $((\frac{1}{2}, \frac{1}{2}))$ and tensor $((1,0) \oplus (0,1))$ representations, respectively. The amplitude for the transition $\alpha \rightarrow \beta + \pi_a$ is proportional to the matrix element $[X_a]_{\beta\alpha}$ ⁴⁾. Therefore, all $I=1$ states coupling to two pions must be included in the tensor representation t . But if an indefinitely rising ρ Regge trajectory⁶⁾ exists coupled to two pions, it will contain an infinite number of such states.

The inadequacy of the saturation scheme (1.7)-(1.8) has been recognized from another point of view as well. By writing a super-convergence relation for the $I=2$ exchange part of the helicity flip amplitude for the process $\pi_a + \alpha \rightarrow \pi_b + \beta$, Weinberg⁷⁾ derived an additional algebraic relation

$$[X_a, [X_b, MJ^+]] = \delta_{ab} \mathcal{L} \quad (1.12)$$

Here J^+ denotes the helicity raising operator and $\mathcal{L}$ is some unspecified matrix. Extending the saturation scheme of Eqs. (1.7), (1.8) to include helicity ± 1 states and imposing the constraint (1.12), one finds that $\psi = \pi/2$. This would make the π and σ and the ρ and A degenerate in mass and would exclude the decay $\rho \rightarrow 2\pi$. Weinberg concluded that these disastrous results could be avoided only by the inclusion of additional particles in the saturation scheme.

As noted above, the number of additional particles required in any realistic saturation scheme is actually infinite. We shall prove this point in the following Section after reviewing the derivation of Eqs. (1.3) and (1.4). Recognizing the fact that infinitely many particles are required in a realistic saturation scheme, we see that it is rather quixotic to try to guess the properties of all the undiscovered states which must be included in such a scheme.

A more reasonable approach is to search for a subset of sum rules which are rapidly convergent. For a given set of external states such sum rules may be approximately saturated with a finite number of intermediate states. We have already encountered three such sum rules, Eq. (1.3) and the $I=2$ parts of Eqs. (1.4) and (1.12):

$$[X_a, [X_b, M^2]]_{I=2} = 0 \quad (1.13)$$

$$[X_a, [X_b, M J^+]]_{I=2} = 0 \quad (1.14)$$

In Section 3 we will derive additional convergent sum rules, from a study of the reaction $\pi + \alpha \rightarrow \pi + \pi + \beta$. These new sum rules involve the commutators of the axial charge with matrices specifying the couplings of the ρ and f Regge trajectories. Some applications of these sum rules are discussed in Section 4. In Section 5 we summarize our principal results and indicate directions in which they may be generalized.

2. THE REACTION $\pi + \alpha \rightarrow \pi + \beta$

In this Section we review the derivation of the sum rules (1.3) and (1.4). In the course of our derivation it will become clear that the matrix elements of $\mathcal{F}$ in Eq. (1.4) are all either infinite or zero. More precisely, we show that for any states α and β with a non-zero coupling to the f Regge trajectory at $t=0$, the matrix element $\mathcal{F}_{\beta\alpha}$ is infinite; if α and β decouple from the f trajectory at $t=0$, then $\mathcal{F}_{\beta\alpha}$ is zero.

To review the techniques for constructing current algebra sum rules, we consider first the Feynman amplitude $M_{\beta b, \alpha a}(s, \lambda)$ for the forward scattering process

$$\pi_a(q) + \alpha(p, \lambda) \rightarrow \pi_b(q') + \beta(p', \lambda) \quad (2.1)$$

We assume that chiral symmetry is realized exactly by the existence of massless pions. The kinematics are described by the variables

$$\begin{aligned}
 s &= (p+q)^2 \\
 t &= (q'-q)^2 = 0 \\
 u &= (p-q')^2
 \end{aligned}
 \tag{2.2}$$

Since all the particles are collinear, there is no difference in the helicity λ of particles α and β .

The amplitude M may be separated into isospin odd and even parts as follows:

$$M_{\beta b, \alpha a}^{(-)}(s, \lambda) = (s-u)^{-1} \left[M_{\beta b, \alpha a}(s, \lambda) - M_{\beta a, \alpha b}(s, \lambda) \right],
 \tag{2.3}$$

$$M_{\beta b, \alpha a}^{(+)}(s, \lambda) = \frac{1}{2} \left[M_{\beta b, \alpha a}(s, \lambda) + M_{\beta a, \alpha b}(s, \lambda) \right].
 \tag{2.4}$$

Crossing symmetry provides the constraint

$$M_{\beta b, \alpha a}(s, \lambda) = M_{\beta a, \alpha b}(u, \lambda),
 \tag{2.5}$$

so both $M^{(-)}$ and $M^{(+)}$ are symmetric under $s \leftrightarrow u$ interchange.

Chiral symmetry implies that the threshold values of $M^{(\pm)}(s, \lambda)$ have the form ⁴⁾

$$\begin{aligned}
 M_{\beta b, \alpha a}^{(-)}(m_\pi^2, \lambda) &= 4i F_\pi^{-2} \epsilon_{abc} [I_c]_{\beta\alpha} \\
 &+ 4 F_\pi^{-2} \sum_{\gamma} \left\{ [X_b(\lambda)]_{\beta\gamma} [X_a(\lambda)]_{\gamma\alpha} \right. \\
 &\quad \left. - [X_a(\lambda)]_{\beta\gamma} [X_b(\lambda)]_{\gamma\alpha} \right\},
 \end{aligned}
 \tag{2.6}$$

$$M_{\beta b, \alpha a}^{(+)}(m_\alpha^2, \lambda) = 2 F_\pi^{-2} \sum_{\gamma} (2m_\gamma^2 - m_\alpha^2 - m_\beta^2) \times \left\{ [X_b(\lambda)]_{\beta\gamma} [X_a(\lambda)]_{\gamma\alpha} + [X_a(\lambda)]_{\beta\gamma} [X_b(\lambda)]_{\gamma\alpha} \right\} \quad (2.7)$$

In these equations $[I_a]_{\beta\alpha}$ and $[X_a(\lambda)]_{\beta\alpha}$ denote matrix elements of the isospin and axial charge operators, respectively. In terms of $X_a(\lambda)$, the coupling of a pion π_a to states α and β is given in any collinear frame by the expression

$$g(\pi_a, \alpha, \beta) = 2 F_\pi^{-1} (m_\alpha^2 - m_\beta^2) [X_a(\lambda)]_{\beta\alpha}, \quad (2.8)$$

where λ denotes the helicity of α and β . [See Ref. 4) for a more thorough discussion.] The notation " γ " in Eqs. (2.6) and (2.7) indicates that the sums over intermediate states include only those states γ for which $m_\gamma^2 = m_\alpha^2$ or $m_\gamma^2 = m_\beta^2$.

The high energy behaviour of $M^{(-)}(s, \lambda)$ is determined by the ρ Regge pole. In the limit $|s| \rightarrow \infty$ ($\text{Im } s > 0$),

$$M_{\beta b, \alpha a}^{(-)}(s, \lambda) \rightarrow i \epsilon_{abc} \left(\frac{1 - e^{-i\pi\alpha_\rho}}{\sin \pi\alpha_\rho} \right) [R_c(\lambda)]_{\beta\alpha} s^{\alpha_\rho - 1} + \dots \quad (2.9)$$

The matrix $R_c(\lambda)$ denotes the $t=0$ residue of the ρ Regge trajectory, and α_ρ is the $t=0$ intercept. Since $\alpha_\rho < 1$, $M^{(-)}(s, \lambda)$ vanishes asymptotically and, therefore, satisfies an unsubtracted dispersion relation. Saturating this dispersion relation with a sequence of narrow resonances, we obtain, using Eq. (2.8), the expression

$$M_{\beta b, \alpha a}^{(-)}(s, \lambda) = 4 F_\pi^{-2} \sum_{\gamma} \frac{(m_\gamma^2 - m_\beta^2)(m_\alpha^2 - m_\gamma^2)}{(m_\gamma^2 - s)(m_\gamma^2 - u)} \times \left\{ [X_b(\lambda)]_{\beta\gamma} [X_a(\lambda)]_{\gamma\alpha} - [X_a(\lambda)]_{\beta\gamma} [X_b(\lambda)]_{\gamma\alpha} \right\}. \quad (2.10)$$

Evaluating the dispersion relation at threshold we find

$$M_{\beta b, \alpha a}^{(-)}(m_a^2, \lambda) = -4F_\pi^{-2} \sum_{\gamma \neq} \left\{ [X_b(\lambda)]_{\beta\gamma} [X_a(\lambda)]_{\gamma\alpha} - [X_a(\lambda)]_{\beta\gamma} [X_b(\lambda)]_{\gamma\alpha} \right\} . \quad (2.11)$$

The notation " $\gamma \neq$ ", indicates that one excludes from the sum all states γ for which $m_\gamma^2 = m_\alpha^2$ or $m_\gamma^2 = m_\beta^2$. It is obvious from Eq. (2.10) that such states make no contribution. Equating Eqs. (2.6) and (2.11) we obtain the result

$$[X_a(\lambda), X_b(\lambda)] = i \epsilon_{abc} I_c , \quad (2.12)$$

which coincides with Eq. (1.3).

The leading Regge trajectory (exclusive of the Pomeron) contributing to the high energy behaviour of $M^+(s, \lambda)$ is the f Regge pole. Thus, in the limit $|s| \rightarrow \infty$ ($\text{Im } s > 0$) the resonance dominated part of $M^+(s, \lambda)$ has the asymptotic form⁸⁾

$$M_{\beta b, \alpha a}^{(+)}(s, \lambda) \rightarrow \delta_{ab} \left(\frac{1 + e^{-i\pi\alpha_f}}{\sin \pi\alpha_f} \right) [F(\lambda)]_{\beta\alpha} S^{\alpha_f} + \dots . \quad (2.13)$$

The matrix $F(\lambda)$ is proportional to the $t=0$ residue of the f Regge trajectory, and α_f is the $t=0$ intercept. The lower order terms omitted in Eq. (2.13) all vanish in the limit $|s| \rightarrow \infty$, provided that all other trajectories with $I=0$ and $G=+1$ and any $I=2$ trajectories have $t=0$ intercepts less than 0.

Consider now a Cauchy integral for the function $M^{(+)}(s, \lambda)$

$$M^{(+)}(s, \lambda) = \frac{1}{2\pi i} \int_C ds' \frac{M^{(+)}(s', \lambda)}{s' - s} , \quad (2.14)$$

around the contour C illustrated in Fig. 1. This contour consists of two pieces: a portion C_1 , which wraps around the real axis and a circle C_2 of radius $\sqrt{(m_\alpha^2 + m_\beta^2)}/2$ centered at $s' = (m_\alpha^2 + m_\beta^2)/2$. Approximating the singularities of $M^{(+)}(s, \lambda)$ by a sequence of resonances, and using Eq. (2.8), we find that the C_1 integral has the form

$$M_{\beta b, \alpha a}^{(+, G_1)}(s, \lambda) = \frac{2}{F_\pi^2} \sum_{\gamma}^{\nu} \frac{(m_a^2 - m_\gamma^2)(m_b^2 - m_\gamma^2)(2m_\gamma^2 - m_a^2 - m_b^2)}{(m_\gamma^2 - s)(m_\gamma^2 - u)} \times \left\{ [X_b(\lambda)]_{\beta\gamma} [X_a(\lambda)]_{\gamma\alpha} + [X_a(\lambda)]_{\beta\gamma} [X_b(\lambda)]_{\gamma\alpha} \right\}. \quad (2.15)$$

The cut-off ν in the sum on γ indicates that only states with $m_\gamma^2 < \nu$ are to be included. For $s \ll \nu$ we can use Eq. (2.13) to estimate the integral around G_2 . This yields the expression

$$M_{\beta b, \alpha a}^{(+, G_2)}(s, \lambda) = \frac{2}{\pi \alpha_f} [F(\lambda)]_{\beta\alpha} \nu^{\alpha_f} + \dots \quad (2.16)$$

Summing Eqs. (2.15) and (2.16) and taking the limit $s \rightarrow m_a^2$, we obtain the following expression for the threshold value of $M^{(+)}(s, \lambda)$:

$$M_{\beta b, \alpha a}^{(+)}(m_a^2, \lambda) = -2 F_\pi^{-2} \sum_{\gamma \neq a}^{\nu} (2m_\gamma^2 - m_a^2 - m_b^2) \times \left\{ [X_b(\lambda)]_{\beta\gamma} [X_a(\lambda)]_{\gamma\alpha} + [X_a(\lambda)]_{\beta\gamma} [X_b(\lambda)]_{\gamma\alpha} \right\} + \delta_{ab} 2(\pi \alpha_f)^{-1} [F(\lambda)]_{\beta\alpha} \nu^{\alpha_f} + \dots \quad (2.17)$$

Equating this expression with the chiral symmetry prediction (2.7), we obtain the finite-energy sum rule ⁹⁾

$$\sum_{\gamma}^{\nu} (2m_\gamma^2 - m_a^2 - m_b^2) \left\{ [X_b(\lambda)]_{\beta\gamma} [X_a(\lambda)]_{\gamma\alpha} + [X_a(\lambda)]_{\beta\gamma} [X_b(\lambda)]_{\gamma\alpha} \right\} = \delta_{ab} F_\pi^2 (\pi \alpha_f)^{-1} [F(\lambda)]_{\beta\alpha} \nu^{\alpha_f} + \dots \quad (2.18)$$

Extracting the $I=2$ part of this expression, one can take the limit $\nu \rightarrow \infty$ to obtain the commutation relation

$$[X_a(\lambda), [X_b(\lambda), M^2]]_{I=2} = 0, \quad (2.19)$$

which coincides with Eq. (1.13). The $I=0$ part of Eq. (2.18) is, however, infinite in the limit $\nu \rightarrow \infty$ unless $[F(\lambda)]_{\beta\alpha}$ is zero. Explicitly comparing Eqs. (1.4) and (2.18), one can write $\mathcal{F}$ in the form

$$\mathcal{F} = F_\pi^2 (\pi d_f)^{-1} F \times \lim_{\nu \rightarrow \infty} \nu^{\alpha_f}, \quad (2.20)$$

so the matrix elements of $\mathcal{F}$ are indeed all infinite (or zero). The infinite matrix elements of $\mathcal{F}$ correspond to all those states α and β with non-zero couplings to the f trajectory at $t=0$. Therefore, any particles which couple to the f trajectory must be connected by $\underline{X}$ to the infinite number of particles which appear as intermediate states in the divergent commutator $[X_a, [X_a, M^2]]_{\beta\alpha}$. The particles α and β thus belong to multiplets containing linear combinations of an infinite number of physical states.

The sum rules (2.12) and (2.18) exhaust the chiral symmetry constraints (2.6) and (2.7) for the reaction $\pi + \alpha \rightarrow \pi + \beta$. By considering the helicity flip amplitudes for this process, it is possible to derive additional sum rules ^{3), 7)}. These sum rules follow from the superconvergence property of the helicity flip amplitudes, and are logically independent of chiral symmetry constraints. In this paper we restrict our attention to the helicity non-flip amplitudes and consider only sum rules which arise from chiral symmetry.

Since high mass contributions to Eqs. (2.12) and (2.19) are damped by a power of the mass, it is reasonable to saturate these sum rules approximately with some finite number of intermediate states. If m_α^2 and m_β^2 are of order m^2 , and if only intermediate states with $m_\gamma^2 < \nu$ are included in Eq. (2.12), the error due to the neglect of states with $m_\gamma^2 \geq \nu$ will be of order $(m^2/\nu)^{1-\alpha_f}$. Similarly, in Eq. (2.19) the error introduced by neglecting intermediate states with $m_\gamma^2 \geq \nu$ would be of order $(m^2/\nu)^{-\alpha_{I=2}}$.

In sharp contrast to Eqs. (2.12) and (2.19) the $I=0$ part of Eq. (2.18) may be divergent. As emphasized in the previous Section, it is unreasonable to approximate this infinite quantity with any finite number of intermediate states.

In Eq. (2.20) we have shown that the infinities arise from the fact that the zero intercept of the f trajectory, α_f , is greater than zero. Thus, any saturation scheme for Eq. (2.18) which employs only a finite number of states, tacitly assumes that $\alpha_f = 0$.

3. THE REACTION $\pi + \alpha \rightarrow \pi + \pi + \beta$

In this Section we shall study the scattering amplitude for the collinear process

$$\pi_a(q) + \alpha(p, \lambda) \rightarrow \pi_b(q') + \pi_c(k) + \beta(p', \lambda) \quad (3.1)$$

Applying the constraints of current algebra, we shall derive new sum rules analogous to those of the previous Section. In particular we will look for convergent sum rules which, like Eqs. (2.12) and (2.19), are amenable to finite approximation schemes and hence can provide useful constraints on low-lying resonances.

The Feynman amplitude $M_{\beta bc, \alpha a}(s_1, s_2, \lambda)$ is a function of the variables

$$s_1 = (p + q)^2 \quad (3.2)$$

$$s_2 = (k + p')^2 \quad (3.3)$$

The momentum transfers $t = (p' - p)^2$ and $(q' - q)^2$ are both fixed to be zero. Let s_1 be held large and fixed with $\text{Im } s_1 > 0$. For $|s_1| \gg |s_2| + m_\alpha^2 + m_\beta^2$ the amplitude $M(s_1, s_2, \lambda)$ may be represented by the Regge asymptotic expansion illustrated in Fig. 2. The coefficient of each term in the Regge expansion involves the scattering amplitude Reggeon $R + \alpha \rightarrow \pi_c + \beta$.

Since the Reggeon "mass", $(q'-q)^2$, is zero, the kinematics for this process are identical to those for the process $\pi + \alpha \rightarrow \pi + \beta$ of the previous Section. The present analysis, therefore, closely parallels that previously given.

In the limit $s_2 \rightarrow m_\beta^2$ the pion π_c will have a vanishing four-momentum and the value of the amplitude will be completely specified by chiral symmetry. Restricting attention to the amplitude $M_{\beta bc, \alpha a}^{(-)}(s_1, s_2, \lambda)$ antisymmetric in a and b , we find from the Adler condition, Ref. 10), and Eq. (2.9) the result

$$M_{\beta bc, \alpha a}^{(-)}(s_1, m_\beta^2, \lambda) = 2i F_\pi^{-1} \epsilon_{abd} \left(\frac{1 - e^{-i\pi\alpha_p}}{\sin \pi\alpha_p} \right) S_1^{\alpha p} \\ \times \left\{ \sum_{\gamma(=\alpha)} [R_d(\lambda)]_{\beta\gamma} [X_c(\lambda)]_{\gamma\alpha} - \sum_{\gamma(=\beta)} [X_c(\lambda)]_{\beta\gamma} [R_d(\lambda)]_{\gamma\alpha} \right\} + \dots \quad (3.4)$$

The notation " $\gamma(=\alpha)$ " and " $\gamma(=\beta)$ " indicates that only those states γ with $m_\gamma^2 = m_\alpha^2$ and $m_\gamma^2 = m_\beta^2$ are to be included in the respective sums. The terms of Eq. (3.4) correspond simply to the s_2 and u_2 channel pole terms pictured in Fig. 3. $[u_2 = (k-p)^2 = m_\alpha^2 + m_\beta^2 - s_2]$.

We now seek to convert the low energy theorem (3.4) into a sum rule. The procedure is to write a Cauchy integral in the variable s_2 , evaluate the Cauchy integral at $s_2 = m_\beta^2$, and equate the result to Eq. (3.4). The contour C for the Cauchy integral is the one illustrated in Fig. 1. Recall that C consists of two parts: C_1 wraps around the real axis and C_2 is a circle of radius $\nu - (m_\alpha^2 + m_\beta^2)/2$ centered at $(m_\alpha^2 + m_\beta^2)/2$. The parameter ν is chosen to be very large compared to m_α^2 and m_β^2 but small compared to $|s_1|$

$$m_\alpha^2 + m_\beta^2 \ll \nu \ll |s_1| \quad (3.5)$$

We will assume that the only singularities in $M(s_1, s_2, \lambda)$ are poles due to narrow width resonances. If $\text{Im } s_1 \gg \nu$, and $\text{Re } s_1 \gg \nu$, then singularities from poles in the channels $(\pi_a \beta)$, $(\pi_b \alpha)$ and $(\pi_b \beta)$ will be far outside the contour C . The only singularities lying close to C are poles on the real axis corresponding to resonances in the channels $(\pi_c \beta)$ and $(\pi_c \alpha)$. These pole terms are illustrated in Fig. 3.

Using Eqs. (2.8) and (2.9) then, we may evaluate the G_1 integral explicitly and write the Cauchy integral in the following form:

$$\begin{aligned}
 M_{\beta bc, da}^{(-)}(s_1, s_2, \lambda) &= 2i F_{\pi}^{-1} \epsilon_{abd} \left(\frac{1-e^{-i\pi\alpha_p}}{\sin \pi\alpha_p} \right) s_1^{\alpha_p} \\
 &\times \sum_{\gamma} \left\{ \frac{(m_{\gamma}^2 - m_{\beta}^2) [X_c(\lambda)]_{\beta\gamma} [R_d(\lambda)]_{\gamma d}}{m_{\gamma}^2 - s_2} \right. \\
 &\quad \left. + \frac{(m_{\gamma}^2 - m_{\alpha}^2) [R_d(\lambda)]_{\beta\gamma} [X_c(\lambda)]_{\gamma d}}{m_{\gamma}^2 - u_2} \right\} + \dots \\
 &+ \frac{1}{2\pi i} \int_{C_2} ds' \frac{M_{\beta bc, da}^{(-)}(s_1, s', \lambda)}{s' - s_2} .
 \end{aligned} \tag{3.6}$$

In the G_1 integrand we have retained only terms of the leading order in s_1 .

If ν is large, satisfying the inequality (3.5), then $M^{(-)}(s_1, s', \lambda)$ may be approximated on C_2 by the double Regge exchange diagram ¹¹⁾ of Fig. 4. The Reggeon exchanged between π_c and β must have natural parity $P(-1)^J = -1$ and G parity -1 . The contribution of the A_1 trajectory, for example, to the G_2 integral is given to the leading orders in s_1 and ν by

$$(\delta_{ac}\delta_{be} - \delta_{ae}\delta_{bc}) \left(\frac{1-e^{-i\pi\alpha_p}}{\sin \pi\alpha_p} \right) \left(\frac{s_1}{\nu} \right)^{\alpha_p} \nu^{\alpha_A} [A_e]_{\beta d} + \dots , \tag{3.7}$$

where $[A_e]_{\beta d}$ is proportional to the $t=0$ residue of the A_1 Regge pole. Taking the limit $s_2 \rightarrow m_{\beta}^2$, and using Eq. (3.7), we write Eq. (3.6) in the form

$$\begin{aligned}
 M_{\beta bc, da}^{(-)}(s_1, m_{\beta}^2, \lambda) &= 2i F_{\pi}^{-1} \epsilon_{abd} \left(\frac{1-e^{-i\pi\alpha_p}}{\sin \pi\alpha_p} \right) s_1^{\alpha_p} \\
 &\times \left\{ \sum_{\gamma(\neq \beta)} [X_c(\lambda)]_{\beta\gamma} [R_d(\lambda)]_{\gamma d} - \sum_{\gamma(\neq d)} [R_d(\lambda)]_{\beta\gamma} [X_c(\lambda)]_{\gamma d} \right\} + \dots \\
 &+ \left(\frac{1-e^{-i\pi\alpha_p}}{\sin \pi\alpha_p} \right) s_1^{\alpha_p} \nu^{\alpha_A - \alpha_p} [\delta_{ac} A_b - \delta_{bc} A_a]_{\beta d} \\
 &+ \text{other Regge terms} .
 \end{aligned} \tag{3.8}$$

The notation " $\gamma(\neq\beta)$ " and " $\delta(\neq\alpha)$ " indicates that states γ with $m_\gamma^2 = m_\beta^2$ and $m_\delta^2 = m_\alpha^2$ are to be excluded from the respective sums.

Now we equate expressions (3.4) and (3.8). Dividing by the common factor $(1 - e^{-i\pi\alpha_\rho}) \sin^{-1} \pi\alpha_\rho \cdot s_1^{\alpha_\rho}$, and taking the limit $s_1 \rightarrow \infty$ along some fixed ray, gives the result

$$\begin{aligned} 2i F_\pi^{-1} \epsilon_{abd} \sum_{\gamma}^{\vee} \{ [X_c(\lambda)]_{\beta\gamma} [R_d(\lambda)]_{\gamma\alpha} \\ - [R_d(\lambda)]_{\beta\gamma} [X_c(\lambda)]_{\gamma\alpha} \} \\ = \nu^{\alpha_A - \alpha_\rho} [\delta_{ac} A_b - \delta_{bc} A_a]_{\beta\alpha} \\ + \text{other Regge terms} . \end{aligned} \quad (3.9)$$

The A_1 trajectory and all other trajectories with $P(-1)^J = -1$ lie lower than the ρ trajectory. Therefore, $\alpha_A - \alpha_\rho < 0$ (and similarly, $\alpha_i - \alpha_\rho < 0$ for the other Regge terms), so in the limit $\nu \rightarrow \infty$ the right-hand side of Eq. (3.9) vanishes. In the same limit the cut-off in the sum on γ on the left-hand side is removed, and we are left with the result

$$[X_c(\lambda), R_d(\lambda)] = 0 . \quad (3.10)$$

Note that the vanishing of the right-hand side of Eq. (3.10) occurs because the ρ trajectory lies higher than any trajectory with $P(-1)^J = -1$. Therefore, analogous expressions may be derived for the other leading positive natural parity trajectories, the f , ω and A_2 . If, for example, we were to consider the amplitude $M_{\beta bc, \alpha a}^{(+)}(s_1, s_2, \lambda)$ symmetric in a and b and repeat the analysis leading to Eq. (3.10), we would obtain the result

$$[X_c(\lambda), F(\lambda)] = 0 . \quad (3.11)$$

Relations involving the ω and A_2 trajectories may be obtained from a similar study of the reaction $K + \alpha \rightarrow \pi + K + \beta$.

Because of the existence of divergent commutators such as Eq. (1.4), relations such as (3.10) and (3.11) must be handled very carefully. Consider, for example, the double commutator $[\bar{X}_a(\lambda), [\bar{X}_b(\lambda), R_c(\lambda)]]$. The usual Jacobi identity cannot be applied to this commutator because the value of the commutator is itself not well-defined; it depends on the order in which the masses of the various intermediate states are allowed to go to infinity.

4. APPLICATIONS

From Eq. (3.9) we see that the commutator (3.10) converges rather slowly - at a rate $\nu^{\alpha_A - \alpha_P} \simeq \nu^{-\frac{1}{2}}$. Consequently, approximate evaluation of the commutator is difficult, since a large number of intermediate states must be included. More rapid convergence - and hence greater ease of approximation - is possible if one evaluates the commutator between states which decouple at $t=0$ from the leading trajectories of negative natural parity.

Such a situation is obtained if we choose α and β to be nucleon states and consider the commutator

$$\langle N_\beta | [X_a, R_b] | N_\alpha \rangle = 0 \quad . \quad (4.1)$$

The coupling of nucleon states to any trajectory of negative natural parity must involve the Dirac matrix γ_5 in the form $\bar{u}(p') \gamma_5 u(p)$ or $\bar{u}(p') \gamma_5 \gamma_\mu u(p)$. The contributions of these couplings to the leading terms of the double Regge expansion for the process $\pi + \alpha \rightarrow \pi + \pi + \beta$ (see Fig. 4) vanish in the limit $t = (p' - p)^2 \rightarrow 0$. Therefore the commutator (4.1) will converge at the rate $\nu^{\alpha_A - 1 - \alpha_P} \simeq \nu^{-\frac{3}{2}}$. Optimistically we may approximate the commutator by including only the lowest lying intermediate states - the N and Δ .

It is convenient to define the following reduced matrix elements of $\bar{X}$ and R :

$$\langle N_\beta | X_a | N_\alpha \rangle = X_{NN} \varphi_\beta^\dagger \tau_a \varphi_\alpha , \quad (4.2)$$

$$\langle N_\beta | R_a | N_\alpha \rangle = R_{NN} \varphi_\beta^\dagger \tau_a \varphi_\alpha , \quad (4.3)$$

$$\langle N_\beta | X_a | \Delta_\alpha \rangle = X_{N\Delta} \varphi_\beta^\dagger \chi_a^\alpha , \quad (4.4)$$

$$\langle N_\beta | R_a | \Delta_\alpha \rangle = R_{N\Delta} \varphi_\beta^\dagger \chi_a^\alpha . \quad (4.5)$$

In these expressions φ denotes a two-component isotopic spinor and χ is a four-component isotopic vector-spinor subject to the constraint

$$\tau_a \chi_a = 0 . \quad (4.6)$$

The spinors are normalized such that

$$\sum_\alpha \varphi_\alpha \varphi_\alpha^\dagger = 1 , \quad (4.7)$$

$$\sum_a \chi_a^\alpha \chi_a^{\beta\dagger} = \delta_{\alpha\beta} - \frac{\tau_a \tau_b}{3} . \quad (4.8)$$

Including only N and Δ intermediate states in the commutator (4.1), we obtain the following constraint on the reduced matrix elements:

$$X_{NN} R_{NN} - \frac{1}{3} X_{N\Delta} R_{N\Delta} = 0 . \quad (4.9)$$

This relation may be tested experimentally. The reactions $\pi^- p \rightarrow \pi^0 n$ ¹²⁾ and $\pi^+ p \rightarrow \pi^0 \Delta^{++}$ ¹³⁾ are both well described by ρ Reggeon exchange. According to Eqs. (2.9), (4.3) and (4.5) the ratio $|R_{NN}/R_{N\Delta}|$ is given by the expression

$$r = \left| \frac{R_{NN}}{R_{N\Delta}} \right| = \left[\frac{d\sigma/dt (\pi^- p \rightarrow \pi^0 n)}{2 d\sigma/dt (\pi^+ p \rightarrow \pi^0 \Delta^{++})} \Big|_{t=0} \right]^{1/2} . \quad (4.10)$$

Experimentally this ratio is found to have the value ¹⁴⁾

$$r_{\text{experimental}} = 0.61 \pm 0.08 \quad (4.11)$$

The value of X_{NN} is given ⁴⁾ directly in terms of the axial vector coupling constant

$$X_{NN} = g_A / 2g_V = 0.62 \quad (4.12)$$

From Eq. (2.8) we obtain the following expression ⁴⁾ for the $\Delta \rightarrow N\pi$ decay rate:

$$\Gamma(\Delta^{++} \rightarrow p\pi^+) = \frac{p^3 |X_{N\Delta}|^2}{\pi F_\pi^2} \quad (4.13)$$

where p denotes the pion momentum in the Δ rest frame. This allows us to extract $|X_{N\Delta}|$ from the measured width of the Δ ,

$$|X_{N\Delta}| = 1.04 \pm 0.05 \quad (4.14)$$

Substituting the experimental values for X_{NN} and $X_{N\Delta}$ in Eq. (4.9), we obtain a prediction for the ratio r ,

$$r_{\text{predicted}} = 0.55 \pm 0.03 \quad (4.15)$$

Comparing this prediction with the experimental value (4.11), we see that their agreement is rather good.

In principle, tests of the sort just discussed can be carried out for other choices of the external states α and β . In practice this is difficult because of the limited data available on the matrix elements of $\underline{R}$. In any case, sum rules of the type (3.10) and (3.11) provide useful constraints on models of the meson and baryon spectrum. We should emphasize that reasonable approximations of these sum rules can be obtained only when the cut-off mass for the intermediate states is large relative to the external masses. The same caveat also applies to the sum rules (1.3), (1.13) and (1.14).

In some cases ³⁾ it has been noted that saturation of the sum rules (1.3) and (1.13) with particles belonging to a single multiplet of $SU(6)$ will yield results characteristic of the symmetry $SU(6)_W$. This feature

is not shared by Eq. (3.10). With external nucleons we have saturated this sum rule with N and Δ states to obtain the constraint (4.9). By contrast, $SU(6)_W$ requires ¹⁷⁾ that $R_{N\Delta} = 0$ while X_{NN} , $X_{N\Delta}$, and R_{NN} may be non-zero.

5. SUMMARY AND DISCUSSION

In summarizing the results of this paper, we should emphasize just what assumptions were necessary at each stage of the work. In Section 2, where Weinberg's derivation of Eqs. (1.3) and (1.4) has been reviewed, the only assumptions were those of Weinberg, namely chiral symmetry and Regge asymptotic behaviour. In particular, our claim that the $I=0$ part of (1.4) is divergent follows from these assumptions alone.

The principal results of Section 3 are the Reggeon sum rules, Eqs. (3.10) and (3.11). The derivation of these equations requires, in addition to chiral symmetry and multi-Regge asymptotic behaviour, the assumption that the only singularities of $M(s_1, s_2, \lambda)$ are simple poles. This assumption should be reasonable for the resonance dominated part of the amplitude, which, according to duality, is what builds the Regge asymptotic behaviour. The results are in any case applicable to dual resonance models, and indeed they may shed some light on how such models can satisfy the constraints of chiral symmetry.

For our application to the $N-\Delta$ system, Eq. (4.9), the assumption that $M(s_1, s_2, \lambda)$ has only simple poles is more difficult to justify, as there may be appreciable cut contributions to the spin non-flip part of the amplitudes for $\pi^- p \rightarrow \pi^0 n$ and $\pi^+ p \rightarrow \pi^0 \Delta^{++}$. In the presence of such terms our derivation of Eq. (3.10) must be re-examined. If Eq. (3.10) is shown to be valid for some particular cut model, then one must extract the cut terms from the data before calculating the ratio r of Eq. (4.10). Our procedure of Section 4 may be viewed as either the simple neglect of all cut terms or as the parametrization of the full amplitude by an effective pole term.

It is clear that the work presented here may be profitably expanded in several directions.

- i) The relative signs of the terms in Eq. (4.9) could be extracted from detailed data on reactions such as $\pi^- p \rightarrow \pi^0 (\pi^- p)$ over a mass

region of the (π^-p) system which spans the Δ^0 resonance. If detailed data were available, one could also study the contributions of higher mass states to the sum rule (4.1).

- ii) Additional Reggeon sum rules may also be derived. An obvious extension is to consider the helicity flip couplings of the ρ and f trajectories and search for superconvergence sum rules analogous to Eq. (1.14). Conceivably one could derive sum rules for the couplings of baryon Regge trajectories as well.
- iii) Using all the sum rules of this paper as well as their possible generalizations, one could investigate the constraints they imply for realistic models of the spectrum of low lying meson and baryon resonances.

ACKNOWLEDGEMENTS

I would like to thank J. Weyers for advice and encouragement throughout the course of this work. I am also pleased to acknowledge the hospitality of the School of Natural Sciences at the Institute for Advanced Study and the Theoretical Study Division of CERN, and I am grateful for discussions with H.D.I. Abarbanel, D. Amati, R. Dashen and J. Yellin.

REFERENCES

- 1) For a general discussion see
S.L. Adler and R.F. Dashen, Current Algebras and Applications to
Particle Physics, W.A. Benjamin, Inc., New York (1968).
- 2) S.L. Adler, Phys.Rev.Letters 14, 1051 (1965), and Phys.Rev. 140, B736
(1965);
W.I. Weisberger, Phys.Rev.Letters 14, 1047 (1965), and Phys.Rev. 143,
1302 (1966).
- 3) F.D. Gilman and H. Harari, Phys.Rev. 165, 1803 (1968).
- 4) S. Weinberg, Phys.Rev. 177, 2604 (1969).
- 5) M. Noga and C. Cronström, Phys.Rev. D1, 2414 (1970);
F. Buccella, C.A. Savoy and E. Celeghini, Nuovo Cimento 7A, 281 (1972)
and further references therein.
- 6) The proof that some matrix elements of $\mathcal{T}$ are infinite does not
depend on the existence of indefinitely rising trajectories. The
argument given here is merely to establish the plausibility of multi-
plets with infinitely many particles.
- 7) S. Weinberg, Phys.Rev.Letters 22, 1023 (1969).
- 8) This is a statement of the hypothesis that Regge poles (exclusive of the
Pomeron) are dual to resonances. See
P.G.O. Freund, Phys.Rev.Letters 20, 1235 (1968) and
H. Harari, *ibid.*, 20, 1395 (1968).
The addition of a Pomeron term of Eq. (2.13) would not significantly
alter any of our subsequent conclusions. The crucial feature of this
equation is the fact that $M^{(+)}(s, \lambda)$ is infinite in the limit $s \rightarrow \infty$;
the rate at which $M^{(+)}(s, \lambda)$ tends to ∞ is not really important.
- 9) This relation has also been derived by H. Kleinert, CERN report TH.1474
(14 March 1972).
- 10) S. Adler, Phys.Rev. 139, B1638 (1965).
- 11) N.F. Bali, G.F. Chew and A. Pignotti, Phys.Rev. 163, 1572 (1967).
- 12) G. Höhler, J. Baacke, H. Schlaile and P. Sonderegger, Phys.Letters 20,
79 (1966).
- 13) R.D. Mathews, Nuclear Phys. B11, 339 (1969).

- 14) We have used data for $\pi^- p \rightarrow \pi^0 n$ at $P_{\text{lab}} = 13.3 \text{ GeV}/c$ [Ref. 15] and $\pi^+ p \rightarrow \pi^0 \Delta^{++}$ at $P_{\text{lab}} = 13.1 \text{ GeV}/c$ [Ref. 16]. The reported values of the forward cross-sections for these reactions are 195 ± 25 at $260 \pm 40 \mu\text{b}/(\text{GeV})^2$, respectively.
- 15) P. Sonderegger et al., Phys.Letters 20, 75 (1966).
- 16) J.H. Scharenguival et al., Nuclear Phys. B36, 363 (1972).
- 17) See, e.g.,
R. Carlitz and M. Kislenger, Phys.Rev. D2, 405 (1970).

FIGURE CAPTIONS

Figure 1 : Contour C for the Cauchy integral.

Figure 2 : Regge asymptotic form of the amplitude $\pi_a + \alpha \rightarrow \pi_b + \pi_c + \beta$.

Figure 3 :

- a) : s_2 channel pole.
- b) : u_2 channel pole

Figure 4 : Double Regge asymptotic form of the scattering amplitude
 $\pi_a + \alpha \rightarrow \pi_b + \pi_c + \beta$.

FIG.1

FIG.2

FIG.3

FIG.4