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ON A CLASS OF REPRESENTATIONS OF THE BILOCAL

CURRENT ALGEBRA ON A LIGHT-LIKE RAY

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A B S T R A C T

A class of representations of the algebra of bilocal currents on a plane perpendicular to a light-like ray is constructed. The saturation of these representations with states of definite momenta conjugate to the configuration space variables on the light-like plane reveals a Fock space nature of the representation space, generated by quark and anti-quark fields obeying Fermi statistics, projected on the characteristic light-like ray. The mass spectrum, however, is not determined by the algebra. Representations on a subspace of the constructed Fock space are discussed for special momentum configurations.

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I. INTRODUCTION

The aim of this work is to study a class of representations of the algebra of light-like bilocal charges on a plane perpendicular to a light ray ¹⁾.

We consider the bilocal currents

$$J_a^{(n)\pm}(x,y) = n^\mu J_{a\mu}^\pm(x,y)$$

$$n = \left(\frac{1}{2}, 0, 0, -\frac{1}{2}\right) \quad n^2 = 0 \quad (I.1)$$

and study their algebraic properties using the mnemonics of free quark field representation

$$J_{a\mu}^\pm(x,y) \sim \bar{\psi}(x) \frac{\lambda_a}{2} \gamma_\mu (1 \pm \gamma_5) \psi(y) \quad (I.2)$$

Configuration space shall be restricted to the plane

$$n \cdot X = 0 \quad \rightarrow \quad X_+ = 0$$

$$X_\pm = \frac{1}{\sqrt{2}} (X_0 \pm X_3) \quad (I.3)$$

Following Gell-Mann and Fritzsche ¹⁾ one finds choosing the light ray joining four points x, y, u, v in the direction of n :

$$\left[J_a^{(n)\pm}(x,y), J_b^{(n)\pm}(u,v) \right] =$$

$$= \left\{ \begin{aligned} & \frac{1}{2} (i f_{abc} + d_{abc}) \sqrt{2} \delta_3^{(n)}(y-u) J_c^{(n)\pm}(x,v) \\ & + \frac{1}{2} (i f_{abc} - d_{abc}) \sqrt{2} \delta_3^{(n)}(v-x) J_c^{(n)\pm}(u,y) \end{aligned} \right\}$$

$$[n \cdot (x, y, u, v)] = 0$$

$$\delta_3^{(n)}(\xi) = \delta(\xi_-) \delta_2(\xi_\perp) \quad (I.4)$$

Introducing the centre-of-mass and relative variables

$$R = \frac{1}{2} (x+y) ; z = x-y = (z, 0) ; R' = \frac{1}{2} (u+v) ; z' = u-v = (z', 0)$$

and momentum vectors

$$k = (k_+, 0, k_\perp) ; k' = (k'_+, 0, k'_\perp)$$

(I.4) can be transformed to commutation relations for the quantities

$$\tilde{j}_a^{(h)\pm}(k, z) = \frac{1}{\sqrt{2}} \int dR_+ d^2R_\perp e^{+i k \cdot R} j_a^{(h)\pm}(x, y)$$

$$[\tilde{j}_a^{(h)\pm}(k, z), \tilde{j}_b^{(h)\pm}(k', z')] = E_{abc}(kz - k'z) \tilde{j}_c^{(h)\pm}(k+k', z+z')$$

$$E_{abc}(A) = \frac{1}{2} \left\{ \begin{aligned} & (i f_{abc} + d_{abc}) \exp\left(\frac{iA}{2}\right) \\ & + (i f_{abc} - d_{abc}) \exp\left(-\frac{iA}{2}\right) \end{aligned} \right\} \quad (I.5)$$

In order to saturate the relations in Eq. (I.5) by one-particle states with definite momentum components

$$P_+, P_\perp$$

we introduce the non-relativistic normalization

$$\langle P^{(h)}, \alpha / P'^{(h)}, \beta \rangle = (2\pi)^3 \int_{\alpha\beta} \delta(P_+ - P'_+) \delta_2(P_\perp - P'_\perp)$$

$$|P, \alpha\rangle_{\text{invariant}} = \sqrt{2P_+} |P^{(h)}, \alpha\rangle$$

α, β denote any quantum numbers other than the relevant momentum components (mass, spin, ...).

Assuming that a saturation of (I.5) with one-particle states exists one obtains

$$\sum_{r, \delta} \left\{ \begin{aligned} & \langle P_2^{(h)} \beta / j_a^{(h)\pm}(z) / P_r^{(h)} \delta \rangle \langle P_r^{(h)} \delta / j_b^{(h)\pm}(z') / P_1^{(h)} \alpha \rangle \\ & - \langle P_2^{(h)} \beta / j_b^{(h)\pm}(z') / \bar{P}_r^{(h)} \delta \rangle \langle \bar{P}_r^{(h)} \delta / j_a^{(h)\pm}(z) / P_1^{(h)} \alpha \rangle \end{aligned} \right\}$$

$$= F_{abc} (kz' - k'z) \langle P_2^{(h)} \beta / j_c^{(h)\pm}(z+z') / P_1^{(h)} \alpha \rangle$$

$$j_a^{(h)\pm}(z) = \frac{1}{\sqrt{2}} j_a^{(h)\pm}(P=0, z); \quad P_r^{(h)} = P_2^{(h)} + k^{(h)} = P_1^{(h)} - k'^{(h)} \\ \bar{P}_r^{(h)} = P_2^{(h)} + k'^{(h)} = P_1^{(h)} - k^{(h)} \quad (I.6)$$

We investigate in the following representations of the commutation relations in Eqs. (I.5) and (I.6).

II. REPRESENTATIONS OF THE BILOCAL CHARGE ALGEBRA ON A LIGHT-LIKE PLANE

From the construction of the algebra (I.5) it is clear that the following form of $\tilde{j}^{(n)}(k, z)$ yields a representation

$$\tilde{j}_a^{(h)\pm}(k, z) = \int d^{(h)} R e^{+ikR} Q_{(h)\pm}^* \left(R + \frac{z}{2} \right) \frac{\partial}{\partial z} Q_{(h)\pm} \left(R - \frac{z}{2} \right) \quad (II.1)$$

provided that the operators $Q_{(n)\pm}$ satisfy the (anti) commutation relations

$$[Q_{(h)\pm}(R), Q_{(h)\pm}^*(R')] = \delta_3^{(h)}(R-R') \quad (II.2)$$

Contrary to the situation on a space-like plane the relativistic spectral conditions on physical states imply that $Q_{(n)\pm}$ cannot be a pure absorption operator. It is this fact which renders the realization of the current commutation relations on a light-like plane by currents which can be saturated without continuum states extremely questionable.

In fact, one finds the following Fourier decomposition of $Q_{(n)\pm}$

$$Q_{(n)\pm}^\alpha(R) = \int d^3 q_+ \left[a_{(n)\pm}^\alpha(q) e^{-i q R} + b_{(n)\pm}^{\alpha*}(q) e^{+i q R} \right]$$

$$\int d^3 q_+ = \int \mathcal{V}(q_+) d q_+ d q_\perp$$

(II.3)

Furthermore (II.2) and (II.3) are compatible only if we choose the anti-commutation relations

$$\begin{aligned} [a_{(n)\pm}^\alpha(q), a_{(n)\pm}^{\alpha'*}(q')]_+ &= [b_{(n)\pm}^\alpha(q), b_{(n)\pm}^{\alpha'*}(q')]_+ = \\ &= \frac{1}{(2\pi)^3} \mathcal{V}(q_+) \mathcal{V}(q'_+) \delta_3^{(\bar{n})}(q - q') \delta_{\alpha\alpha'} \\ \delta_3^{(\bar{n})}(q) &= \delta(q_+) \delta_2(q_\perp) \end{aligned}$$

(II.4)

All other anti-commutators vanish. The currents we are discussing are not normal ordered so that all relations hold modulo c number terms.

If the quark operators act on a larger space than just configuration space the algebra (I.5) can be realized in the following form

$$\hat{j}_a^{(n)\pm}(k, z) = \int d^4 R e^{+i k R} Q_{(n)\pm}^*(R + \frac{z}{2}) \frac{\lambda_a}{2} e^{i A z} Q_{(n)\pm}(R - \frac{z}{2})$$

(II.5)

where A is an arbitrary self-adjoint operator not acting on configuration space.

The representation of the algebra realized in Eq. (II.5) implemented by the requirements of Lorentz invariance yields a class of representations realized on configuration space and an arbitrary number of internal four-vectors ²⁾ $\xi_1, \dots, \xi_d$ in the following way

$$\begin{aligned}
 \tilde{j}_a^{(n)\pm}(k, z) &= \int d^{(n)}R e^{+ikR} \int d^4\xi_1 \dots d^4\xi_d \times \\
 &\times Q_{(n)\pm}^* \left(R + \frac{z}{2}, \xi_1 + \alpha_1 \frac{z}{2}, \dots, \xi_d + \alpha_d \frac{z}{2} \right) \frac{\lambda_d}{2} \times \\
 &\times Q_{(n)\pm} \left(R - \frac{z}{2}, \xi_1 - \alpha_1 \frac{z}{2}, \dots, \xi_d - \alpha_d \frac{z}{2} \right) \\
 \left[Q_{(n)\pm}(R, \underline{\xi}), Q_{(n)\pm}^*(R', \underline{\xi}') \right]_+ &= \delta_3^{(n)}(R-R') \delta_{4d}(\underline{\xi}-\underline{\xi}') \\
 \underline{\xi} &= (\xi_1, \dots, \xi_d)
 \end{aligned} \tag{II.6}$$

$\alpha_1, \dots, \alpha_d$ are arbitrary real constants which can be normalized to 1 for $\alpha \neq 0$.

The light-like vector n can be factorized by means of the Pauli matrices

$$(n^\mu \sigma_\mu)_{\neq 0} = \ell_\sigma \cdot \ell_\sigma^*$$

Hence $Q_{(n)\pm}$ depend linearly on ℓ . Since A in Eq. (II.5) therefore cannot depend on n or ℓ the existence of a representation of the Lorentz group implies that A transforms like a four-vector

$$U(\Lambda) A^\mu U^{-1}(\Lambda) = \Lambda^\mu_\nu A^\nu$$

and thus the choice of representations in Eq. (II.6).

The representations so constructed give the following structure for the currents

$$j_{\beta\alpha}^{(n)\pm}(z) \rightarrow \frac{1}{\sqrt{2}} j_a^{(n)\pm}(0, z) = \sum_{\beta, \alpha} j_{\beta\alpha}^{(n)\pm} \left(\frac{\lambda_d}{2} \right)_{\beta\alpha} \tag{II.7}$$

characteristic for particle (anti-particle) creation with subsequent annihilation as well as pair creation and pair annihilation

$$: J_{\beta\alpha}^{(n)\pm}(z) := \int d^3 q_+ d^3 q'_+ d\xi \times$$

$$\left\{ \begin{aligned} & \times e^{i \frac{q+q'}{2} z} a_{\beta}^{(n)\pm *}(q', \xi + \frac{z}{2}) a_{\alpha}^{(n)\pm}(q, \xi - \frac{z}{2}) \\ & - e^{-i \frac{q+q'}{2} z} b_{\alpha}^{(n)\pm *}(q, \xi - \frac{z}{2}) b_{\beta}^{(n)\pm}(q', \xi + \frac{z}{2}) \\ & - e^{i \frac{q-q'}{2} z} b_{\alpha}^{(n)\pm *}(q, \xi - \frac{z}{2}) a_{\beta}^{(n)\pm *}(q', \xi + \frac{z}{2}) \\ & + e^{-i \frac{q-q'}{2} z} b_{\beta}^{(n)\pm}(q', \xi + \frac{z}{2}) a_{\alpha}^{(n)\pm}(q, \xi - \frac{z}{2}) \end{aligned} \right\}$$

$$: J : \text{ normally ordered products} \quad (\text{II.8})$$

$\xi \pm \frac{z}{2}$ stands for $(\xi_1 \pm \alpha_1 \frac{z}{2}, \dots, \xi_d \pm \alpha_d \frac{z}{2})$

The $a^{(n)}$, $b^{(n)}$ satisfy the anti-commutation relations

$$\begin{aligned} [a_{\alpha}^{(n)\pm}(q, \xi), a_{\alpha'}^{(n)\pm *}(q', \xi')]_+ &= [b_{\alpha}^{(n)\pm}(q, \xi), b_{\alpha'}^{(n)\pm *}(q', \xi')]_+ \\ &= \left(\frac{1}{2\pi}\right)^3 \delta_{\alpha\alpha'} \delta(q_+) \delta(q'_+) \delta_3^{(\frac{1}{2})}(q-q') \delta_{dd}(\xi-\xi') \end{aligned}$$

(II.9)

with all other anti-commutation vanishing.

A representation of (I.6) is thus realized in the Fock space generated by

$$a_{\alpha}^{(n)\pm}, b_{\beta}^{(n)\pm}$$

The question naturally arises whether a subspace of the so constructed Fock space can saturate the algebra of Eq. (I.6) at least for a suitable choice of the momenta $p_2, p_R, \bar{p}_R, p_1$ ³⁾. Although we cannot give a complete answer it is obvious that whenever the pair creation and pair annihilation terms have to contribute a subspace saturating the algebra will come close to the total Fock space and probably coincides with the latter if trivially conserved quantum numbers are fixed.

We want to emphasize at this point that the Fock representation according to (II.9) does not mean that there are one-particle quark states.

The mass spectrum is determined by the dynamical operator

$$P_- = \frac{(M^2 + P_\perp^2)}{2P_+}$$

$$P_- = P_- (a^{(n)\pm}, b^{(n)\pm}, a^{(n)\pm*}, b^{(n)\pm*})$$

which is not restricted by the algebraic relations (I.5) or (II.9).

A favourable situation for a "small" representation is to choose

$$a) \quad P_{2+} \geq P_{r+} \geq P_{1+} \iff P_{2+} \geq \bar{P}_{r+} \geq P_{1+}$$

$$b) \quad P_{2+} \leq P_{r+} \leq P_{1+} \iff P_{2+} \leq \bar{P}_{r+} \leq P_{1+}$$

a) or b) imply that it is legitimate to consider subspace with definite number of quarks and anti-quarks, because pair creation and subsequent absorption (or vice versa) are excluded.

Using the notation

$$\left\{ \hat{T}_a^{(n)\pm} (P_2, z, P_1) \right\}_{\beta\alpha} = \langle P_2^{(n)} \beta / \hat{T}_a^{(n)\pm}(z) / P_1^{(n)} \alpha \rangle$$

the following choice of the operators $\hat{T}_a^{(n)\pm}$ yields a representation of (I.6) in the case a) or b):

$$\hat{T}_a^{(n)\pm} = \left\{ \begin{array}{l} \frac{\lambda_a}{2} \left(\exp \left[- (P_2^{(n)} - P_1^{(n)}) A + z B \right] \right) \cdot E_1^\pm + \\ + \left(\frac{-\bar{\lambda}_a}{2} \right) \cdot \left(\exp \left[(P_2^{(n)} - P_1^{(n)}) A' + z B' \right] \right) \cdot E_2^\pm \end{array} \right\} \quad (II.10)$$

A, B and A', B' are two arbitrary pairs of conjugate anti-self-adjoint vector operators, $E_{1,2}^\pm$ denote projection operators satisfying the relations

$$[A_\mu, B_\nu] = i g_{\mu\nu} \mathbb{1} ; [A'_\mu, B'_\nu] = i g_{\mu\nu} \mathbb{1}$$

$$E_1^\pm \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} E_1^\pm ; E_2^\pm \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} A' \\ B' \end{pmatrix} E_2^\pm$$

$$E_1^\pm \cdot E_1^\mp = E_2^\pm E_2^\mp = E_1^\pm E_2^\pm = E_1^\pm E_2^\mp = E_2^\pm E_1^\pm = E_2^\pm E_1^\mp = 0$$

(II.11)

The commutation relations for A, B; A', B' in (II.11) can always be represented in an $\mathbb{R}^4$ space ⁴⁾

$$\begin{aligned} A_\mu &\rightarrow (-i) \xi_\mu & B_\mu &\rightarrow \frac{\partial}{\partial \xi_\mu} \\ A'_\mu &\rightarrow (-i) \eta_\mu & B'_\mu &\rightarrow \frac{\partial}{\partial \eta_\mu} \end{aligned} \quad (\text{II.12})$$

Invariant reduced form factors of local currents obtained from the representation (II.10) are of the form

$$\begin{aligned} F_\pm(t) &\sim \langle p_2 / j^\pm(z=0) / p_1 \rangle \\ F_\pm(t) &= \int d^4 \xi \left\{ \begin{aligned} &c_1 \varphi_{1\pm}^*(p_2, \xi) e^{i(p_2-p_1)\xi} \varphi_{1\pm}(p_1, \xi) + \\ &+ c_2 \varphi_{2\pm}^*(p_2, \xi) e^{-i(p_2-p_1)\xi} \varphi_{2\pm}(p_1, \xi) \end{aligned} \right\} \end{aligned}$$

c_1, c_2 : Clebsch-Gordan coefficients (II.13)

Due to the projection operators $E_{1,2}^\pm$ in (II.10) the normalization of the wave functions $\varphi_{1,2}^\pm(p, \xi)$

$$N_{1,2}^\pm = \int d^4 \xi \varphi_{1,2}^{\pm*}(p, \xi) \varphi_{1,2}^\pm(p, \xi)$$

is arbitrary and can be adapted to the relevant coupling constants $g_V, g_A, \dots$

The representation in (II.10) clearly fails if either

$$p_{r+} > p_{1+} + p_{2+} \quad \text{or} \quad \bar{p}_{r+} > p_{1+} + p_{2+}$$

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