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COVARIANT PERTURBATION THEORY AND CHIRAL SUPERPROPAGATORS

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A B S T R A C T

We use a covariant formulation of perturbation theory for the non-linear chiral invariant pion model to define chiral superpropagators leading to S matrix elements which are independent of the choice of the pion field co-ordinates. The relation to the standard definition of chiral superpropagators is discussed.

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The chiral invariant non-linear pion model has recently been studied within the context of superpropagator methods ^{1),2)}. One of the most interesting applications of such methods is based on the fact that the superpropagator approach gives a prescription how to fix the undetermined parameters that occur in the usual perturbation theory for non-renormalizable Lagrangians. In most cases there is considerable freedom in the choice of a superpropagator reflecting the ambiguity in the definition of time ordering for an infinite sum of products of free field propagators ^{3),4),5)}. Once a particular superpropagator has been chosen one would, however, expect that the resulting S matrix elements are determined uniquely. In particular, for the non-linear pion model, S matrix elements should be independent of the choice of the pion field in terms of which the Lagrangian is defined. Although it is not clear whether the equivalence theorem ⁶⁾ applies to the undetermined subtraction constants in the perturbation theory for non-renormalizable Lagrangians, it seems natural to require that the superpropagator method leads to co-ordinate independent S matrix elements. That this requirement presents a problem has been observed by Lehmann and Trute ²⁾ in their treatment of $\pi\pi$ scattering where the subtraction constants do depend on the parametrization of the pion field.

In this letter we want to demonstrate that by using a co-ordinate independent formulation of perturbation theory as developed in Ref. 7), one is able to define superpropagators without working in any particular pion reference frame. This implies that the above-mentioned subtraction constants are co-ordinate independent.

In order to arrive at chiral invariant and co-ordinate independent expressions one is naturally led to use a formalism that exhibits the covariance with respect to chiral and co-ordinate transformations at every stage of the calculations. Such a formalism has been set up in Ref. 7) and, in the following, we will briefly sketch its main features.

The Green's functions of the theory are obtained by functional differentiation of the generating functional

$$Z(J) = N^{-1} \int \prod_i \pi^i(x) \sqrt{g(\pi)} \exp \left\{ i S(\pi) + i \int J_i(x) \pi^i(x) dx \right\} \quad (1)$$

with respect to external sources $J_i(x)$, where

$$S(\pi) = \frac{1}{2} \int dx g_{ij}(\pi) \partial_\mu \pi^i \partial^\mu \pi^j \quad (2)$$

$g(\pi)$ is the determinant of the metric $g_{ij}(\pi)$ in curved isospace with constant curvature ⁸⁾ $\frac{1}{\pi^2}$, and N is a normalization constant chosen so that $Z(0) = 1$.

A classical field $\varphi^i(x)$ is introduced ⁹⁾ as the solution of the equation of motion

$$\frac{\delta S}{\delta \pi^i(x)} + J_i(x) = 0 \quad (3)$$

corresponding to an extremal point of the exponent in (1). Equation (3) can be turned into an integral equation for $\varphi^i(x)$ in which the free Green's function with Feynman boundary conditions occurs. The exponent in (1) as well as the measure ¹⁰⁾ $\prod_x \sqrt{g(\pi)}$ are then expanded covariantly around the classical field $\varphi^i(x)$, which means that only covariant derivatives appear and distances in isospace are measured along geodesics. Finally, by expanding exponentials one reduces the functional integral to a series of functional integrals of Gaussian type which can be calculated in the standard way. One obtains a series of functionals depending, in a covariant way, on the classical field $\varphi^i(x)$ which is itself a functional of the external sources $J_i(x)$ via Eq. (3). Each functional in this series corresponds to a class of Feynman diagrams with a definite number of loops and a definite topological loop structure, but with an arbitrary number of tree diagrams attached.

In this approach the "superpropagator" approximation consists in summing all diagrams with two vertices and an arbitrary number of internal lines connecting them. The contributions $W_n(J)$ to the generating functional of connected Green's functions $W(J)$, which contain all these diagrams, are of the form

$$W_n(J) = \frac{i^{n+1}}{2(n!)^2} S_{a_1 \dots a_n} \left(\sum_{P(b_i)} G_{a_1 b_1} \dots G_{a_n b_n} \right) S_{b_1 \dots b_n} \quad (4)$$

$n \geq 3$

where we have introduced dreibein fields $e_a^k(\pi)$ ($a=1,2,3$) via

$$g_{ij}(\pi) = e_{ai}(\pi) e_{aj}(\pi), \quad e_a^i e_{bi} = \delta_{ab} \quad (5)$$

and

$$S_{a_1 \dots a_n} = e_{a_1}^{k_1} \dots e_{a_n}^{k_n} S_{;k_1 \dots k_n} \quad ,$$

$S_{;k_1 \dots k_n}$ being the n th covariant functional derivatives of $S(\pi)$ with respect to the π^i . $G_{ab}(x, x')$ is defined by

$$\int d^4x'' S_{ab}(x, x'') G_{bc}(x'', x') = \delta_{ac} \delta(x-x') \quad (6)$$

and the sum in (4) is over the $n!$ permutations of the $b_1, \dots, b_n$. The functional $W_n(J)$ contains arbitrary tree insertions in the loop lines because G_{ab} is a functional of the classical field φ^i by way of its definition (6). In Eq. (4) and in the following (except where space-time arguments are explicitly written) we use the condensed notation whereby indices a_i stand both for isospin and a space-time argument x_{a_i} and a summation over a_i implies integration over x_{a_i} .

In order to calculate the functionals $W_n(J)$ one has to know the vertices $S_{a_1 \dots a_n}$.

We first calculate

$$\frac{\delta S}{\delta \pi^k} = S_{;k} = - (D_\mu)_{ke} \partial^\mu \pi^e \quad (7)$$

with

$$(D_\mu)_{ke} = (g_{ke} \partial_\mu^{(k)} + \Gamma_{k, \ell m} \partial_\mu \pi^m) \delta(x_k - x_e) \quad (8)$$

where $\partial_\mu^{(k)}$ indicates differentiation with respect to x_k and $\Gamma_{k, \ell m}(\pi)$ are the Christoffel symbols for the metric $g_{ik}(\pi)$.

Using

$$(\mathbb{D}_\mu)_{kl;r} = R_{klrt} \partial_\mu \pi^t \quad (9)$$

and

$$(\partial_\mu \pi^l)_{;m} = (\mathbb{D}_\mu)^l{}_m \quad (10)$$

where R_{klrt} is the Riemann curvature tensor, one gets

$$S_{ikl} = - (\mathbb{D}_\mu)_{kl} (\mathbb{D}^\mu)^r{}_l - R_{klrm} \partial_\mu \pi^r \partial^\mu \pi^m \quad (11)$$

Since isospace has constant curvature F_π^{-2} , we have

$$R_{klrm} = F_\pi^{-2} (g_{kl} g_{rm} - g_{km} g_{rl}) \delta(x_k - x_l) \delta(x_k - x_r) \delta(x_k - x_m) \quad (12)$$

and therefore the covariant derivatives of the Riemann tensor vanish. It is at this point that chiral invariance comes into the game and allows us to write down immediately the general vertices, namely

$$\begin{aligned} S_{ik_1 \dots k_{2n}} &= 4^{n-1} \partial_\mu \pi^r R_{nk_1 k_2 r} R^r{}_{k_3 k_4} \dots R^t{}_{k_{2n-1} k_{2n}} u \partial^\mu \pi^u + \\ &+ 4^{n-1} (\mathbb{D}_\mu)^n{}_{k_1} R_{nk_2 k_3 r} R^r{}_{k_4 k_5} \dots R^t{}_{k_{2n-2} k_{2n-1}} u (\mathbb{D}^\mu)^u{}_{k_{2n}} \quad (13a) \\ &\quad (n \geq 2) \end{aligned}$$

$$S_{ik_1 \dots k_{2n+1}} = 4^n \partial_\mu \pi^r R_{nk_1 k_2 r} R^r{}_{k_3 k_4} \dots R^t{}_{k_{2n-1} k_{2n}} u (\mathbb{D}^\mu)^u{}_{k_{2n+1}} \quad (13b) \\ (n \geq 1)$$

Complete symmetrization of all indices is understood in Eqs. (13a) and (13b) because only the completely symmetrized expressions contribute to $W_n(J)$.

Before applying the method as described above to a specific example, let us first comment on the difference between the covariant and the usual approach.

The equivalence theorem ⁶⁾ asserts that S matrix elements calculated from the sum of all Feynman diagrams with a fixed number of internal loops must be independent of the parametrization of the pion field ^{*)}. Moreover, it is clear that diagrams with different analytical properties cannot combine to give co-ordinate independent S matrix elements, unless the individual contributions are already co-ordinate independent on shell. To consider a definite example, one would expect that the diagrams in Fig. 1, which are two three-loop contributions to $\pi\pi$ scattering, give separately co-ordinate independent S matrix elements. The internal lines in the diagrams stand for free massless propagators and the vertices are given by the ordinary derivatives $S_{,k_1k_2k_3}$ and $S_{,k_1k_2k_3k_4}$ in the usual formulation and the respective covariant derivatives in the covariant formulation.

The diagrammatic representation of the amplitudes in Fig. 1 is, however, misleading in the non-covariant approach in the following sense. It is not difficult to convince oneself that in general the vertex functionals $S_{,k_1\dots k_n}$ ($n \geq 3$) will contain terms of the form $T_{k_1\dots k_n} \square \delta(x)$, where $T_{k_1\dots k_n}$ does not contain derivatives and transforms in a complicated way under pion field redefinitions. This term, when acting on the free propagator, replaces the free Green's function by a δ function, in other words, the line representing the propagator is contracted ^{**)}. Therefore, diagram 1b contains contributions which have the same analytic properties as the amplitude in Fig. 1a. That extra contributions like the ones coming from diagram 1b have to be taken along in order to guarantee co-ordinate independent S matrix elements is exemplified by the work of Lehmann and Trute ²⁾ who consider only diagrams of type 1a (corresponding to the conventional superpropagator approximation) and obtain co-ordinate dependent results for the subtraction constants.

It is the main virtue of covariant perturbation theory that the covariant vertices $S_{,k_1\dots k_n}$ do not contain terms with $\square \delta(x)$ for all $n \geq 3$, as can be seen from Eqs. (13a) and (13b). By using covariant vertices

^{*)} With a possible ambiguity for the renormalization constants due to the non-renormalizability of the theory.

^{**)} Tadpole diagrams are eliminated by normal ordering, which in both approaches amounts to setting $G_0(0) = 0$.

$S; k_1 \dots k_n$ instead of ordinary vertices $S, k_1 \dots k_n$ in the superpropagator approximation one automatically includes all contributions from type b diagrams in the standard perturbation expansion where certain lines are contracted as explained above. Therefore, the diagrammatic representation in the covariant perturbation theory unambiguously reflects the analytic properties of the specific amplitude one is considering. In particular, diagrams of type a in the covariant framework lead to co-ordinate independent S matrix elements as will be shown in a forthcoming publication where the present formalism will be applied to $\pi\pi$ scattering.

Although the covariant perturbation expansion finds its most natural application in cases where the space of field co-ordinates is curved, it is in principle also useful when the field metric has been produced by a field co-ordinate transformation from the Euclidean metric. Consider the simplest example of a theory of free massless scalar fields and assume that amplitudes are calculated using a parametrization of the fields which makes the Lagrangian highly non-linear. In the usual non-covariant approach it would be an extremely tedious task to verify by explicit calculation that all S matrix elements are trivial. In the covariant approach this is obvious without further calculations : in flat space the Riemann tensor vanishes, therefore $S; k_1 \dots k_n = 0$ ($n \geq 3$), no matter what field parametrization is used, and $S_{a_1 a_2} = -\delta_{a_1 a_2} \square \delta(x_{a_1} - x_{a_2})$ which implies that the exact Green's functions of the theory as defined in Eq. (6) are the Green's functions for free massless scalar fields.

In order to give an explicit example we calculate the sum of all two-point functions with one external leg at each vertex and any number of internal lines between the two vertices (Fig. 2). Omitting the free propagator $G_0(x) = i/[4\pi^2(x^2 - i\epsilon)]$, one obtains the following result

$$\hat{G}_{ij}(x-y) = \delta_{ij} \int dx' dy' G_0(x-x') S(x'-y') G_0(y'-y) \quad (14)$$

with

$$S(x) = \frac{16}{3} (2\pi)^4 \sum_{n=1}^{\infty} \left(\frac{4}{F\pi^2} \right)^{2n} \frac{(2n-1)(2n+1)}{2n(2n)!} [i G_0(x)]^{2n+3} \quad (15)$$

The first observation is that the Green's function $\tilde{G}_{ij}$ is independent of the pion field parametrization. This is not required by general principles, but is a consequence of the specific off-shell behaviour implied by the functional method. What is more interesting is that the superpropagator $S(x)$ is an entire function of $G_0(x)$ of order 1, as one would expect if the interaction Lagrangian is a localizable field in Jaffe's sense¹¹⁾. However, Lehmann and Trute have shown that the requirement of localizability to all orders in the usual perturbation expansion uniquely determines the parametrization of the pion field to be the exponential one. It is therefore remarkable that the covariant perturbation theory leads to a unique superpropagator, no matter what co-ordinate system is chosen for the pion field. This remark has of course nothing to do with the usual ambiguities of time ordering which lie outside the present approach.

Finally, we want to note that the covariant superpropagator approach is in principle also applicable to the Einstein Lagrangian which governs the gravitational interaction. Whether such a program is technically feasible is still another question.

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REFERENCES

- 1) A.P. Hunt, K. Koller and Q. Shafi - Imperial College London Preprint ICTP 69/19 (1970) ;
B.W. Keck and J.G. Taylor - Phys.Rev. D3, 1846 (1971).
- 2) H. Lehmann and H. Trute - DESY Preprint 72/8 (1972).
- 3) H. Lehmann and K. Pohlmeyer - Commun.Math.Phys. 20, 101 (1971).
- 4) "Non-Polynomial Lagrangians, Renormalization and Gravity" - Lectures from the Coral Gables Conference (1971), Gordon and Breach, New York.
- 5) F. Krause and M. Scheunert - Bonn University Preprint PI-2-105 (1972).
- 6) S. Coleman, J. Wess and B. Zumino - Phys.Rev. 177, 2239 (1969).
- 7) J. Honerkamp - Nuclear Phys. B36, 130 (1972).
- 8) K. Meetz - J.Math.Phys. 10, 589 (1969).
- 9) D.G. Boulware and L.S. Brown - Phys.Rev. 172, 1628 (1968).
- 10) J. Honerkamp and K. Meetz - Phys.Rev. D3, 1996 (1971) ;
J.M. Charap - Phys.Rev. D3, 1998 (1971) ;
J. Gerstein, R. Jackiw, B.W. Lee and S. Weinberg - Phys.Rev. D3, 2486 (1971).
- 11) A. Jaffe - Phys.Rev. 158, 1454 (1967).

FIGURE CAPTIONS

Figure 1 a) and b) : three-loop contributions to $\pi\pi$ scattering.

Figure 2 Feynman diagram contributing to the superpropagator with two external lines $\tilde{G}_{ij}(x-y)$.

a)

b)

FIG.1

FIG.2