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KINEMATICAL CUT-OFF FOR THERMODYNAMIC SINGLE-PARTICLE
SPECTRA USING A PHASE-SPACE METHOD

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ABSTRACT

Thermodynamic momentum spectra do not obey energy conservation; therefore a kinematical cut-off procedure has to be added. We describe here a method which replaces the thermodynamic formulae by phase space near the kinematical limits. This method is actually used in calculating the thermodynamic single-particle spectra.

1. INTRODUCTION

The following discussion is to be understood in the context of the thermodynamical model¹⁾, where fireballs with a given velocity (characterized by a Lorentz factor γ) decay according to a thermodynamic black-body momentum distribution. In order to restore energy-momentum conservation, which is violated by the exponential tail of this thermodynamic distribution, we have to find the kinematical limits for particles emitted from a fireball with given velocity γ and to introduce some smooth cut-off procedure at this limit.

Notation: We use $E = \sqrt{p^2 + m^2}$ and p as convenient and in parallel.

2. THE KINEMATICAL LIMITS

The kinematical limits for secondary particles produced by the decay of fireballs depend on the quantum numbers of the fireballs, the kind of the produced particle and the mass of the fireball. We consider two different situations

- i) peripheral production of fireballs
- ii) central production of fireballs.

Furthermore, we define at which condition we switch in calculating the kinematical limits from the one to the other. Only kinematical limits for secondary particle production in pp collisions are considered, the extension of the method to other initial states is straightforward.

2.1 The kinematical limits for peripheral production of fireballs

The situation considered is defined in Fig. 1. The fireball with mass $M(\gamma)$ is produced in a pp collision. It decays into the particle considered with mass m and one or several recoil particles with invariant mass m_R . The energy of particle m in the M -rest frame resulting from the decay $M(\gamma) \rightarrow m + m_R$ is²⁾

$$\varepsilon(\gamma, m) = \frac{M^2 + m^2 - m_R^2}{2M} \quad (1)$$

The maximum of $\varepsilon(\gamma, m)$ is reached for $M^2(\gamma) \rightarrow \max$ and $m_R^2 \rightarrow \min$. At the minimum, m_R consists of the minimum number of particles m_i , required because of conservation laws for Q , S , etc., in the decay of M , with zero relative momenta

$$m_R = \sum m_i \quad (2)$$

Fig. 1 Decay of a fireball $M(\gamma)$ produced peripherally with Lorentz factor γ .

The fireball M is produced in the c.m.s. of a collision with c.m.s. energy $E_{cm} = \sqrt{s}$, together with a group of particles with invariant mass m' .

$$E_{cm} \rightarrow M + m' \quad (3)$$

The c.m.s. energy of the fireball becomes

$$\mathcal{E}(M) = \frac{E_{cm}^2 + M^2 - m'^2}{2E_{cm}} \quad (4)$$

and its Lorentz factor γ is

$$\gamma(M) = \frac{\mathcal{E}(M)}{M} = \frac{E_{cm}^2 + M^2 - m'^2}{2ME_{cm}} \quad (5)$$

If γ and m' are given, the mass $M(\gamma)$ of the fireball follows from Eq. (5):

$$M(\gamma) = \gamma E_{cm} - \sqrt{E_{cm}^2 (\gamma^2 - 1) + m'^2} \quad (6)$$

$M(\gamma)$ becomes maximal if $m' \rightarrow \min$. In pp collisions we have

$$m'_{min} = m_p \quad (7)$$

Therefore the maximum fireball mass at a given γ is

$$M(\gamma) = \gamma E_{cm} - \sqrt{E_{cm}^2 (\gamma^2 - 1) + m_p^2} \quad (8)$$

The maximum $\varepsilon(\gamma, m)$ in the M-rest frame is obtained by inserting expressions (8) and (2) into Eq. (1). This is then the kinematical limit for m in the peripheral situation for given γ of the decaying fireball.

2.2 The kinematical limits for central production of fireballs

The situation is defined in Fig. 2. Only one fireball is formed at rest in the c.m.s. It decays into $(m_R + m_p)$ in the backward direction and m in the forward direction. The c.m.s. energy of m becomes

$$\varepsilon(m) = \frac{E_{cm}^2 + m^2 - (m_R + m_p)^2}{2E_{cm}} \quad (9)$$

This becomes maximal for $m_R = m_R(\min)$.

Fig. 2 Decay of a single fireball produced centrally in the c.m.s.

2.3 The transition from peripheral to central production of fireballs

If, in the peripheral case, we let γ decrease from its maximum value γ_0 , the kinematical limit $\varepsilon(\gamma, m)$ increases up to a maximum reached at some value $\bar{\gamma}$ and then decreases again. This maximum value of $\varepsilon(\gamma, m)$ equals $\varepsilon(m)$ given by Eq. (9), because the kinematical situation is the same as in the central case and because the central limit $\varepsilon(m)$ is the absolute maximum for given m_R . Thus we use the peripheral limit for all $\gamma \geq \bar{\gamma}$ and switch there to the constant central limit $\varepsilon(m) = \varepsilon(\bar{\gamma}, m)$, which is used for all $\gamma \leq \bar{\gamma}$. In other words, we switch at the point where in the peripheral case m_R and m_p have the same velocity in the backward

direction. At this point the fireball moves with the Lorentz parameter $\bar{\gamma}$. The situation is defined in Fig. 3.

Fig. 3 Decay of the fireball $M(\gamma)$ into m and m_R such that m_R moves backward with the same velocity as m_p .

We determine first the mass M for which in the decay

$$\begin{array}{l} E_{cm} \rightarrow M + m_p \\ \quad \quad \quad \searrow \\ \quad \quad \quad m + m_R \end{array} \quad (10)$$

the particles m_R and m_p get the same Lorentz factors $\gamma_R = \gamma_p$. This is best done in the M -rest frame. In this frame we get

$$\gamma_R = \frac{E_R}{m_R} = \frac{M^2 + m_R^2 - m^2}{2Mm_R} \quad (11)$$

In the decay $E_{cm} \rightarrow M + m_p$ the energy of m_p seen from the rest frame of M is²⁾

$$E_p = \frac{p_M \cdot p_p}{M} = \frac{E_{cm}^2 - M^2 - m_p^2}{2M} \quad (12)$$

where p_M and p_p are the momentum four-vectors of M and m_p . The γ -parameter of m_p in the M -frame becomes

$$\gamma_p = \frac{E_p}{m_p} = \frac{E_{cm}^2 - M^2 - m_p^2}{2Mm_p} \quad (13)$$

The condition $\gamma_R = \gamma_p$ gives

$$(E_{cm}^2 - M^2 - m_p^2) m_R = (M^2 + m_R^2 - m^2) m_p \quad (14)$$

from which it follows that

$$M^2(\bar{\gamma}) = \frac{m_R}{m_R + m_p} E_{cm}^2 + \frac{m_p}{m_R + m_p} m^2 - m_R m_p \quad (15)$$

The Lorentz factor $\bar{\gamma}$ of this fireball lies in the "pionization" region:

$$\bar{\gamma} = \frac{\varepsilon[M(\bar{\gamma})]}{M(\bar{\gamma})} = \frac{E_{cm}^2 + M^2(\bar{\gamma}) - m_p^2}{2E_{cm} M(\bar{\gamma})} \xrightarrow{E_{cm} \rightarrow \infty} \frac{2m_R + m_p}{2\sqrt{m_R} \sqrt{m_R + m_p}} \quad (16)$$

For a particle m produced from a fireball moving with the Lorentz parameter γ in the c.m.s. we calculate the kinematical limits according to the peripheral case if $\gamma > \bar{\gamma}$ and according to the central case if $\gamma \leq \bar{\gamma}$. For pp collisions, Table 1 gives the minimum recoil masses m_R used in expressions (1), (9), and (15). In all cases we use at least two particles in m_R . The case with only one recoil particle is already partly contained in the explicitly-considered isobar contributions¹⁾ to the thermodynamic spectra of particle m_i :

Furthermore, a one-particle recoil gives a δ -function-like contribution to the spectra, which would lead possibly to peaks in the spectra if the integration over all allowed γ -values is, as at present, performed numerically¹⁾. Omitting deliberately such two-body decays has however the consequence that our spectra cannot be expected to exhibit the kinematical bumps near the kinematical limits connected with two-particle decay of $M(\gamma)$ or of E_{cm} .

3. THE PHASE SPACE CUT-OFF FUNCTION

At the kinematical limits worked out in the previous section, the energy spectrum of the particle considered vanishes. It does so, however, in general, not abruptly but in a smooth manner governed by the available phase-space volume which

goes to zero at the kinematical limit. In this section we determine this behaviour.

3.1 Replacing the thermodynamic spectra by phase space near the kinematic limits

We consider the decay of a fireball of mass M into the wanted particle m and n further particles with masses μ_i whose invariant mass is the recoil mass m_n , see Fig. 4. From the microscopic point of view the momentum spectrum of particle m

Fig. 4 Decay of a fireball M into a particle m with momentum p and n recoil particles μ_i with invariant mass m_n .

is a superposition of momentum spectra resulting from 2-body, 3-body, $\dots$, N -body decay of M , where all possible combinations of particles and resonances, into which M might decay, have to be taken into account. The momentum spectrum in each single decay mode will be of approximate phase-space form. The momentum spectrum $f_m(p, T)$ according to the thermodynamic model is a good approximation to this superposition as long as we are far away from the kinematical limit ϵ_0

$$\epsilon_0 = \frac{M^2 + m^2 - (\sum \mu_i)^2}{2M}$$

$$\sum_{i=1}^n \mu_i = m_R \quad (\text{See Table 1}) \quad (17)$$

If ϵ_1 is the energy at which the creation of one additional pion is allowed,

$$\varepsilon_1 = \frac{M^2 + m^2 - (\sum \mu_i + m_\pi)^2}{2M}, \quad (18)$$

then only n+1-body decay is possible in the region where particle m has an energy E in the interval $\varepsilon_1 < E < \varepsilon_0$.

In the region $\varepsilon_2 < E < \varepsilon_1$ with

$$\varepsilon_2 = \frac{M^2 + m^2 - (\sum \mu_i + 2m_\pi)^2}{2M} \quad (19)$$

also n+2-body decay is possible and so on.

Within the limits $\varepsilon_2 < E \leq \varepsilon_0$ the spectrum can be approximated by a sum of n+1-body phase space and n+2-body phase space (the latter contributing only in $\varepsilon_2 < E \leq \varepsilon_1$). According to Table 1 we have, in all cases considered, $n \geq 2$. In $\varepsilon_2 < E \leq \varepsilon_0$ we replace the thermodynamic spectra by n+1 and n+2-body phase space. In the region $E < \varepsilon_2$ the number of produced particles can be five or higher; therefore we assume the validity of the thermodynamic spectra in this region.

3.2 The n+1-body phase space in the region $\varepsilon_1 < E < \varepsilon_0$

We use the invariant phase space of Srivastava and Sudarshan²⁾

$$R_{n+1}(M, m, \mu_1, \mu_2, \dots, \mu_n) \quad (20)$$

This phase space is of dimension $(\text{mass})^{2n-4}$. If we want to use it as continuation of the thermodynamic spectra we have to introduce dimensionless probabilities

$$\begin{aligned} P_{n+1}(M, m, \mu_1, \dots, \mu_n) &\equiv \\ &\equiv m_0^4 \left[\prod_{i=0}^n K_i \right] R_{n+1}(M, m, \mu_1, \dots, \mu_n) \end{aligned} \quad (21)$$

where $i = 0$ refers to particle m, m_0 is a normalization mass whose numerical value is not relevant and K_i is defined^{*)} as

$$\begin{aligned} K_i &= 2 \bar{E}_i V_0 / (2\pi)^3 \quad ; \quad \bar{E}_i = \langle E_i = \sqrt{p_i^2 + m_i^2} \rangle \\ V_0 &= 4\pi / (3 m_\pi^3) \end{aligned} \quad (22)$$

*) See explanation at the end of this section.

We use for R_{n+1} the recursion relation²⁾

$$R_{n+1} = \int dm_n^2 R_2(M, m_n, m) R_n(m_n, \mu_1 \dots \mu_n) \quad (23)$$

where because of the invariance each R is calculated in its rest frame. R_2 is the two-body phase space for $M \rightarrow m + m_n$

$$\begin{aligned} R_2(M, m_n, m) &= \\ &= \int \frac{d^3 p_n}{2E_n} \frac{d^3 p}{2E} \delta(M - \sqrt{p_n^2 + m_n^2} - \sqrt{p^2 + m^2}) \delta^3(\vec{p} + \vec{p}_n) \end{aligned} \quad (24)$$

We get the p -dependence of R_2 by integrating only over $\vec{p}_n$

$$dR_2 = dp \frac{\pi p^2 \delta(M - \sqrt{p^2 + m_n^2} - \sqrt{p^2 + m^2})}{\sqrt{p^2 + m_n^2} \cdot \sqrt{p^2 + m^2}} \quad (25)$$

Inserting into Eq. (23) we get the p -dependence of R_{n+1}

$$\begin{aligned} dR_{n+1}(M, m_n, \mu_1 \dots \mu_n) &= \\ &= \pi dp \int dm_n^2 p^2 \frac{\delta(M - \sqrt{p^2 + m_n^2} - \sqrt{p^2 + m^2})}{\sqrt{p^2 + m_n^2} \cdot \sqrt{p^2 + m^2}} R_n(m_n, \mu_1 \dots \mu_n) \end{aligned} \quad (26)$$

The integration over m_n^2 yields

$$dR_{n+1} = dp \frac{2\pi p^2}{\sqrt{p^2 + m^2}} R_n\left(m_n = \sqrt{(M - \sqrt{p^2 + m^2})^2 - p^2}, \mu_1 \dots \mu_n\right) \quad (27)$$

As R_n is evaluated in the rest frame of the invariant recoil mass $m_n(p)$ and used in the interval

$$\begin{aligned} \sum \mu_i &\leq m_n(p) \leq \sum \mu_i + 2m_\pi, \\ \varepsilon_2 &\leq \sqrt{p^2 + m^2} \leq \varepsilon_0, \end{aligned} \quad (28)$$

the average kinetic energy of the particles μ_i is small $\langle T_i \rangle \leq 2m/n$. Therefore, all particles μ_i can be considered in non-relativistic approximation. In this case R_n can easily be calculated

$$R_n(m_n, \mu_1, \dots, \mu_n) = \int \frac{d^3 p_1 \dots d^3 p_n}{2E_1 \dots 2E_n} \delta(m_n - \sum E_i) \delta^3(\sum \vec{p}_i) \quad (29)$$

We get with

$$E_i = \mu_i + \frac{p_i^2}{2\mu_i} \approx \mu_i \quad (30)$$

and according to Milburn³⁾

$$R_n = \frac{(2\pi)^{\frac{3n-3}{2}}}{2^n \prod_1^n \mu_i} \cdot \left[\frac{\prod_1^n \mu_i}{\sum_1^n \mu_i} \right]^{3/2} \cdot \frac{(m_n - \sum \mu_i)^{\frac{3n-5}{2}}}{(\frac{3n-5}{2})!} \quad (31)$$

Inserting everything into Eq. (27) we obtain, remembering expression (21),

$$\begin{aligned} dP_{n+1}(M, m, \mu_1, \dots, \mu_n) &= dp \left[m_c^4 \frac{2V_c}{(2\pi)^{7/2}} \right] \times \\ &\times \left[\frac{V_c}{(2\pi)^{3/2}} \right]^n \left[\frac{\prod_1^n \mu_i}{\sum_1^n \mu_i} \right]^{3/2} \times \frac{p^2 [m_n - \sum_1^n \mu_i]^{\frac{3n-5}{2}}}{(\frac{3n-5}{2})!} \end{aligned} \quad (32)$$

where $m_n(p) = \sqrt{(M - \sqrt{p^2 + m^2})^2 - p^2}$.

This expression is dimensionless. By going over from R_{n+1} to P_{n+1} we have related it to the Fermi formulation of phase space, which is the one related to the thermodynamic model.

In the Fermi statistical model, as in the thermodynamic model, the weight of a phase-space cell is

$$\frac{V_c d^3 p}{h^3} \equiv \frac{V_c d^3 p}{(2\pi)^3} \quad (33)$$

In invariant phase space it is

$$\frac{K d^3 p}{2\sqrt{p^2 + m^2}} \quad (34)$$

with our choice [Eq. (22)] of K_i we have thus correctly related the invariant phase space to the thermodynamic spectra²⁾; therefore, we obtain a reasonable cut-off function for the thermodynamic spectra.

Note that in the phase-space integration (29) all $\bar{E}_i \approx \mu_i$, therefore

$$K_i \approx 2\mu_i V_c / (2\pi)^3 \quad \text{for } i = 1, 2, \dots, n$$

while for the particle m , whose kinematical limits we consider and which is not non-relativistic,

$$K_c = 2 \frac{V_c}{(2\pi)^3} \sqrt{p^2 + m^2}$$

so that the factor $1/\sqrt{p^2 + m^2}$ in Eq. (27) is cancelled.

3.3 The behaviour of the phase space near the kinematic boundary

The momentum dependence of the phase space (32) has the form

$$\frac{dP_{n+1}}{dp} = \text{const} \cdot p^2 \left[m_n(p) - \sum_1^n \mu_i \right] \frac{3n-5}{2} \quad (35)$$

We obtain the maximum momentum $p_0 = \sqrt{\varepsilon_0^2 - m^2}$ from Eqs. (17). Representing $m_n(p)$ as a power series at the point $p = p_0$ and keeping only terms with $(p_0 - p)$ we get

$$m_n(p) - \sum_1^n \mu_i = \frac{p_0 M}{\varepsilon_0 \sum \mu_i} (p_0 - p) + \dots \quad (36)$$

and

$$\frac{dP_{n+1}}{dp} = \text{const} \cdot p_0^2 \left(\frac{p_0 M}{\varepsilon_0 \sum \mu_i} \right) \frac{3n-5}{2} \cdot (p_0 - p) \frac{3n-5}{2} + \dots \quad (37)$$

3.4 The kinematic cut-off at ε_1 with dP_{n+1}/dp only

In this case all numerical factors are irrelevant. We use Eq. (35) with

$$m_n(p) = \sqrt{(M - \sqrt{p^2 + m^2})^2 - p^2} \quad (38)$$

Let the function $f_m(p, T)$ represent the thermodynamic spectrum in the rest frame of M . We get the function with cut-off as follows

$$\hat{f}_m(p, T) = \begin{cases} f_m(p, T) & \text{for } E \leq \varepsilon_1 \\ \frac{f_m(p, T) p^2 [m_n(p) - \sum \mu_i]^{\frac{3n-5}{2}}}{p_1^2 [m_n(p_1) - \sum \mu_i]^{\frac{3n-5}{2}}} & \text{for } \varepsilon_1 \leq E \leq \varepsilon_c \end{cases} \quad (39)$$

It is to be noted that in expression (39) the functions $f_m(p, T)$ and dP_{n+1}/dp are normalized to have the same value at $p = p_1$. The first derivatives of the two functions at this point are however expected to differ. This will lead to a reasonable shape of the curves with cut-off in the majority of cases. In a few situations where

$$\left. \frac{d}{dp} f_m(p, T) \right|_{p=p_1} \text{ very different from } \left. \frac{d}{dp} \left(\frac{dP_{n+1}}{dp} \right) \right|_{p=p_1} \quad (40)$$

the formula (39) leads however to unreasonable curves. We improve this situation by adding one more phase-space contribution, i.e. we go back to ε_2 and continue by pure phase space from there to ε_0 . Furthermore, the value and derivative of the thermodynamic and of the phase-space spectrum will be made equal at ε_2 .

3.5 Smooth cut-off at ε_2 with dP_{n+1}/dp and dP_{n+2}/dp

We add dP_{n+1}/dp and dP_{n+2}/dp and extract all common factors

$$\begin{aligned} dP^{(2)}/dp &= \\ &= C p^2 \left[\frac{(m_n - \sum \mu_i)^{\frac{3n-5}{2}}}{(\frac{3n-5}{2})! \sum \mu_i} + \frac{V_0 m_\pi^{3/2}}{(2\pi)^{3/2}} \frac{(m_n - \sum \mu_i - m_\pi)^{\frac{3n-5}{2} + \frac{3}{2}}}{(\frac{3n-5}{2} + \frac{3}{2})! (\sum \mu_i + m_\pi)} \right] \end{aligned} \quad (41)$$

$\sum \mu_i$ is here either equal to m_R (peripheral) or m_R (central); see Table 1. The most frequent case is $n = 2$; there we have

$$\left(\frac{3n-5}{2}\right)! = \frac{1}{2}! = \frac{1}{2}\sqrt{\pi} \quad ; \quad \left(\frac{3n-2}{2}\right)! = 2 \quad (42)$$

Taking the term

$$\left[\left(\frac{3n-5}{2}\right)! \sum \mu_i \right]^{-1} \quad (43)$$

out of the square bracket in expression (41) and neglecting all constant factors we get

$$\begin{aligned} dP^{(2)}/dp &= \\ &= p^2 \left[(m_n - \sum \mu_i)^{\frac{3n-5}{2}} \Theta(\varepsilon_0 - E) + g(n, \varepsilon_1) (m_n - \sum \mu_i - m_\pi)^{\frac{3n-2}{2}} \right] \quad (44) \end{aligned}$$

with

$$g(n, \varepsilon_1) = \left(\frac{m_\pi}{2\pi}\right)^{3/2} \frac{\sum \mu_i}{\sum \mu_i + m_\pi} V_0 \Theta(\varepsilon_1 - E) \times \begin{cases} \frac{\sqrt{\pi}}{4} & \text{for } n=2 \\ \frac{32}{105\sqrt{\pi}} & \text{" } n=3 \\ \frac{7\sqrt{\pi}}{128} & \text{" } n=4 \\ \vdots & \end{cases} \quad (45)$$

and $E \equiv \sqrt{p^2 + m^2}$,

$$m_n = \sqrt{(M - \sqrt{p_0^2 + m^2})^2 - p^2} \quad ; \quad V_0 = 1.59 \times 10^{-6} \text{ [MeV}^3\text{]} \quad (46)$$

$\Theta(x)$ is the usual step function, 0 for $x < 0$, 1 for $x \geq 0$. The first term of Eq. (44) becomes zero at ε_0 , the second already at ε_1 . For all $E \geq \varepsilon_2$ Eq. (44) will replace the thermodynamic formula $f_m(p, T)$. Requiring that the two function values be equal at ε_2 we get

$$\tilde{f}_m(p, T) = \begin{cases} f_m(p, T) & \text{for } E \leq \varepsilon_2 \\ f_2(p, T) \equiv \frac{f_m(p_2, T)}{(dP^{(2)}/dp)_{p_2}} \cdot \frac{dP^{(2)}}{dp}(p) & \text{for } \varepsilon_2 \leq E \leq \varepsilon_0 \end{cases} \quad (47)$$

If according to Eq. (19) ε_2 becomes $\leq m$ we use in expression (47) a fake value $\varepsilon_2' = m + \delta$, where δ is a few MeV.

In order to obtain a smooth transition from the thermodynamic momentum spectrum to the phase-space spectrum, we require now in addition to the equality of the two functions at ε_2 , that also their derivatives be equal there. That will of course distort the phase-space cut-off [without, however, changing its power behaviour (37) at ε_0], but we believe that the thermodynamic spectra, which contain strong interaction dynamics, are more reliable than pure phase space. This justifies our imposing thermodynamic normalization and distortion upon phase space.

To make the derivatives equal to ε_2 we multiply $f_2(p,T)$ by an exponential function and define the smooth cut-off function F_2 by

$$F_2(p,T) \equiv \exp[-\alpha(\varepsilon - \varepsilon_2)] \cdot f_2(p,T) \quad (48)$$

With the condition

$$\left. \frac{\partial}{\partial \varepsilon} f_m(p,T) \right|_{\varepsilon=\varepsilon_2} = \left. \frac{\partial}{\partial \varepsilon} F_2(p,T) \right|_{\varepsilon=\varepsilon_2}$$

we get for the parameter α

$$\alpha = \frac{\left. \frac{\partial}{\partial \varepsilon} f_2(p,T) \right|_{\varepsilon=\varepsilon_2} - \left. \frac{\partial}{\partial \varepsilon} f_m(p,T) \right|_{\varepsilon=\varepsilon_2}}{f_m(p_2, T)} \quad (49)$$

Replacing $f_2(p,T)$ by $F_2(p,T)$ in Eq. (47) gives a smooth cut-off procedure for the thermodynamic spectra. This method has been introduced into the latest version of the computer program calculating thermodynamic single particle spectra⁴⁾.

Figure 5 shows how this works in an extreme case where at ε_2 the derivatives of the thermodynamic and of the phase-space spectra are very different. Such cases are rare; they arise only if the mass M of the decaying fireball is only slightly above the minimum required. The weight by which such cases contribute to the whole spectrum of particle m , is small. This is easily seen by realizing that the procedure described here is applied to each fireball with mass $M(\gamma)$ and that the final c.m.s. spectrum of the particle considered is obtained by an integration over all possible values of γ , resulting in a continuous superposition of spectra, each Lorentz transformed (with γ) from the rest frame of $M(\gamma)$ to the c.m. frame of the collision.

Fig. 5 Phase space cut-off at p_2 in a case of small fireball mass:
 $M = 1600$ MeV. The curves are:

- AB: thermodynamic spectrum with $T = 122$ MeV
- C : phase space cut-off function $f_2(p_1T)$
- D : smooth cut-off function $F_2(p_1T)$
- AD: thermodynamic spectrum with smooth cut-off
as used in the computer program.

3.6 Calculating the thermodynamic spectra near the kinematic limit with a lowered effective temperature

We calculate¹⁾ the thermodynamic spectra usually with temperatures $T(\lambda)$ determined from the maximum kinematically-possible energy densities $\varepsilon(\lambda)$

$$\varepsilon(\lambda) = \frac{\varepsilon_0 \gamma_0}{(\gamma_0 - 1)|\lambda| + 1} \quad (50)$$

where ε_0 is the energy density corresponding to the rest energy of the incoming

particle^{*)} (say a proton), and $|\lambda| = (\gamma - 1)/(\gamma_0 - 1)$ measures the velocity of the decaying fireball. Considering the production of heavy particles m_i which have, because of conservation laws, to be created together with other particles $\sum m_j$, we have in the whole kinematically-allowed region rather large energy densities $\varepsilon(\lambda)$ because even at the threshold for m_i -production it still is

$$\varepsilon(\lambda)_{\min} = \frac{m_p + m_i + \sum m_j}{V_0} \equiv \frac{M_{\min}}{V_0} \quad (51)$$

Therefore the temperature remains rather high up to the kinematic limit, but at this point the spectra have to be cut down to zero. We already get a better behaviour of the uncut thermodynamic spectra if the temperature near the kinematic limit is lowered. The particle m_i is created together with the particles $\sum m_j$ from the same fireball. In this case the rest mass of $\sum m_j$ is not really available to provide a part of the kinetic energy of m_i . Therefore, we lower the energy density by the amount corresponding to the rest mass of these particles

$$\varepsilon_i(\lambda) = \varepsilon(\lambda) - \frac{\sum m_j}{V_0} \quad (52)$$

and then calculate the temperature from this new $\varepsilon_i(\lambda)$. The particle m_p is not to be added in Eq. (52); it is already taken into account in using the function $\varepsilon^{(1B)}(T)$.

The above correction is important only near to the kinematical limit, where it lowers the temperature. Far away from the production threshold of the particle considered, the temperature is not significantly changed and no correction to spectra and production rates occurs.

This correction can be understood as a "cooling effect", due to the evaporation of the particles $\sum m_j$ enforced by the conservation law by which the particle m_i is tied up with $\sum m_j$. We apply it only to the momentum distribution of particle m_i and not to the total production rate, which is already proportional to (roughly) $\exp[-(1/T)(m_i + \sum m_j)]$. We are aware of being inconsistent here; it is the price we have to pay for applying thermodynamic methods near threshold, where strict energy conservation is important, but in conflict with the notion of temperature. This inconsistency, which disappears rapidly with increasing distance from threshold, improves the fits to spectra of $\bar{p}$, etc., at low collision energies. As to the relative importance with which this inconsistency enters into the total spectrum of particle m_i , the same remark as at the end of the preceding subsection applies.

*) This ε_0 has nothing to do with the ε_0 (kinematical limit) of the previous section; we wish to remain close to the notation of earlier papers and therefore do not use a new name here.

Table 1

Minimum recoil masses m_R for
particle production in pp collisions

Emitted particle	m_R peripheral	m_R central
π	$m_p + m_\pi$	$2m_p$
K	$m_Y + m_\pi$	$m_p + m_Y$
$\bar{K}$	$m_p + m_K$	$2m_p + m_K$
N	$2m_\pi$	$m_p + m_\pi$
$\bar{N}$	$2m_p$	$3m_p$
Y	$m_K + m_\pi$	$m_p + m_K$
$\bar{Y}$	$m_p + m_Y$	$2m_p + m_Y$

Note that in pp collisions the decaying fireball of mass $M(\gamma)$ has baryon number 1 in the peripheral case, and 2 in the central one.

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