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INCLUSIVE SINGLE PARTICLE SPECTRA AND THE
STRONG THERMODYNAMIC BOOTSTRAP SOLUTION

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ABSTRACT

The thermodynamical model was based so far on a bootstrap condition requiring asymptotic equality of the logarithms of the hadronic mass spectrum and of the level density of the fireballs, resulting in a family of exponential solutions, of which one was chosen on the basis of plausibility arguments.

Through the work of Frautschi, Frautschi and Hamer and of Nahm, the bootstrap condition has been sharpened and now leads to a unique asymptotic solution slightly different from the one chosen before.

We discuss in this paper the consequences of using this strong bootstrap solution in the thermodynamical model for particle production. If all other assumptions remain unmodified, the following changes occur:

- the decay chain of fireballs becomes approximately manageable and is included in the analysis;
- scaling and limiting fragmentation hold exactly;
- the rapidity distribution has a central plateau (with a small constant dip in the middle);
- multiplicities grow logarithmically.

Numerical differences against the old version are, however, small even at ISR energies, where we compare our predictions with experimental data.

1. INTRODUCTION

The thermodynamical model of strong interactions at high energies is composed of two intimately mixed, but very different, ingredients:

- i) a statistical-thermodynamical bootstrap model of hadron structure¹⁾ [Ref. (1) will be quoted as (I)];
- ii) a superposition of moving "fireballs", whose decay properties follow from (i), but whose velocity distribution depends on the collision kinematics and cannot be calculated from thermodynamics²⁾ [Ref. (2) will be quoted as (II)].

Logically, (i) is a closed theory with rather little experimental input, while (ii) contains, at least at the present state, two empirical velocity distribution functions plus a number of "nearly known parameters", i.e. parameters whose approximate values are *a priori* known from considerations outside the model. Naturally, any change and improvement of (i) will have its consequences for (ii), while the reverse is not true, because (i) is almost entirely determined by the requirement of completeness and internal consistency³⁾. Only almost, however. The little freedom which remains has been considerably narrowed by the recent work of Frautschi⁴⁾, Frautschi and Hamer⁵⁾, and Nahm⁶⁾. The apparently small change in the asymptotic behaviour of the hadronic mass spectrum found by these authors has important -- and welcome -- consequences for the asymptotic behaviour of "boiling hadron matter" and thus for the procedure (ii).

In this paper, we first discuss the modifications of the bootstrap model (Section 2), then their consequences for the description of inclusive one-particle spectra (Section 3), we further describe the asymptotic behaviour of these spectra -- limiting fragmentation and scaling -- for infinite collision energy (Section 4), then we present the results of fits to experiments with the new version of the model (Section 5), in which the number of fit parameters has been reduced while the fits have improved. On the basis of fits at 19-70 GeV predictions for up to 1500 GeV are made and compared with CERN-ISR experiments in Section 6. The paper ends with a discussion in Section 7.

Our units are $\hbar = c = k = 1$. We use as far as possible the same notation and definitions as in (II) and in our other papers quoted in the list of references. For a general introduction to the subject a set of CERN lecture notes³⁾ may be useful [Ref. (3) will be quoted as "CERN Lectures"].

2. THE SOLUTION OF THE STRONG THERMODYNAMIC BOOTSTRAP CONDITION

In (I) a statistical-thermodynamical bootstrap model of hadrons was constructed along the following lines:

- Highly excited hadronic matter (a fireball) is described as a gas of hadrons enclosed in a natural interaction volume $V_0 \approx (4\pi/3) m_\pi^{-3}$.
- The strong interactions between these hadrons are represented by including *all* resonant states as new particles in their own right and by allowing free absorption and creation of all particles.
- Consistency and completeness required that not only the known, but *all existing resonances* be included as particles. In particular, the fireballs, which are the object of description, have to be considered as heavy resonances; hence they are themselves particles participating in building-up fireballs.
- Let $\rho(m) dm$ be the number of different hadrons in the mass interval $\{m, dm\}$; we call $\rho(m)$ the hadronic mass spectrum; further, let $\sigma(E, V_0) dE$ be the number of states a fireball can have in the energy interval $\{E, dE\}$. Since the total energy E of a fireball equals its rest mass m and since fireballs are resonances, consistency requires that somehow $\rho(m)$ and $\sigma(m, V_0)$ become the same function for large m . This is the bootstrap condition.

The above reasoning is essentially that of (I); it can be motivated in more detail (CERN Lectures). Bootstrap is not postulated, but enforced by the requirement of consistency and completeness. It almost completely determines the structure of the model.

The question remains: How has one to formulate this condition? In (I), where everything was written in the language of the Gibbs canonical ensemble, it was natural to require only that the thermodynamical quantities related to $\rho(m)$ and $\sigma(m, V_0)$ should become asymptotically equal, leaving thus some freedom for them to differ (at lower masses they cannot be equal). The appropriate thermodynamical quantity is the entropy, namely the logarithm of the number of states. Thus, in (I) the "logarithmic bootstrap condition" was stated:

logarithmic bootstrap condition

$$\frac{\ln \rho(m)}{\ln \sigma(m, V_0)} \xrightarrow{m \rightarrow \infty} 1 \quad (2.1)$$

and it was shown, that it can be satisfied if, and only if, ρ and σ both grow exponentially with the same exponential factor $\exp(m/T_0)$, differing only by functions of lower than exponential order. It was further shown that

$$\rho(m) \rightarrow c m^a \exp(m/T_0) \quad (2.2)$$

with $-7/2 \leq a \leq -5/2$ was a solution of expression (2.1) and that $\sigma(m, V_0)$ would differ from $\rho(m)$ only by a polynomial-bounded function, so that the proposed solutions

actually satisfied a stronger condition than (2.1), namely the "weak bootstrap condition":

weak bootstrap condition

$$\frac{m^b \rho(m)}{\sigma(m, V_0)} \xrightarrow{m \rightarrow \infty} 1 \quad \text{for some } b > 0. \quad (2.3)$$

All solutions (2.2) have the property that T_0 is a universal upper limiting temperature for all matter in thermodynamical equilibrium. Its value can be approximately read off from the lower part of the empirical mass spectrum; it lies somewhere between 130 and 170 MeV. Existence and numerical value of T_0 account quantitatively for the long known limiting behaviour of transverse momenta.

Out of the set of solutions $-7/2 \leq a \leq -5/2$, the end point $-5/2$ was chosen in (I) because it seemed the most plausible one: it is the only solution for which the mean particle number in the fireball diverges (logarithmically) for $T \rightarrow T_0$ ($E \rightarrow \infty$) in apparent agreement with the experiments. Besides there were other, more formal, reasons to prefer that solution, which then was considered as "the solution" in all subsequent work until Frautschi challenged it⁴⁾. He reformulated the model in the microcanonical picture; that is, entirely in phase space. He was thus able to incorporate exact energy-momentum conservation; he also eliminated the one-particle state (where the fireball consists of just itself). As a consequence of these improvements he discovered that a still stronger bootstrap condition:

strong bootstrap condition

$$\frac{\rho_{in}(m)}{\rho_{out}(m)} \xrightarrow{m \rightarrow \infty} 1 \quad (2.4)$$

should be postulated and could be solved; here $\rho_{in}(m)$ is the same as $\rho(m)$ above, while $\rho_{out}(m) \rightarrow m^{-3/2} \sigma(m, V_0)$. The factor $m^{-3/2}$ comes from momentum conservation [neglected in (I)] and makes the strong condition (2.4) possible. He then showed that with this strong condition the same spectrum of solutions (2.2) was found, except that now the end-point $a = -5/2$ -- just the favoured solution so far -- was excluded. Shortly afterwards it was shown by Frautschi and Hamer⁵⁾ that the condition (2.4), narrowed by the requirement

$$\frac{\rho_{in}(m)}{\rho_{out}(m)} = 1 \quad \text{for } m \geq M < \infty \quad (2.5)$$

and used in a numerical iterative procedure, leads to a ≈ -3 and $T_0 \approx m_\pi$, provided one makes the usual assumption about the interaction volume, namely $V_0 = (4\pi/3)m_\pi^{-3}$ and about the input mass spectrum, namely the known lower mass particles. This was the first time that the value of T_0 -- in fact of all parameters of the asymptotic mass spectrum $\rho(m)$ -- could be computed inside the model; until then only speculative arguments [see (I)] could be given for T_0 being of order m_π . At the same time, and independently, Nahm⁶⁾ succeeded in proving analytically that if

$$\frac{\rho_{in}(m)}{\rho_{out}(m)} = 1 + O(e^{-\lambda m}) \text{ for some } \lambda > 0 \quad (2.6)$$

of which the Frautschi-Hamer condition can be considered a special case, the value of a must be exactly 3. He also derived a sum rule from which T_0 can be determined to lie about in the interval $140 \leq T_0 \leq 170$ MeV, with a probable value near to 150 MeV. In both the paper of Frautschi-Hamer as well as that of Nahm, it turned out that including or excluding exotics had no effect on a and little on T_0 .

As a shorthand notation we shall call

$$\begin{aligned} \rho(m) &\Rightarrow c m^{-5/2} \exp(m/T_0) && \text{the "weak bootstrap solution"} \\ \rho(m) &\Rightarrow c m^{-3} \exp(m/T_0) && \text{the "strong bootstrap solution"}. \end{aligned} \quad (2.7)$$

The difference looks insignificant, but in fact amounts to an important improvement of the thermodynamic part of the model.

It is very easy to show [CERN lectures] that the partition function $Z(V, T)$, which for the weak solution diverges logarithmically, now tends to a constant when $T \rightarrow T_0$. More precisely

$$\begin{aligned} \ln Z(V, T \rightarrow T_0) &\xRightarrow{\text{weak}} \text{const.} \cdot \ln \left(\frac{T_0}{T_0 - T} \right) \\ \ln Z(V, T \rightarrow T_0) &\xRightarrow{\text{strong}} \text{const.} \end{aligned} \quad (2.8)$$

Furthermore, the energy density ε is given by

$$\varepsilon = \frac{T^2}{V} \frac{\partial}{\partial T} \ln Z(V, T) \quad (2.9)$$

hence

$$\varepsilon_{\text{weak}} \Rightarrow \text{const.} \cdot \frac{T_0^2}{T_0 - T} ; \quad \varepsilon_{\text{strong}} \Rightarrow \text{const.} \cdot \frac{T_0^{3/2}}{\sqrt{T_0 - T}} \quad (2.10)$$

and finally, as the mean particle number N in the *first* generation of the fireball decay is given by

$$N \approx \ln Z(V_0, T) \quad (2.11)$$

it follows that for large energy densities

$$N_{\text{weak}} \Rightarrow \text{const.} \cdot \ln E \quad (2.12)$$

$$N_{\text{strong}} \Rightarrow \text{const.}$$

Furthermore, Frautschi⁴⁾ has shown that $N_{\text{strong}} \approx 2.4$ and that decay into one heavy and one light particle is favoured over other decays.

These results have two obvious and -- in spite of the seemingly insignificant change of the power of m in front of an exponential function -- far reaching consequences:

- from (2.10) the relation between energy density and temperature will be changed; this effect is, however, numerically unimportant at accelerator energies;
- the decay chain of an *ideal* fireball now looks very different. In the weak solution each fireball of mass m (large) decayed into $N \sim \ln m$ subfireballs and the whole decay chain was a complicated interplay of Lorentz transformations and isotropic thermodynamic decays with rapidly decreasing average fireball masses (Fig. 1a). Very rough and uncertain estimates lead to expect that the total multiplicity of observable final particles would grow with a small power of the mass of the first parent fireball. Thus $N_{\text{final}} \approx M^\alpha$, where $\alpha \approx \frac{1}{2}$ or so. But then energy conservation requires^{*)} $\langle N_{\text{final}} \rangle \cdot \langle E_{\text{final}} \rangle \gtrsim M$ so that the mean energy $\langle E_{\text{final}} \rangle$ of the observed final particles had to grow faster than $M^{1-\alpha}$ in contradiction to the well-established limitation of transverse momenta.

In the strong solution this discrepancy disappears, because the decay chain looks now different (Fig. 1b). The average decay produces $N_{\text{strong}} \approx 2.4$ particles; one of these is heavy ($m_1, m_2, \dots m_i \dots$) except at the end of the chain, while the other 1.4 particles with average mass $\bar{m} \approx 2m_\pi$ are light: most of them pions, sometimes resonances or kaons, rarely baryons.

Averaged over the decay chain, the heavy masses m_i will suffer random direction recoil. They will thus appear to have very small kinetic energy E_{kin} in the

*) Equality holds for a sharp distribution.

rest frame of the first parent fireball M. The energy balance per decay step is then on the average

$$(m_i + E_{kin,i}) \rightarrow (m_{i+1} + E_{kin,i+1}) + 1.4 \left(\bar{m} + \frac{3}{2} T \right) . \quad (2.13)$$

If the decay chain ends after n steps we have

$$1.4 \, n \left(\bar{m} + \frac{3}{2} T \right) \cong M . \quad (2.14)$$

Putting $\bar{m} \approx 2m_\pi$ and $T \approx m_\pi$ yields a rough estimate for the number n of steps (number of generations), valid for sufficiently high excitation:

$$n(M) \approx \frac{M}{5m_\pi} = \frac{\varepsilon V_0}{5m_\pi} \approx 1.4 \frac{\varepsilon}{\varepsilon_0} , \quad (2.15)$$

where $\varepsilon_0 = m_p/V_0$ is the proton rest mass density.

Thus a fireball born in a region of local energy density ε will decay via

$$n(\varepsilon) \approx 1.4 \frac{\varepsilon}{\varepsilon_0} \approx 0.2 \frac{M}{m_\pi} \quad (2.16)$$

generations into

$$\langle N_{final}(\varepsilon) \rangle \approx 1.4 \, n \approx 2 \frac{\varepsilon}{\varepsilon_0} \approx 0.28 \frac{M}{m_\pi} \quad (2.17)$$

final particles with an average energy per particle

$$\langle E_{final} \rangle \approx \bar{m} + \frac{3}{2} T . \quad (2.18)$$

Now the inequality $\langle N_{final} \rangle \cdot \langle E_{final} \rangle \geq M$ is fulfilled with an average energy per particle which agrees with the observed transverse momenta and limiting temperature.

Note that strictly speaking the energy $\bar{m} + 3T/2$ is to be measured in the rest frame of the heavy mass m_1 ; but these masses have a very narrow velocity distribution in the rest frame of M; they move so slowly that their velocity can be neglected. Thus also the final particles have, to a good approximation, a thermal momentum distribution in the M-frame with an average energy given by Eq. (2.18).

The result (2.17), that the decay of an ideal fireball leads to a final multiplicity roughly proportional to its mass, seems to disagree with the known very slow increase of multiplicity with collision energy. The present result applies, however, only to *ideal fireballs* with no internal collective motions; if it is properly combined with collision kinematics, it will lead to a slow increase of the final multiplicity in *real collision* processes. If we keep to our old assumption of paper (II) that the velocity weight functions are energy independent, then the final multiplicity will grow logarithmically with the collision energy and the single particle inclusive spectra will exhibit scaling and limiting fragmentation behaviour with a "central plateau". This will be discussed in Section 4 of this paper.

3. CHANGES OCCURRING IN QUANTITIES RELEVANT FOR THE CALCULATION OF INCLUSIVE SINGLE PARTICLE SPECTRA

The inclusive secondary particle distributions according to the thermodynamical model are modified if we replace the weak thermodynamic bootstrap solutions by the strong one. Changed are mainly the relation between energy density and temperature and the functions $N_I(T)$ and $\bar{m}_I(T)$ giving the average multiplicity and mass of particles belonging to family I (i.e. to a family of excited hadrons all having the same baryon number and strangeness). These changes are caused by the different asymptotic mass spectrum in integrals extending to infinity.

The hadronic mass spectrum according to the strong thermodynamic bootstrap can be parametrized as follows:

$$\rho(m) = \frac{a m^b}{(m_0 + m)^{b+3}} \cdot \exp(m/T_0) . \quad (3.1)$$

A reasonable fit to the experimentally known mass spectrum is obtained with the parameters

$$\begin{aligned} a &= 1.48 \times 10^6 \text{ (MeV)}^2 \\ b &= 0.5 \\ m_0 &= 500 \text{ MeV} \\ T_0 &= 147.5 \text{ MeV} . \end{aligned} \quad (3.2)$$

The resulting curve is so similar to the one in Fig. 1 of paper (II), that it is not worth while to produce here a new figure. It must be emphasized that this fit is not unique. Equally good agreement is obtained with $130 \leq T_0 \leq 170 \text{ MeV}$ and correspondingly changed parameters a , b and m_0 . If one wishes only a reasonable extrapolation for $m \geq 1 \text{ GeV}$, the following set of parameters can be considered as equally good (only asymptotic part):

$$\begin{aligned}
 b &= 0 \\
 m_0 &= 0 \\
 a &= 3.42 \times 10^5 \text{ (MeV)}^2 \\
 T_0 &= 148.9 \text{ MeV} .
 \end{aligned}
 \tag{3.3}$$

These two examples show the unsafety of the extrapolation from the small region of known masses. The physics of the model is, however, not very sensitive to such uncertainties.

As described in Appendix II of paper (II), the energy density of a fireball without conserved quantum numbers as a function of the temperature is given by

$$\mathcal{E}^{(0)} = \frac{T^2}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{\infty} m^2 \rho(m, n) \left[3K_2\left(\frac{nm}{T}\right) + \frac{nm}{T} K_1\left(\frac{nm}{T}\right) \right] dm ,$$

(3.4)

where the K_i are Hankel functions and

$$\rho(m, n) = \rho_{\text{Bose}}(m) - (-1)^n \rho_{\text{Fermi}}(m)$$

$\rho_{\text{Bose}}(m)$ and $\rho_{\text{Fermi}}(m)$ are the hadronic mass spectra of Bosons and Fermions. We split the integral (3.4) into one integral from zero to M (≈ 1 GeV) and one integral from M to ∞ . The first integral reduces to a sum and is calculated numerically using all known hadron masses in the region $m \leq M$ (zero width approximation). In the second integral we use the asymptotic mass spectrum (3.1) with the parameters (3.2) and perform the integration numerically (only $n = 1$ contributes).

In p-p collisions, because of the conserved through-going baryon, the energy density $\epsilon^{(1B)}(T)$ has to be used, which is defined by [see Eq. (5.5) and (II, AII)]

$$\mathcal{E}^{(1B)}(T) \equiv \mathcal{E}^{(0)}(T) + \left[\frac{3}{2} T + \bar{m}_B(T) \right] / (V/2) . \tag{3.5}$$

The two resulting curves $\epsilon^{(0)}(T)$ and $\epsilon^{(1B)}(T)$ are qualitatively so similar to those given in Fig. 2 of paper (II) that we do not show them here. The differences can be seen better in comparing Table 1 at the end of this paper with Table (AII;1) of paper (II). Note that here again the actual values in the table have to be considered as preliminary: they depend on the choice of parameters in the fit to the hadronic mass spectrum.

In paper (II), Appendix II, the mean (excited) baryon mass was defined:

$$\bar{m}_B(T) = \frac{\int m \rho(m) m^{3/2} \exp(-m/T) dm}{\int \rho(m) m^{3/2} \exp(-m/T) dm} . \tag{3.6}$$

Using for $\rho_B(m)$ the form

$$\rho_B(m) = 4 \delta(m - m_p) + a_B \Theta(m - m_{0B}) \frac{\exp(m/T_0)}{m^3} \quad (3.7)$$

we obtain for $\bar{m}_B(T)$ in place of Eq. (A.II.16) of paper (II)

$$\bar{m}_B(T) = \frac{4 m_p^{5/2} \exp(-\frac{m_p}{T}) + \frac{a_B}{\sqrt{T}} \sqrt{\pi} [1 - \operatorname{erf}(\sqrt{T} m_{0B})]}{4 m_p^{3/2} \exp(-\frac{m_p}{T}) + a_B \left[\frac{2 \exp(-T m_{0B})}{\sqrt{m_{0B}}} - 2 \sqrt{\pi T} (1 - \operatorname{erf}(\sqrt{T} m_{0B})) \right]} \quad (3.8)$$

where

$$\tau \equiv \frac{T_0 - T}{T T_0} \quad (3.9)$$

$$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (3.10)$$

The average multiplicity of particles belonging to family I and not subject to a conservation law is defined [see (II), Appendix I and this paper text after equ. (4.13)] as

$$N_I(T) = \frac{V_0 T}{2\pi^2} \int_0^\infty m^2 \rho_I(m) K_2\left(\frac{m}{T}\right) dm \quad (3.11)$$

We use the approximation

$$K_2\left(\frac{m}{T}\right) \cong \sqrt{\frac{\pi T}{2m}} e^{-\frac{m}{T}} \quad (3.12)$$

and

$$\begin{aligned} \rho_I(m) &= z_i \delta(m - m_i) + a_I \Theta(m - m_{0I}) \frac{\exp(m/T_0)}{m^3} \\ z_i &= (2I_i + 1)(2J_i + 1) \end{aligned} \quad (3.13)$$

and obtain

$$N_I(\tau) = z_i V_0 \left(\frac{m_i T}{2\pi} \right)^{3/2} \exp\left(-\frac{m_i}{T}\right) + a_I V_0 \left(\frac{T}{2\pi} \right)^{3/2} F\left(-\frac{3}{2}, \tau m_{0I}\right) \quad (3.14)$$

with

$$F\left(-\frac{3}{2}, \tau m_{0I}\right) = \frac{2 \exp(-\tau m_{0I})}{\sqrt{m_{0I}}} - 2\sqrt{\pi\tau} \left[1 - \operatorname{erf}(\sqrt{\tau m_{0I}}) \right] \quad (3.15)$$

The parameters used in Eqs. (3.7) and (3.13) are

$$\begin{aligned} a_B &= 0.0256 \text{ (GeV)}^2 & m_{0B} &= 1.236 \text{ GeV} \\ a_K &= 0.0515 \text{ (GeV)}^2 & m_{0K} &= 0.725 \text{ GeV} \\ a_Y &= 0.022 \text{ (GeV)}^2 & m_{0Y} &= 1.382 \text{ GeV} \end{aligned} \quad (3.16)$$

Again, the values a_B , a_K and a_Y depend on the fit to the mass spectrum. To the changes relevant for calculating single particle spectra which occur in $\varepsilon(T)$, $\bar{m}(T)$ and $N_I(T)$ a very important one is still to be added: the new structure of the decay chain discussed in Section 2. We shall come back to the joint effect of these and some other improvements in the next Section.

4. ASYMPTOTICS OF INCLUSIVE SINGLE PARTICLE SPECTRA IN THE THERMODYNAMICAL MODEL

The asymptotic behaviour of inclusive single particle spectra in our model derives from the following three assumptions [see (II) and Ref. 3]

- local energy conservation;
- energy-independence (for $s \rightarrow \infty$) and continuity of the velocity distribution functions $F(\lambda)$ $F_0(\lambda)$, in particular at $\lambda = 0$;
- local validity of the thermodynamic description.

On the other hand, the asymptotic behaviour of spectra can be described in terms of the model-independent concepts of scaling, limiting fragmentation and central plateau. These concepts constitute categories of description rather than theories or hypotheses. Of course, they have been invented -- or better: extracted from experimental knowledge and properties of models -- with the suspicion that they might describe actual properties of asymptotic cross-sections, a suspicion which so far has been strengthened by most experiments, but has also been challenged [V. Lindern et al.⁷⁾].

It is interesting and worth while to classify models according to such general concepts. This has been done in several review articles [Frazer et al.⁸⁾, Van Hove⁹⁾, Berger¹⁰⁾, and Ranft and Ranft¹¹⁾].

In the review by Frazer et al., as well as in an article by De Tar¹²⁾, the thermodynamical model is discussed and confronted with other models. The authors come to the conclusion that it leads to scaling with a vanishing central plateau and to limiting fragmentation and that the total multiplicity increases faster than $\ln s$. These conclusions were valid for the model in its old version with the weak bootstrap solution. The strong bootstrap solution with the above unaltered assumptions leads to a central plateau and to a logarithmic increase of multiplicity. In order to discuss this in detail, we recall briefly the concepts of scaling, central plateau, and limiting fragmentation.

4.1 Scaling, central plateau, limiting fragmentation

4.1.1 Scaling

Consider the reaction $a + b \rightarrow i + \text{anything}$. Scaling is defined [Feynman¹³⁾] in the c.m. system as the following property of the invariant differential cross-section: it is said to scale if

$$E^* \frac{d^3\sigma^*}{d^3p^*} \equiv f(p_{\parallel}^*, p_{\perp}, s) \xrightarrow{s \rightarrow \infty} g(x, p_{\perp}) \quad (4.1)$$

for all $|x| \geq x_0(s) > 0$

where E^* , $p_{\parallel}^*$ are c.m. energy and longitudinal momentum of the observed particle "i", $p_{\perp}$ is its transverse momentum (invariant under longitudinal Lorentz boosts); x and $x_0(s)$ are defined as

$$\begin{aligned} x &\equiv 2p_{\parallel}^* / \sqrt{s} \quad \text{with} \quad s = E_{c.m.}^2 \\ x_0(s) &= \text{const.} \cdot m_i / \sqrt{s} \\ -\sqrt{\alpha(s, m_i, m_R)} &\leq x \leq \sqrt{\alpha(s, m_i, m_R)} \end{aligned} \quad (4.2)$$

$$\alpha(s, m_i, m_R) = \frac{1}{s^2} [s - (m_i + m_R)^2] [s - (m_i - m_R)^2] \leq 1.$$

The limiting values which x can reach, $\pm\sqrt{\alpha}$, depend on the mass m_i of the particle considered and of the minimum recoil mass m_R required for production of particle i . For $s \rightarrow \infty$ these limits tend to ± 1 . Note that other definitions of x are in current use, for instance $x = p_{\parallel}^* / p_{\text{max}}^*$ (here $-1 < x \leq 1$ for all particles; scaling may be approached faster in this definition of x) or $x = p_{\parallel}^* / p_{\text{incoming}}^*$; all these become equal in the limit $s \rightarrow \infty$.

The above definition (4.1) of scaling is cautious: limitation to $|x| \geq x_0$ permits a s -dependent singularity of $g(x, p_\perp)$ at $x = 0$. Indeed, all momenta $p_\parallel^*$ which, for $s \rightarrow \infty$, grow less fast than $\sim \sqrt{s}$ will eventually be mapped onto the point $x = 0$. This region, however, is very interesting, as various models may give predictions which, for $s \rightarrow \infty$, differ only at $x = 0$. Therefore, the variable x is unsuitable to discuss such differences between models. Before we turn to the better suited rapidity variable, let us note that sometimes scaling is understood as meaning $g(x, p_\perp)$ to be continuous at $x = 0$ [then the above cautious formulation is unnecessary and $x_0(s)$ may be replaced by zero]. Of course this is a new and stronger assumption which one calls "scaling with central plateau"¹²⁾; the reason for this name will become clear soon.

The longitudinal rapidity y^* is defined by any of the three following equations

$$\begin{aligned} y^* &= \tanh^{-1} \left(\frac{p_\parallel^*}{E^*} \right) = \frac{1}{2} \ln \left(\frac{E^* + p_\parallel^*}{E^* - p_\parallel^*} \right) \\ y^* &= \cosh^{-1} \left(\frac{E^*}{\mu} \right) = \ln \left(\frac{E^* + \sqrt{E^{*2} - \mu^2}}{\mu} \right) \\ y^* &= \sinh^{-1} \left(\frac{p_\parallel^*}{\mu} \right) = \ln \left(\frac{p_\parallel^* + \sqrt{p_\parallel^{*2} + \mu^2}}{\mu} \right) \end{aligned} \quad (4.3)$$

where $\mu \equiv \sqrt{p_\perp^2 + m^2}$ = "longitudinal mass".

With $p_\parallel^* = x\sqrt{s}/2$ the relation between y^* and x is most easily inferred from the last equation (4.3):

$$\begin{aligned} y^* &= \sinh^{-1} \left(\frac{x\sqrt{s}}{2\mu} \right) = \ln \left(\frac{\sqrt{s}}{2\mu} \right) + \ln \left(x + \sqrt{x^2 + 4\mu^2/s} \right) \\ x &= \frac{2\mu}{\sqrt{s}} \sinh y^* \end{aligned} \quad (4.4)$$

$$dy^* = \frac{dp_\parallel^*}{E^*} = \frac{dx}{\sqrt{x^2 + 4\mu^2/s}} .$$

The last relation implies

$$E^* \frac{d^3\sigma^*}{d^3p^*} = \frac{d^3\sigma^*}{dy^* d^2p_\perp} = \sqrt{x^2 + 4\mu^2/s} \frac{d^3\sigma^*}{dx d^2p_\perp} . \quad (4.5)$$

Note that the relation between x and y^* is not one-to-one because the $p_{\perp}$ -dependent longitudinal mass enters.

The longitudinal rapidity has two advantages:

i) A Lorentz transformation along the collision axis adds only a constant to y^* . Let $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$ characterize such a transformation, then with

$$\begin{aligned} E^* &= \mu \cosh y^* & p_{\parallel}^* &= \mu \sinh y^* \\ \gamma &= \cosh Y & \beta\gamma &= \sinh Y \end{aligned} \quad (4.6)$$

we obtain the transformed variables

$$\begin{aligned} E &= \mu \cosh (y^* + Y) & p_{\parallel} &= \mu \sinh (y^* + Y) \\ y &= y^* + Y & dy &= dy^* \end{aligned} \quad (4.7)$$

Therefore a longitudinal Lorentz transformation leaves the shape of the differential cross-section [Eq. (4.5)] unchanged, it merely shifts the distribution by an amount $Y = \cosh^{-1} \gamma$.

Example: in a pp collision the y^* distribution is symmetric around $y^* = 0$ and extends to $\pm y_{\max}^*$. This limit is reached only when $p_{\perp} = 0$; from Eqs. (4.2) and (4.4) it follows

$$y_{\max}^* = \ln \frac{\sqrt{s}}{2m} + \ln (\sqrt{\alpha} + \sqrt{\alpha + 4m^2/s}) \xrightarrow{s \rightarrow \infty} \ln \frac{\sqrt{s}}{m} \quad (4.8)$$

It depends -- via $\alpha(s, m, m_R)$ -- on the kind of particle considered. Transformation to the target rest frame (laboratory) with

$$Y_0 = \cosh^{-1} \gamma_0 = \ln \frac{\sqrt{s}}{2m_p} + \ln (1 + \sqrt{1 - 4m_p^2/s})$$

will shift this distribution to the right such that

$$\begin{aligned} y_{\text{lab}}^{\min} &= \ln \left(\frac{m}{m_p} \cdot \frac{\sqrt{s} + \sqrt{s + 4m_p^2}}{\sqrt{\alpha s} + \sqrt{\alpha s + 4m^2}} \right) \xrightarrow{s \rightarrow \infty} \ln \frac{m}{m_p} \\ y_{\text{lab}}^{\max} &\xrightarrow{s \rightarrow \infty} \ln (s/m m_p) \end{aligned} \quad (4.9)$$

ii) The second advantage is, that in the rapidity distribution the region around $x = 0$ becomes stretched out: from Eq. (4.4) we see that $dy^* \rightarrow dx/x$ for $s \rightarrow \infty$. In fact, a constant interval $-\xi \leq x \leq \xi$ will, in the y^* representation, appear as $-\ln(\xi\sqrt{s}/\mu) \leq y^* \leq \ln(\xi\sqrt{s}/\mu)$ when $s \rightarrow \infty$. Therefore any interval, which in the y^* representation expands less than $\ln(\xi\sqrt{s}/\mu)$ with some $\xi > 0$, will shrink to $x = 0$ in the x representation when $s \rightarrow \infty$. In other words: if in the y^* distribution some structure can be seen (e.g. a bump or a dip), which stays fixed or moves out to $\pm\infty$ less fast than $\ln(\xi\sqrt{s}/\mu)$, no matter how small the constant $\xi > 0$ is chosen, then $g(x, p_\perp)$ is discontinuous at $x = 0$. On the other hand, if $g(x, p_\perp)$ is continuous at $x = 0$, then a $\xi(\Delta) > 0$ can be found such that for $s \rightarrow \infty$ the variation of $d^3\sigma^*/dy^* d^2p_\perp$ remains smaller than Δ in the interval $|y^*| < \ln(\xi(\Delta)\sqrt{s}/\mu)$. This region, which expands logarithmically with s , is called the "central plateau"¹²⁾.

4.1.2 Central plateau, pionization

From the above discussion it follows that existence of a continuous $g(x, p_\perp)$ implies a y^* distribution which for $s \rightarrow \infty$ remains, except for a logarithmic dilation, similar to itself. In that case -- unless $g(0, p_\perp) = 0$ -- the total multiplicity must grow logarithmically with s ; this is true already for the central plateau region which contains all c.m. longitudinal momenta $p_\parallel^*$ growing less fast than $\sqrt{s}$. This range of momenta is often called the "pionization" region; in this definition it coincides with the central plateau region; Van Hove⁹⁾ calls any region which in the y^* representation expands less fast than $\ln(\xi\sqrt{s}/\mu)$ an "intermediate region".

There is no standard use of the word "pionization" in the literature. For instance, Chou and Yang¹⁴⁾ define pionization as follows: if, for $s \rightarrow \infty$, the fraction f of produced hadrons having a c.m. energy less than a given W does not go to zero, then there is pionization. Particles with c.m. energy less than a constant W have also longitudinal momenta less than a given constant and hence an $|x| < \text{const}/\sqrt{s}$ or $|y^*| < y_0 = \text{const}$. If the total multiplicity increases beyond limit when $s \rightarrow \infty$, and if f does not go to zero, then the spectrum $d^3\sigma^*/dy^* d^2p_\perp$ increases beyond limit inside a fixed interval $|y^*| < y_0$. Thus with this definition of pionization there is not a central plateau, but an ever increasing peak around $y^* = 0$.

In order to avoid confusion, we shall not use the word pionization; indeed it will turn out that the properties of the thermodynamical model in the strong bootstrap solution can be described with the notions scaling and central plateau.

4.1.3 Limiting fragmentation

Limiting fragmentation is defined in the rest frame of either the target or of the projectile particle [Benecke et al.¹⁵⁾]: there is limiting fragmentation if

$$E \frac{d^3\sigma}{d^3p} = F(\vec{p}, s) \xrightarrow{s \rightarrow \infty} G(\vec{p}) \quad (4.10)$$

where E and $\vec{p}$ are energy and momentum of the observed particle (in the target or projectile frame). Note that the convergence will not be uniform; the meaning of Eq. (4.10) is, that for any given $|\vec{p}|$ an $s(p)$ can be found such that $|G(\vec{p}) - F(\vec{p}, s)|$ is arbitrarily small for $s > s(p)$.

It is easy to see that this is equivalent to scaling. Let us choose the target frame and fix there $\vec{p} = (p_{\parallel}, \vec{p}_{\perp})$. Transforming to the c.m. frame gives $p_{\parallel}^* = \gamma_0(p_{\parallel} - \beta_0 E)$ and, with $2m\gamma_0 = \sqrt{s}$ we have $x = 2p_{\parallel}^*/\sqrt{s} = (p_{\parallel} - \beta_0 E)/m \rightarrow (p_{\parallel} - E)/m = \text{const.}$ Hence Eq. (4.10) states that for fixed x the invariant differential cross-section becomes independent of s , which is indeed scaling.

4.2 Limiting fragmentation and scaling for thermodynamic spectra

For notation and details we refer to paper (II) and/or to the "CERN lectures". We consider here, for the case of pp collisions, the most general contribution to single particle spectra namely where some conservation law enforces the creation of a particle "b" in addition to the observed one "a". If no such conservation law is acting (e.g. in pion production) then in the following formulae all factors bearing a subscript "b" become equal to one.

In the old version of the model (weak bootstrap solution) the single particle inclusive spectra were:

i) *newly produced a and b* (e.g. $K + \bar{K}$)

$$W_a(\vec{p}) = Q_a(\gamma_0) \int_{-1}^1 d\lambda F(\lambda) [F(\lambda) N(T)]_b L(\lambda, \gamma_0) \{f_a(p', T)\} \quad (4.11)$$

ii) *throughgoing $a \rightarrow a' + b$* (e.g. $p \rightarrow \Lambda + K$)

$$W_{a'}(\vec{p}) = \int_{-1}^1 d\lambda F_0(\lambda) \frac{[F(\lambda) N(T)]_b}{N_a(T)} L(\lambda, \gamma_0) \{f_{a'}(p', T)\}. \quad (4.12)$$

Here the variables have the following meaning:

$$\lambda = \text{Sign}(\beta) \frac{\gamma - 1}{\gamma_0 - 1} \quad (4.13)$$

is related to the c.m.s. velocity of the fireball with Lorentz factor γ from which particle "a" is emitted; $\gamma_0 = \sqrt{s}/2m_p$ is the Lorentz factor of the c.m.s.

$F(\lambda)$ and $F_0(\lambda)$, both normalized to two over the interval $-1 \leq \lambda \leq 1$ and symmetric for pp collisions, are the "velocity weight functions". They are supposed to be independent of the collision energy and of the type of particle created; $F(\lambda)$ is for newly created, $F_0(\lambda)$ for throughgoing particles [see (II), Section 2.4];

$N_i(T)$ with $i = a, b$ is a formal normalization factor arising automatically from imposing the conservation law on the partition function [see (II), Appendix 1]; physically $N_i(T)$ is the number of particles of family i [i.e. any particle with the same baryon number and strangeness as i (excited or not)] which would be produced in the first generation *if there were no conservation law* [see Eq. (3.11)];

$T = T[\varepsilon(\lambda, \gamma_0)]$ is the local temperature depending on the local energy density $\varepsilon(\lambda, \gamma_0)$; for $\varepsilon(\lambda, \gamma_0)$ see below;

$f_i(p', T)$ is the isotropic Planck-type momentum spectrum of particle i coming from a black-body radiation of temperature T and at rest in the λ frame;

$L(\lambda, \gamma_0)$ is a Lorentz boost operator along the collision axis which transforms $f_i(p', T)$ from the λ frame into the frame of observation (laboratory, c.m.s. etc.), thus generating an anisotropic momentum distribution;

the integration over λ can be interpreted in various ways: either as a sum over many fireballs created in each single collision or, if a single collision produces only one or two fireballs (e.g. diffraction dissociation), as an average sum over many collisions, or as any intermediate possibility^{*)};

finally, $Q_a(\gamma_0)$ was an over-all normalization factor which, in the old version, globally took care of the increase of multiplicity due to the decay chain of the weak bootstrap solution (Fig. 1a).

Here a significant change occurs when we now consider the strong bootstrap solution with its simple decay chain (Fig. 1b). Instead of an $\sqrt{s}$ -dependent factor $Q_a(\gamma_0)$ in front of the integral, we will now have a multiplicity factor depending no longer on the collision energy but only on the masses of decaying fireballs: each generation of decay will yield a spectrum $f_i(p', T)$ and if we neglect cooling along the decay chain, the correct normalization of the observed final spectrum will be obtained by multiplying $f_i(p', T)$ by the "number of contributing generations". For pions this number of contributing generations is expected to be roughly equal to the number $n(\varepsilon)$ of generations given by Eq. (2.16); for heavier particles it might be only a fraction of $n(\varepsilon)$, because cooling along the decay chain will suppress the contributions from later generations.

*) This freedom of interpretation will disappear when two (or more) particle correlations are considered.

Thus in the strong bootstrap solution, the factor $Q_a(\gamma_0)$ has to be replaced by a factor proportional to

$$n(\varepsilon) \approx 1.4 \frac{\varepsilon}{\varepsilon_0}. \quad (4.14)$$

We shall write this factor as

$$q_a(\varepsilon) \equiv q_a \cdot \frac{\varepsilon(\lambda, \gamma_0)}{\varepsilon_0} \quad (4.15)$$

where q_a is a constant of order one taking globally into account the complicated details of the decay chain, which we do not try to explain; this factor q_a , which only depends on the type of particle, rests outside the integral (4.11) while the factor $\varepsilon(\lambda, \gamma_0)/\varepsilon_0$ multiplies the spectrum $f_a[p', T(\varepsilon)]$ *inside* the integral, because for each value of the integration variable λ , the integrand represents the contribution from a particular fireball.

The energy density $\varepsilon(\lambda, \gamma_0)$ is assumed to follow from local energy conservation [see paper (II), or Ref. 3; our present ε is identical with $\varepsilon^{(1,B)}$ defined in Eq. (3.5)]

$$\varepsilon_0 \gamma_0 = \varepsilon \gamma \quad (4.16)$$

where $\varepsilon_0 = m_p/V_0$ is the rest energy density of the proton before the collision and ε that of excited hadronic matter moving collectively with Lorentz factor γ . Using Eq. (4.13) we have

$$\varepsilon(\lambda, \gamma_0) = \varepsilon_0 \frac{\gamma_0}{\gamma} = \varepsilon_0 \frac{\gamma_0}{1 + |\lambda|(\gamma_0 - 1)} \quad (4.17)$$

Let us fix a value $\lambda_0 < 1$. We consider then collision energies such that

$$\gamma_0 \lambda_0 \gg 1 \quad \text{i.e.} \quad \frac{\sqrt{s}}{2m} \gg \frac{1}{\lambda_0} \quad (4.18)$$

In this case Eq. (4.17) becomes

$$\varepsilon(\lambda, \gamma_0) = \varepsilon_0 / |\lambda| \quad \text{for} \quad |\lambda| > \lambda_0 \quad (4.19)$$

so that for $|\lambda| > \lambda_0$ all quantities under the integrals in Eqs. (4.11) and (4.12) which depend via $T(\epsilon)$ on the energy density, depend in fact only on λ and no longer explicitly on the primary energy γ_0 .

The limiting behaviour of $T(\lambda, \gamma_0)$ is shown in Fig. 2.

Limiting fragmentation becomes now obvious if we transform to the target (or projectile) frame: the velocity four vector

$$V_{\lambda}^{(lab)} \equiv (\gamma_{\lambda}, \beta_{\lambda} \gamma_{\lambda})^{(lab)} \quad (4.20)$$

of a λ fireball emerging from the target ($\lambda < 0$) and seen from the target rest frame (i.e. lab frame) is given by

$$\begin{aligned} \gamma_{\lambda}^{(lab)} &= \gamma \gamma_0 (1 - \beta \beta_0) \\ [\beta_{\lambda} \gamma_{\lambda}]^{(lab)} &= \gamma \gamma_0 (\beta + \beta_0) \end{aligned} \quad (4.21)$$

where β_0, γ_0 are velocity and Lorentz factor of the c.m.s. seen from the target frame, while β, γ are the same quantities of the λ frame seen from the c.m.s.

For $|\lambda| > \lambda_0$ and $\gamma_0 \rightarrow \infty$ the forward movement of the c.m.s. and the backward movement of the λ frame ($\lambda < 0$) compensate such that

$$\begin{aligned} \gamma_{\lambda}^{(lab)} &\Rightarrow \frac{1 + \lambda^2}{2|\lambda|} \\ [\beta_{\lambda} \gamma_{\lambda}]^{(lab)} &\Rightarrow \frac{1 - \lambda^2}{2|\lambda|} \end{aligned} \quad (4.22)$$

so that the four velocity $V_{\lambda}^{(lab)}$ of a "fragment of the target" becomes independent of γ_0 . This velocity $V_{\lambda}^{(lab)}$ is to be used in the Lorentz boost $L(\lambda, \gamma_0)$ from the λ frame to the target rest frame (lab), if the particle spectrum is to be observed there. Looking back at the representations (4.11) and (4.12) and interpreting them as spectra in the target rest frame, we see that because of Eq. (4.22) in both of them the integrand depends only on λ and on $\epsilon_0/|\lambda|$ for all $\lambda < -\lambda_0$, while for all λ in $-\lambda_0 \leq \lambda < 1$ an explicit γ_0 dependence remains. Hence the lab. spectrum can be written in the general form

$$\begin{aligned}
 W_{(\vec{p})}^{(lab)} = & \int_{-1}^{-\lambda_0} G(\lambda) L^{(lab)}(\lambda) \{ \phi(p', \lambda) \} d\lambda \\
 & + \int_{-\lambda_0}^{\lambda_0} G(\lambda) L^{(lab)}(\lambda, \gamma_0) \{ \phi(p', \lambda, \gamma_0) \} d\lambda \\
 & + \int_{\lambda_0}^1 G(\lambda) L^{(lab)}(\lambda, \gamma_0) \{ \phi(p', \lambda) \} d\lambda
 \end{aligned} \tag{4.23}$$

where $G(\lambda)$ is some combination of $F(\lambda)$, $F_0(\lambda)$ and $N(T)$, while $L^{(lab)}(\lambda, \gamma_0)$ is the boost operator which is γ_0 independent for $\lambda < -\lambda_0$, and $\phi(p', \lambda, \gamma_0)$ is a combination of the Planck-type momentum spectrum $f(p', T)$ with the multiplicity factor $q(\epsilon)$ given by Eq. (4.15). Note that this combination only depends on $\epsilon(\lambda, \gamma_0)$ and becomes, because of Eq. (4.19), independent of γ_0 for all $|\lambda| > \lambda_0$. Hence only the integral from $-\lambda_0$ to λ_0 contains a γ_0 dependent spectrum $\phi(p', \lambda, \gamma_0)$. The first integral (from -1 to $-\lambda_0$) gives the limiting fragmentation spectrum in the lab. frame, the second yields a central plateau (if any) and the last would become γ_0 independent in the projectile frame; it is the limiting fragmentation spectrum of the projectile. Obviously, the two fragmentation regions contribute constant multiplicity when we let $\gamma_0 \rightarrow \infty$ for λ_0 fixed.

Thus the hypothesis that $F(\lambda)$ and $F_0(\lambda)$ are (or become for $s \rightarrow \infty$) energy independent, is equivalent to limiting fragmentation and thus to scaling.

The constant x_0 , introduced in the discussion of scaling, and the constant λ_0 defined here have similar significance: they exclude the central region. The relation between x and λ is complicated, because x measures the final longitudinal momentum, while λ measures the velocity of the emitting fireball; the particles created from this fireball have already in the λ frame thermic longitudinal (and transversal) momenta which then are Lorentz transformed by $L(\lambda)$ into the final ones. Thus to one single λ value corresponds a whole set of x values obtained by Lorentz transforming a thermic distribution in the λ frame. If one neglects this distribution and considers the emitted particle m on the average at rest in the λ frame, then its c.m.s. longitudinal momentum is, for $|\lambda| > \lambda_0$ approximately

$$\langle p_n^* \rangle_\lambda = \gamma (\langle p_n' \rangle + \beta \langle E' \rangle) \approx \beta \gamma (m + \frac{3}{2} T) = \lambda \gamma_0 (m + \frac{3}{2} T) \tag{4.24}$$

so that the x distribution belonging to a given λ centres around

$$\langle x \rangle_\lambda = \frac{2 \langle p_{||}^* \rangle_\lambda}{\sqrt{s}} = \frac{2 \langle p_{||}^* \rangle_\lambda}{2 m_p \gamma_0} \approx \lambda \cdot \frac{m + \frac{3}{2} T(\lambda)}{m_p} . \quad (4.25)$$

In each individual case x and λ can be anything between very different or equal, depending on what the thermic longitudinal momentum in the λ frame was.

The result of this section is then, that in the strong bootstrap solution the thermodynamic spectra strictly obey scaling and limiting fragmentation; in the old weak bootstrap solution, they did so only if the over-all energy dependent normalization factor $Q(\gamma_0)$ was disregarded, i.e. there scaling and limiting fragmentation was true for the shape only.

Another result of this section is that the multiplicity factor $\varepsilon/\varepsilon_0$ multiplying the Planck spectrum under the integral amounts to a redefinition of $F(\lambda)$ in the fragmentation region $|\lambda| > \lambda_0$, because there $\varepsilon/\varepsilon_0 = |1/\lambda|$ so that $F(\lambda)^{(new)} \approx |\lambda F(\lambda)^{(old)}|$.

4.3 The central plateau with dip; logarithmic multiplicity

We return to Eq. (4.23) and consider now the integral from $-\lambda_0$ to λ_0 , which describes the contribution from the central part of the collision. As λ_0 can be fixed arbitrarily, it depends entirely on our choice, what we call fragmentation region ($|\lambda| > \lambda_0$) and what central region ($|\lambda| < \lambda_0$), provided we first fix λ_0 and then consider only collisions with $\sqrt{s} \gg 2m_p/\lambda_0$.

We discuss the contribution of the central part in the c.m. system for $\lambda_0 \ll 1$:

$$W_{(\vec{p})}^{(cent)} = \int_{-\lambda_0}^{\lambda_0} G(\lambda) L^{c.m.}(\lambda, \gamma_0) \{ \varphi(p', \lambda, \gamma_0) \} d\lambda \quad (4.26)$$

$G(\lambda)$ is a combination of factors $F(\lambda)$, $F_0(\lambda)$ and $N(T)$, all of which are supposed to be continuous at $\lambda = 0$, in particular T will tend to T_0 ; hence $G(\lambda) = G(0)$ is a constant, which can be taken out of the integral.

The spectrum in the rest frame of the decaying fireball is explicitly written

$$\begin{aligned} \phi(p', \lambda, \gamma_0) &= \frac{\varepsilon}{\varepsilon_0} \cdot \omega_{rest} \cdot \left[\exp \frac{E'}{T} \pm 1 \right]^{-1} \\ E' &= \sqrt{p_{||}'^2 + p_{\perp}'^2 + m^2} \equiv \sqrt{p_{||}'^2 + \mu^2} ; \quad T \approx T_0 . \end{aligned} \quad (4.27)$$

The dependence on γ_0 enters only via the multiplicity factor [see Eq. (4.16)]

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\gamma_0}{\gamma} . \quad (4.28)$$

It is convenient to work with the invariant cross section $E^* d^3\sigma^*/d^3p^*$ which is obtained multiplying Eq. (4.26) by E^* and by a constant (containing the total cross-section). Thus

$$\left[\frac{E^* d^3\sigma^*}{d^3p^*} \right]^{(centr)} = \text{const} \cdot \gamma_0 \int_{-\lambda_0}^{\lambda_0} E^* L^{c.m.}(\lambda, \gamma_0) \left[\frac{1}{e^{E'/T} \pm 1} \right] \frac{d\lambda}{\gamma} . \quad (4.29)$$

The invariance property of the cross-section implies [see (II), Appendix V)]

$$E^* L^{c.m.}(\lambda, \gamma_0) \left[\frac{1}{e^{E'/T} \pm 1} \right] = \frac{E'}{e^{E'/T} \pm 1} \quad (4.30)$$

where $E' = \sqrt{p_{\parallel}^{\prime 2} + \mu^2}$ has to be expressed in c.m. variables. This is most easily done using the longitudinal rapidity. We denote by

$$\left. \begin{aligned} y' &= \cosh^{-1} (E'/\mu) = \text{rapidity in } \lambda \text{ frame} \\ y^* &= \cosh^{-1} (E^*/\mu) = \text{rapidity in c.m. frame} \\ \eta &= \cosh^{-1} \gamma = \text{rapidity of } \lambda \text{ frame seen from c.m. frame} \end{aligned} \right\} \quad (4.31)$$

Additivity of rapidities gives $y' = y^* - \eta$ so that

$$\begin{aligned} \left[\frac{E^* d^3\sigma^*}{d^3p^*} \right]^{(centr)} &= \left[\frac{d^3\sigma^*}{dy^* d^2p_{\perp}} \right]^{(centr)} = \\ &= \text{const} \cdot \gamma_0 \int_{-\lambda_0}^{\lambda_0} \frac{d\lambda}{\gamma} \left[\frac{\mu \cosh(y^* - \eta)}{\exp\left(\frac{\mu}{T} \cosh(y^* - \eta)\right) \pm 1} \right] . \end{aligned} \quad (4.32)$$

It remains to express λ and γ in terms of η . We have, with $\gamma_0 \gg 1$,

$$\lambda = \pm \frac{\delta-1}{\delta_0}$$

$$d\lambda = \pm \frac{d\delta}{\delta_0} = |\sinh \eta| \cdot \frac{d\eta}{\delta_0} . \quad (4.33)$$

Therefore the factor γ_0 in front of the integral is cancelled. Note that in the old version (weak bootstrap solution) the factor $\varepsilon/\varepsilon_0 = \gamma_0/\gamma$ was missing so that there the factor $1/\gamma_0$ in the differential $d\lambda$ remained uncanceled and consequently the contribution of the central region vanished like $1/\gamma_0 = 2m_p/\sqrt{s}$, with a resulting constant over-all multiplicity, entirely due to fragmentation^{*}).

We return to Eq. (4.32). The limits λ_0 become with $\lambda_0\gamma_0 \gg 1$

$$\eta_0 = \cosh^{-1} \delta_{\max} = \cosh^{-1} \lambda_0 \delta_0 \cong \ln 2\lambda_0 \delta_0 \quad (4.34)$$

hence, finally

$$\left[\frac{d^3\sigma^*}{dy^* d^2p_\perp} \right]^{(centr)} = \text{const} \int_{-\eta_0}^{\eta_0} d\eta |\tanh \eta| \left[\frac{\mu \cosh(y^* - \eta)}{\exp\left(\frac{\mu}{T} \cosh(y^* - \eta)\right) \pm 1} \right] . \quad (4.35)$$

The qualitative behaviour is easy to understand by inspection of Eq. (4.35), where, for the sake of simplicity, we consider the situation for $\eta_0 \gg 1$.

The function in square brackets is an even, positive function of its argument $y^* - \eta$; it vanishes rapidly when $|y^* - \eta|$ grows larger than 2. The other factor, $|\tanh \eta|$ is zero for $\eta = 0$ and nearly equal to 1 for $|\eta| > 2$. We have therefore to consider essentially two situations, namely either $y^* \approx 0$, in which case the factor $|\tanh \eta|$ matters, or $|y^*| \gtrsim 4$ where the rapid decrease of the square bracket forces η to be of order y^* , so that $\tanh \eta \approx 1$. Clearly, the whole integral vanishes rapidly, and for the same reason, when $|y^*|$ passes beyond $\sim \eta_0 + 2$. The integral is an even function of y^* . All this is illustrated in Fig. 3.

We thus evaluate the integral for $y^* = 0$ and for $1 \ll |y^*| \ll \eta_0$; in both cases we neglect the ± 1

i) $y^* = 0$ (see Fig. 3):

$$\left[\frac{d^3\sigma^*}{dy^* d^2p_\perp} \right]_{y^*=0}^{(centr)} = \text{const} \cdot 2\mu \int_0^{\eta_0} d\eta \sinh \eta \cdot \exp\left(-\frac{\mu}{T} \cosh \eta\right) . \quad (4.36)$$

^{*}) Increasing multiplicities (required by energy sum rules) came from the factors $Q(\gamma_0)$.

As the integrand goes rapidly to zero with increasing η , we can, without error, integrate to ∞ instead of integrating to η_0 and obtain

$$\left[\frac{d^3 \sigma^*}{dy^* d^2 p_\perp} \right]_{y^*=0}^{(centr)} \approx \text{const} \sqrt{\frac{2\mu T}{\pi}} K_{1/2}\left(\frac{\mu}{T}\right) = \text{const} \cdot T e^{-\mu/T} \quad (4.37)$$

where $K_{1/2}(x) = \sqrt{\pi/2x} e^{-x}$ is a modified Hankel function. This result is independent of the collision energy $\sqrt{s}$ for $s \rightarrow \infty$ ($T \rightarrow T_0$).

ii) $1 \ll |y^*| \ll \eta_0$ (see Fig. 3)

Here, $|\tanh \eta| = 1$ for the whole η region in which the square bracket contributes; only the interval $\eta \in \{y^* \pm 2\}$ will be important and all the rest is negligible; introducing the variable $\xi = y^* - \eta$ we can therefore integrate without error from $-\infty$ to $+\infty$ and obtain

$$\begin{aligned} \left[\frac{d^3 \sigma^*}{dy^* d^2 p_\perp} \right]_{1 \ll |y^*| \ll \eta_0}^{(centr)} &\approx \text{const} \cdot \mu \int_{-\infty}^{+\infty} d\xi \cosh \xi \exp\left(-\frac{\mu}{T} \cosh \xi\right) \\ &= \text{const} \cdot 2\mu K_1\left(\frac{\mu}{T}\right) \end{aligned} \quad (4.38)$$

where $K_1(x)$ is the modified Hankel function of order 1. Note that the result is independent of y^* (plateau) and of the collision energy $\sqrt{s}$.

Thus the general picture is as illustrated by a numerical result in Fig. 4:

the central (non-fragmentation) part of the invariant cross-section has an *elongating* central plateau of width $2\eta_0 \approx 2 \ln(2\lambda_0 \gamma_0)$ with a small central dip of *constant* width ≈ 4 . The energy independent ratio of heights (plateau to dip) is given by the ratio of Eq. (4.38) to Eq. (4.37)

$$\frac{H_{\text{plateau}}}{H_{\text{dip}}} \approx \sqrt{\frac{\pi\mu}{2T_0}} \frac{K_1(\mu/T_0)}{K_{1/2}(\mu/T_0)} \gtrsim \sqrt{\frac{\pi\mu}{2T_0}} \quad (4.39)$$

Note that this result is approximative, because for $\mu \lesssim 2T_0$ the -1 in the Planck spectrum should not be neglected.

The thus predicted dip will become observable only under favourable conditions at high energies; roughly speaking: the whole rapidity distribution must be much larger than the dip itself -- ISR energies will just be sufficient. For a fully developed dip these conditions are:

- as the discussion of the integral (4.35) has shown, the contribution of the central region (defined by η_0) starts to decrease from $y^* \geq \eta_0 - 2$ and is practically zero for $y^* \geq \eta_0 + 2$. Hence it dies out within a width of ≈ 4 . The whole rapidity distribution (including fragmentation) has a kinematical limit $y_{\max}^*$ and we must expect that the decline sets in at $y_{\max}^* - 4$. In order to exclude that the decrease due to the kinematical limit leaks back into the central plateau, we must require $y_{\max}^* - 4 \geq \eta_0 - 2$ hence condition
1) $y_{\max}^* \geq \eta_0 + 2$
- next, η_0 itself must be big enough so that the decline of the central plateau, which starts at $\eta_0 - 2$ lies outside the dip which itself extends to ≈ 2 . This gives condition
2) $\eta_0 \geq 4$
- finally the dip must not be compensated by a velocity function $F(\lambda)$ which increases too fast when $\lambda \rightarrow 0$ (in deriving the dip, it was assumed that $F(\lambda)$ is constant between 0 and λ_0). This gives condition [see Eq. (4.39)]
3) $F(0)/F(\lambda_0) \leq H_{\text{plateau}}/H_{\text{dip}} \approx \sqrt{\pi\mu/2T_0}$.

Using the definitions Eq. (4.8) and (4.34) for $y_{\max}^*$ and η_0 respectively, we find from condition (1)

$$y_{\max}^* - \eta_0 = \ln \frac{m_p}{\lambda_0 m} \geq 2 \quad \text{or} \quad \frac{1}{\lambda_0} \geq 7 \frac{m}{m_p} \quad (4.40)$$

and from condition (2)

$$\ln 2\lambda_0 \gamma_0 \geq 4 \quad \text{or} \quad \gamma_0 \geq \frac{25}{\lambda_0} \quad (4.41)$$

which gives the final condition for the collision energy

$$\gamma_0 \geq 25 \cdot \frac{7m}{m_p} . \quad (4.42)$$

This condition is just fulfilled for pions emerging from ISR collisions. Of course, if $p_{\perp}$ is very large, the dip increases in depth and may be visible a little earlier.

To explore condition (3) we note that at ISR energies we have to exploit fully inequality (4.40), which means that the whole contributing range of λ has to be admitted. For the production of pions, only values $\lambda \lesssim m_p / (m_p + m_{\pi})$ contribute, but inequality (4.42) is then still satisfied. Now we turn to condition (3): our present parametrization [see Eqs. (5.2) and (5.3), and Table 3] is

$$F(\lambda) = \text{const} \cdot \exp(-0.755 \lambda) \quad (4.43)$$

so that for $\lambda_0 = m_p / (m_p + m_{\pi}) \approx 7/8$ we obtain from condition (3)

$$\mu = \sqrt{p_{\perp}^2 + m^2} \gtrsim 2.5 T_0 \quad (4.44)$$

thus, for pions

$$p_{\perp} \gtrsim 200 \text{ MeV}/c \quad (4.45)$$

This is illustrated in Fig. 5, where for $p_{\perp} = 0$ the dip is almost invisible, but becomes pronounced for larger $p_{\perp}$.

In the x -representation the dip has a decreasing width proportional to $1/\sqrt{s}$. To its constant extension $y^* \approx 2$ belongs, according to Eq. (4.4), an extension into positive x of

$$\Delta x_{\text{dip}} \approx \frac{2\mu}{\sqrt{s}} \cdot 3.6 \quad (4.46)$$

which at full ISR energies gives $\Delta x_{\text{dip}} \approx 0.1 \mu$. Thus from there to $x = 0$ the cross-section should decrease; as it also decreases (exponentially) towards larger x , there should be a maximum at $x_0 \approx \Delta x_{\text{dip}} \approx 0.1 \mu$. For 1500 GeV/c primary momentum we expect the maximum

$$\begin{aligned} \text{for } p_{\perp}^2 = 0.04 & : \text{ at } x_0 \approx 0.024 \\ \text{for } p_{\perp}^2 = 0.16 & : \text{ at } x_0 \approx 0.042 \\ \text{for } p_{\perp}^2 = 0.64 & : \text{ at } x_0 \approx 0.08 \end{aligned} \quad (4.47)$$

these are the values corresponding to the computed curves shown in Figs. 8 and 9. The maxima in the curves appear at the appropriate places; unfortunately there are no experimental points yet. It would be interesting to have this prediction confirmed, because the dip is an unavoidable consequence of the assumption that $F(\lambda)$ is energy dependent and continuous at $\lambda = 0$. As the dip originates from the factor $|\tanh \eta|$ in the integrand of Eq. (4.35), it would disappear if this factor were missing. This would imply a redefinition of our weight function $F(\lambda)$ such that

$$F(\lambda) \longrightarrow \bar{F}(\lambda, \gamma_0) = \frac{F(\lambda)}{|\tanh \eta|} \quad (4.48)$$

In terms of λ and γ_0 this amounts to the following change:

$$F(\lambda) \longrightarrow \bar{F}(\lambda, \gamma_0) = F(\lambda) \cdot \frac{(\gamma_0 - 1) + 1/\lambda}{\sqrt{(\gamma_0 - 1)^2 + 2(\gamma_0 - 1)/\lambda}} \quad (4.49)$$

Let $\gamma_0 \gg 1$. In the fragmentation region, $\lambda \gg 1/\gamma_0$, nothing is changed, while in the central region, $\lambda \ll 1/\gamma_0$, a singularity appears; expanding Eq. (4.49) yields

$$F(\lambda) \rightarrow \bar{F}(\lambda, \gamma_0) \begin{cases} F(\lambda) & \text{for } \lambda \gg 1/\gamma_0 \\ \frac{F(\lambda)}{\sqrt{2\lambda\gamma_0}} & \text{for } \lambda \ll 1/\gamma_0 \end{cases} \quad (4.50)$$

As discussed above, the dip would become clearly observable only at and above ISR energies. In other words: only at ISR energies we reach a region where a singular energy dependence in $F(\lambda)$ at $\lambda \rightarrow 0$ might become observable; at lower energies, even much more accurate data than those used in the fits would not have forced us to abandon the assumption of energy independence and continuity of $F(\lambda)$.

If the predicted dip turns out to be absent experimentally, then $F(\lambda)$ has to be replaced by $\bar{F}(\lambda, \gamma_0)$. In that case it would be more convenient to work directly in the rapidity representation and define a rapidity weight function $H(\eta, \gamma_0)$ which then would not vanish at $\eta = 0$.

While this is perfectly possible, it would have the consequence that our treatment of associated production in which powers of $F(\lambda)$ appear has to be modified: $\bar{F}(\lambda, \gamma_0)$ has a $1/\sqrt{\lambda}$ singularity, hence powers ≥ 2 of $F(\lambda)$ are no longer

integrable and all corresponding production cross-sections would become infinite. Hence powers ≥ 2 of $F(\lambda)$ must not appear. This might cause great trouble, had not Letessier and Tounsi^{*)} made the important observation, that one should require "locality of conservation laws" in a stricter sense than we did so far (and still do here) -- with the result that then automatically only the first power of $F(\lambda)$ appears. In that case $\bar{F}(\lambda, \gamma_0)$ may have any type of integrable singularity at $\lambda = 0$.

We leave to another publication the detailed discussion of what must be done if there is no dip experimentally. For the present paper the assumption is that $F(\lambda)$ be energy-independent and continuous at $\lambda = 0$.

Multiplicities:

It follows that the total multiplicity must grow logarithmically with the collision energy, because the central plateau has -- apart from the small energy independent central dip -- constant, energy-independent height and it elongates logarithmically (the fragmentation regions have constant multiplicities).

* * *

To come back once more to the arbitrariness of the subdivision of the cross-section into one central part ($|\lambda| < \lambda_0$) and two fragmentation parts ($|\lambda| > \lambda_0$): if we choose a fixed λ_0 , then for all $\sqrt{s} \gg 2m_p/\lambda_0$ the fragmentation multiplicity stays constant and the central multiplicity grows proportionally to $2 \ln 2\lambda_0\gamma_0 = \ln(s\lambda_0^2/m_p^2)$; however, with growing $\sqrt{s}$ we are allowed to make λ_0 smaller and smaller; if we choose $\lambda_0 = A \cdot m_p/\sqrt{s}$ with some $A \gg 1$, then the central plateau will have a constant width $\approx 2 \ln A$ while the fragmentation region increases logarithmically. In the x -representation a constant A shrinks like $x(A) \sim \text{const}/\sqrt{s}$ to zero and everything looks like fragmentation there. It is thus seen that it is largely a matter of taste whether one speaks of fixed limited fragmentation and an increasing central part or of extending fragmentation and a fixed central part, which, for $s \rightarrow \infty$ would become negligible against the rest. [In the sense of Chou and Yang¹⁴⁾ there is no pionization in our model, while other authors would say that there is pionization.] Before closing this section, let us summarize:

Our old assumption of paper (II) that the velocity weight functions $F(\lambda)$, $F_0(\lambda)$ are continuous and (for $s \rightarrow \infty$) energy independent, leads, together with the strong bootstrap solution, to scaling, limiting fragmentation, logarithmically elongating central plateau (with a small constant central dip) and logarithmically increasing multiplicities.

*) We thank Drs. J. Letessier and A. Tounsi (Inst. de Phys. Nucl. Faculté des Sciences, Tour 16, Paris V) for communicating their important result before publication.

We emphasize that so far the thermodynamical model does not provide us with any strong intrinsic argument, why the velocity weight functions should be energy independent and why they should be continuous at $\lambda = 0$. As pointed out in detail in paper (II), we constructed them with the explicit aim to make them as little energy dependent as possible, merely *describing the geometric situation of a collision* -- or so we hoped. The claim that they might in fact be energy independent is and remains a hypothesis, fully equivalent to scaling, limiting fragmentation and elongating central plateau with a constant central dip. If any of these should turn out to be wrong, it would be most natural to admit for $F(\lambda)$ and $F_0(\lambda)$ an explicit s dependence other than that of Eq. (4.49). So far there seems to be no need for it.

5. FIT TO EXPERIMENTAL DATA

The expressions for the thermodynamic single particle spectra are changed against the ones described in paper (II), Ranft¹⁶⁾ and Grote et al.¹⁷⁾. The following changes occur:

- i) Use of the temperature energy relation as obtained from the strong bootstrap solution [see formulae (3.4), (3.5) and Table 1].
- ii) Use of the functions $\bar{m}_B(T)$ and $N_I(T)$ according to the strong bootstrap solution; see Eqs. (3.8) and (3.14).
- iii) The decay chain multiplicity function $Q[\varepsilon(\lambda, \gamma_0)]$ is introduced in all spectra of newly produced particles. According to Eq. (4.15) we use

$$q(\varepsilon(\lambda, \gamma_0)) = q \cdot \frac{\varepsilon(\lambda, \gamma_0)}{\varepsilon_0} \quad (5.1)$$

The right hand side of (5.1) is only known up to multiplicative constants q_i . They will be absorbed into the normalization factors Q_i which are (in principle) *constant* (and not, as in the old version, increasing like $s^{\frac{1}{4}}$).

- iv The empirical cut-off procedure near the kinematic limit as described by Ranft¹⁶⁾ is replaced by the phase space cut-off method as described by Hagedorn and Ranft¹⁸⁾. This new method does not use any arbitrary parameters.
- v) We simplify the parametrization of the velocity weight functions $F(\lambda)$ and $F_0(\lambda)$,

$$F(\lambda) = N \exp(-a\lambda) \quad (5.2)$$

$$F_0(\lambda) = N_0 \exp(b\lambda) \quad (5.3)$$

where the number of free parameters in $F_0(\lambda)$ has been reduced from three to one. N and N_0 are normalization factors determined from the conditions^{*})

$$\int_0^1 F_i(\lambda) d\lambda = 1 \quad ; \quad F_i(\lambda) = F_i(-\lambda). \quad (5.4)$$

In Table 2 the contributions to the spectra of π^- , K^- , $\bar{p}$, π^+ and p are given, which were considered in the present fit.

The parameters a and b are the only two essential parameters. Besides a and b we introduced in the fit two nearly known parameters θ and v defined as follows

$$T_0 = \theta \cdot T_0' ; \quad T_0' = 147.5 \text{ MeV} \quad (5.5)$$

$$V = v \cdot 2V_0 . \quad (5.6)$$

T_0 is to be used wherever it appears explicitly [Eqs. (3.8), (3.9), (3.14), but not in $\rho(m)$]; V is the effective interaction volume of the two colliding protons, V_0 is the "nucleon volume" $4\pi/(3m_\pi^3)$.

Remarks to Eqs. (5.5) and (5.6):

- Eq. (5.5) with the free parameter θ helps to eliminate partly the uncertainties in determining T_0 from the known part of the mass spectrum. If θ comes out a little different from one, then $T_0 = \theta T_0'$ is a better value for the limiting temperature. In principle this new T_0 should be used in the fit to the mass spectrum [Eq. (3.1)] as a *given* constant and the other parameters be re-determined. The new fit to the mass spectrum should then be used to recalculate the relevant formulae of our model and to make a new fit to the data. A new θ and a new T_0 is thus obtained and the whole procedure can start again - and so on. It is easy to see that the iteration will converge; the final T_0 is then compatible with both, the hadronic mass spectrum and inclusive single particle momentum distributions. As we found $\theta = 1.05$, we did not care to carry through another iteration step. The first approximation thus yields

$$T_0 = 155 \text{ MeV} .$$

Note, however, that this value will again slightly change if the model becomes more refined (e.g. by taking into account cooling along the decay chain or/and explicit decays like $\rho \rightarrow 2\pi$).

- In Eq. (5.6) we deviate from the notation of paper (II) by introducing an extra factor 2.

^{*}) In this form valid only for pp collisions.

Furthermore we introduce in the fits normalization parameters. They are necessary because of the errors in absolute normalization of different experiments, because of the unknown multiplicative constants which appear in the decay chain multiplicity functions (5.1) and because of the uncertainties in absolute normalization of the expressions used to calculate the spectra. The absolute normalization of the thermodynamic spectra can be determined using energy sum rules for inclusive spectra as was done in paper II. This procedure involves however the calculation of integrals over the spectra of *all* kinds of secondaries and has not been repeated with the new formulae. All normalization parameters are defined in Table 2 using the notation of Ref. 17. The parameters $A(m^*)$ are determined from the fits of π^+ and p spectra only. C_π^- is determined from the π^- spectra only. We use in the fits data on π^- , K^- , $\bar{p}$, π^+ and p production at primary proton momenta of 19.2 GeV/c¹⁹⁾, 30 GeV/c²⁰⁾ and 70 GeV/c²¹⁾.

At the primary momentum of 19.2 GeV/c ($E_{cm} = 6.15$ GeV) the production of $\bar{p}$ (and $\bar{K}$) is very near to threshold. Therefore the thermodynamic momentum spectra are not expected to provide a good description of the actual spectra. If, for instance, there are two fireballs formed of each about 3 GeV total energy, which decay into one through-going nucleon and a nucleon-antinucleon pair, we have nearly no energy available for the kinetic energy of the resulting particles.

In this situation one has to recall an otherwise neglected effect: if several particles are emitted from one single fireball, each emission carries away energy and thereby reduces the energy density and hence the temperature of the emitting fireball. Because of the locality of conservation laws, both particles are emitted from the same fireball in pair and associated production and the cooling effect becomes important near threshold. We incorporate the cooling effect in the crude manner explained by Hagedorn and Ranft¹⁸⁾: let m_i be the particle observed and $\sum_j m_j$ the total mass to be created together with m_i , then the reduced energy density is

$$\varepsilon'(\lambda) = \varepsilon(\lambda) - \sum m_j / V_0, \quad (5.7)$$

where $\varepsilon(\lambda)$ was the uncorrected one. From ε' we calculate the temperature $T' \equiv T(\varepsilon')$ and use it in the spectrum of m_i [but *not* in the factors $N_I(T)$ explained in Section 4.2 and defined by Eq. (3.14)].

If we correct in this way the momentum spectra, the total production rates would also change. They, however, are correctly given by the Boltzmann factor with the initial temperature T *before* applying the correction. Therefore we re-normalize the corrected spectra so that they retain their old normalization, by multiplying the corrected spectra by the factor

$$B \equiv \exp \left[-\frac{m_i}{T(\varepsilon)} + \frac{m_i}{T(\varepsilon')} \right]. \quad (5.8)$$

This procedure is inconsistent but it helps; in this way we can apply thermodynamics near threshold and extrapolate $\bar{p}$ and $\bar{K}$ spectra from 20 GeV to higher energies.

This is not the only threshold effect; we must allow the normalization factors Q_i to vary strongly near threshold; far away from threshold we expect them to become constant and all of the same order: $Q_p^- \approx Q_K^- \approx Q_\pi^-$. The values found in the fits at different energies (shown in Table 3) are consistent with this conjecture, which is furthermore supported by Fig. 6 showing the $\bar{p}/\pi^-$ ratio predicted at ISR energies when $Q_p^- = Q_\pi^- = 1.65$ is assumed^{*}). Of course, the values given in Table 3 reflect not only a systematic behaviour but also experimental normalization errors.

We do not reproduce here the spectra at 19, 30 and 70 GeV/c resulting from the new fits; they are as good as the old ones¹⁷⁾ (partly better, in particular $\bar{p}$).

6. PREDICTIONS AT ISR ENERGIES

Taking the parameters obtained in the low energy fits, we run the computer program for ISR energies and obtain the following predictions, which we compare to experimental results [Bertin et al.²²⁾ and Ratner et al.²³⁾]

- the rapidity distribution $d^3N/(dy^* d^2p_\perp)$ for π^+ (Fig. 5); for the 20 GeV/c curve the normalization parameter Q_π^- (19.2 GeV/c) has been used, at 300 and 1500 GeV/c we take Q_π^- (30 GeV/c) which probably is the safest. It is seen that at 20 GeV/c the spectra are still far from scaling; of course scaling behaviour sets in earlier in the fragmentation region [which was already studied elsewhere²⁴⁾] than in the central part, but as the rapidity distribution elongates $\sim \ln(\sqrt{s}/m)$, scaling in the fragmentation region cannot be seen in this figure, while it becomes obvious in the x representation (see Fig. 8):
- the $\bar{p}/\pi^-$ ratio (Fig. 6) as discussed above. Note that it is strongly $p_\perp$ dependent (between $p_\perp = 0$ and 1 GeV/c it changes by 2 orders of magnitude).
- the π^+/π^- ratio (Fig. 7). The difference between π^+ and π^- spectra are in this model due to the large (small) isobar contribution to π^+ (π^-) as explained in paper II. For $x \rightarrow 0$, i.e. in the central part, this has no effect, hence there $\pi^+/\pi^- \rightarrow 1$; for $x \rightarrow 1$ only π^+ is possible.

^{*}) If, as the cooling along the decay chain would suggest, we assume $Q_p^- < Q_\pi^-$ (by a factor ≈ 2) the agreement becomes perfect.

- the x -distribution of $E(d^3N/d^3p)$ for π^+ , π^- and p (Figs. 8, 9 and 10). The calculated spectra yield the production rate per interaction; in order to compare them with data, the cross-sections have to be divided by the total inelastic cross-section which we assumed to be energy independent $\sigma_{inel} = 31.5$ mb. The predictions agree well with the data, but are consistently a little too high. Whether this is due to a further slight decrease of the normalization factors Q_π^- from 30 to 1500 GeV/c -levelling off at $Q_\pi^- \sim 1$ for high energies would make the agreement perfect and be satisfactory theoretically - or to some other reason, we do not know. The amount of experimental data and their precision is not yet sufficient to motivate speculations about such details. Note that here scaling behaviour is clearly visible for larger x ; the central region has not yet reached the limit. The small central dip discussed in Section 4.3 is seen on the computed $\pi^\pm$ spectra for larger transverse momenta.

- the $p_\perp^2$ distribution of $E(d^3N/d^3p)$ for π^+ and p *)). We do not show the π^- distribution, because it is similar to π^+ . The main features of the data are well represented: a pronounced change of slope for pions and an almost constant slope for protons. For $p_\perp^2 \gtrsim 0.6$ (GeV/c)² the calculated π^+ curves stay above the data by about a factor of three (same for π^-). This deviation is, however, exaggerated by the over-all normalization. If, as discussed above, one admits that the normalization factor $Q_\pi^- \rightarrow 1$ for high energies, all π curves are lowered by about 40% and the over-all agreement looks better. Nevertheless, there remains a small systematic deviation from the data, which first occurred at 12.5 GeV/c¹⁶⁾ and persisted all the way up to ISR energies. We do not yet know to what it is due. Our guess is, that once two- and three-body decays (e.g. $\rho \rightarrow 2\pi$, $\omega \rightarrow 3\pi$) are taken explicitly into account, the value of T_0 may decrease and with it T and the cross-section at large $p_\perp^2$. That resonance decay has a tendency to simulate a too high temperature, has been shown recently by Hamer²⁵⁾ who claims that T_0 could be as low as 110 MeV. The mass spectrum suggests that it should lie in the range $130 \lesssim T_0 \lesssim 170$ MeV. The procedure to determine the new T_0 after incorporating new explicit processes in the model has been discussed in Section 5.

7. DISCUSSION

The present version of the model, based on the strong bootstrap solution^{4,5,6)}, gives at 19-30 GeV/c as good fits as the old one and, if extrapolated to ISR energies (just running the computer program at 1500 GeV/c) yields predictions which show scaling behaviour and agree reasonably with the data. The changes in the model have not been introduced *ad hoc* to please believers in scaling, they are the straightforward consequences of the strong bootstrap solution. On the other hand, the old version of the model is, at the time of writing this paper, still in use as a computer program and it has been extensively used by ISR physicists. Its predictions

*) Figures 11 and 12.

are not much different numerically from those of the present version and the agreement with the data is as good as here. What then are the differences?

i) the temperature energy relation and other similar formulae are slightly changed. But although the type of singularity of $\epsilon(T)$ for $T \rightarrow T_0$ has changed from $(T_0 - T)^{-1}$ to $(T_0 - T)^{-\frac{1}{2}}$, the numerical consequences for spectra are nearly negligible, because once the collision energy is large, the then also large energy densities lead always to temperatures T in the vicinity of T_0 [no matter what the detailed form of divergence of $\epsilon(T)$ is] and in the spectra only T enters. Thus, here the whole difference amounts to a few percent change in T , which is absorbed by a slight change of T_0 whose value is not too well known.

ii) the decay chain is very different with the main result that most decays are two-body decays and that the number of generations is proportional to the mass of the decaying fireball; consequently the multiplicity per decaying fireball is also proportional to its mass and the momenta are limited. The mass of the decaying fireball is proportional to ϵ/ϵ_0 ; incorporating the multiplicity per fireball amounts therefore to multiply the Planck-spectrum by ϵ/ϵ_0 . The typical particle spectrum in the old and new version is then (neglecting the normalization factors Q):

$$\begin{aligned} \text{old: } W(\vec{p}) &= \int F_{\text{old}}(\lambda) L(\lambda, \gamma_0) \{f_m(p', T)\} d\lambda \\ \text{new: } W(\vec{p}) &= \int F_{\text{new}}(\lambda) L(\lambda, \gamma_0) \left\{ \frac{\epsilon}{\epsilon_0} f_m(p', T) \right\} d\lambda . \end{aligned} \quad (7.1)$$

With the help of Eq. (4.17) it is seen that this amounts to a re-definition of $F(\lambda)$:

$$F_{\text{old}}(\lambda) = F_{\text{new}}(\lambda) \frac{\gamma_0}{1 + |\lambda|(\gamma_0 - 1)} \xrightarrow[\lambda \neq 0]{\gamma_0 \rightarrow \infty} \frac{F_{\text{new}}(\lambda)}{|\lambda|} . \quad (7.2)$$

Postulating energy independence of $F_{\text{new}}(\lambda)$, we see that all results would be strictly identical with the old version if there we had admitted an energy dependence such that $F_{\text{old}}(\lambda, \gamma_0)$ would, for $\gamma_0 \rightarrow \infty$, converge to an energy independent limiting function $F_{\text{new}}(\lambda)/\lambda$ except at $\lambda = 0$, where it diverges proportional to γ_0 . When we first determined $F_{\text{old}}(\lambda)$ from experiments (paper II) and even later when 70 GeV/c data became available, such a behaviour at $\lambda = 0$ could not be detected; the energy dependent factors $Q(s)$ helped of course to mask such an energy dependence at $\lambda = 0$, if it should be there. Even at 1500 GeV/c present experimental error limits do not allow to affirm energy independence of $F_{\text{new}}(\lambda)$.

In fact the numerical differences between the predictions of the old and new version are of the order of the experimental errors.

iii) the factors $Q(s)$ of the old version, which were introduced there for want of better and determined from an energy sum rule (paper II) to be proportional to $s^{\frac{1}{4}}$, are now constant. These factors led in the old version to a $s^{\frac{1}{4}}$ increase of multiplicity, while the present version leads to a logarithmic increase. However, by adapting constants, $As^{\frac{1}{4}}$ and $B \ln s + C$ can be made very similar over a large range of s and therefore again the numerical differences between the old and the new model are insignificant.

iv) the only radical difference is in the asymptotic behaviour, where the old version gave scaling and limiting fragmentation up to a normalization factor $\sim s^{\frac{1}{4}}$ while the new version gives exact scaling and limiting fragmentation; the old version gave a vanishing central plateau, the new one a non-vanishing and logarithmically elongating one with a small, energy independent central dip (in y^*).

We conclude with a few remarks:

- We now are a little less secure on the actual value of T_0 than we believed to be a few years ago; it should lie somewhere in $130 \lesssim T_0 \lesssim 170$ MeV with a probable value around 150 MeV. As the mass spectrum is - and never will be - sufficient to determine T_0 much better, the momentum spectra themselves have to be consulted; but here resonance decay simulates a higher temperature than there actually was. Hence the determination of T_0 from the transverse momentum distributions depends within $\approx \pm 10\%$ on what decay processes are made explicit in the model when one fits experimental data.
- The present version of the model may have to be revised again. In a recent preprint Carlitz²⁶⁾ points out some difficulties arising from the large energy fluctuations in the strong bootstrap situation. Taking into account a recent erratum to his preprint, his result, derived from a microcanonical analysis, agrees with that of Hagedorn²⁷⁾: the relative fluctuations in energy density are asymptotically

$$\frac{\Delta \varepsilon}{\varepsilon} \Rightarrow \frac{\text{const}}{\sqrt{V}} \left(\frac{1}{T_0 - T} \right)^{1/4} \quad (7.3)$$

which, together with Eq. (2.10) leads to

$$\frac{\Delta \varepsilon}{\varepsilon} \Rightarrow \text{const} \cdot \sqrt{\frac{\varepsilon}{V}} \quad (7.4)$$

Therefore, if V is constant (e.g. $= V_0$) then $\Delta\epsilon/\epsilon$ diverges proportional to $\sqrt{\epsilon}$. This means, that the canonical description should not be used above a certain energy density in a physical situation where not the temperature but the energy density is given. If one then goes over to a microcanonical description and calculates the momentum spectrum of a light particle emitted from a heavy fireball having an exponential level density, the result is only approximately a Planck spectrum²⁸⁾; if one makes a best fit to it using a Planck spectrum with an effective temperature, this temperature lies slightly *above* T_0 while in the canonical description it would lie below. We have already discussed similar effects in paper (II), Appendix IV. At that time the model gave a logarithmically increasing number of decay products already in the first generation and the upward shift of the effective temperature for two body decay could be neglected except in genuine two body decays, which were treated exactly. In the new version nearly all decays are two-body decays and therefore the canonical description should be replaced by a microcanonical one (phase space) -- at least in principle. These questions will be discussed in detail in a forthcoming paper by Frautschi and Hamer²⁹⁾. As the present paper shows, the canonical description, while unsuitable in principle, may still remain a very useful approximation even at ISR energies; it has the advantage of simplicity. The most attractive modification of the model might be to keep the present formalism, but to supplement it by an effective temperature function calculated from the microcanonical analysis. This function $T_{\text{eff}}[M(\lambda), p', k]$ would vary with the mass $M(\lambda)$ of the decaying fireball, with the momentum of the emitted particle and along the decay chain, being a function of the sequential number k at which the decay in question occurs (cooling effect).

- one might ask: what, then, is the predictive power and the value of a model, which, after the changes described in this paper, gives as good fits and predictions as before? We hope to have made clear that the changes, while of principal importance for the asymptotic behaviour at $s \rightarrow \infty$, are numerically unimportant even at ISR energies. The changes have made the model more consistent and they have allowed to eliminate seven parameters.
- our model predicts scaling and limiting fragmentation, logarithmic increase of multiplicity due to a logarithmic expanding central plateau (with a dip); thus it contains all the much beloved features en vogue. We stress again that all this is (taken the pure thermodynamic part for granted) a necessary consequence of the plausible but within the model so far unprovable hypothesis that $F(\lambda)$ and $F_0(\lambda)$ be energy independent and continuous at $\lambda = 0$ (paper II).

- in the "Atlas of Particle Spectra" ¹⁷⁾ we reported on a discrepancy by a factor of five in the absolute normalization of predicted and measured pion spectra at 70 GeV/c. We cautiously formulated and justified our suspicion that this probably meant an error in experimental normalization. As there is no such discrepancy from 12-30 GeV/c and again none at ISR energies (old as well as new version) we feel our suspicion strengthened. This implies that the spectra predicted in the Atlas may be used with more confidence than we expected at the time of its publication.

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Table 1

Temperature dependence of $\varepsilon(T)$ [Eq. (3.4)] of $\bar{m}_B(T)$ [Eq. (3.8)] and of $\varepsilon^{(1,B)}(T)$ [Eq. (3.5)]. p_0 is the primary momentum of a proton hitting a proton at rest, which would produce $\varepsilon^{(1,B)}(T)$ at the centre of the collision.

T/T_0	$T(\text{MeV})$	$\varepsilon^{(0)}(T) (\text{MeV}^4)$	$\bar{m}_B(T) (\text{MeV})$	$\varepsilon^{(1,B)}(T) (\text{MeV}^4)$	$p_0 (\text{GeV}/c)$
0.0	0.0	0.0	$m_P = 938.2$	3.44×10^8	< 1
0.2	29.5	1.27×10^5	938.2	3.56×10^8	< 1
0.3	44.2	1.56×10^6	938.8	3.656×10^8	< 1
0.4	59	7.61×10^6	943.0	3.812×10^8	< 1
0.5	73.7	2.579×10^7	958.5	4.130×10^8	< 1
0.6	88.5	7.39×10^7	996.5	4.829×10^8	< 1
0.62	91.4	9.03×10^7	1008.1	5.051×10^8	< 1
0.64	94.4	1.101×10^8	1021.4	5.314×10^8	< 1
0.66	97.3	1.342×10^8	1036.7	5.625×10^8	< 1
0.68	100.3	1.481×10^8	1045.1	5.803×10^8	1.0
0.70	103.2	1.991×10^8	1073.7	6.441×10^8	1.7
0.72	106.2	2.429×10^8	1096.0	6.976×10^8	2.2
0.74	109.1	2.970×10^8	1121.4	7.625×10^8	2.9
0.76	112.1	3.643×10^8	1150.3	8.418×10^8	3.8
0.78	115.0	4.488×10^8	1183.3	9.398×10^8	5.1
0.80	118.0	5.560×10^8	1221.3	1.062×10^9	6.7
0.82	120.9	6.940×10^8	1265.5	1.218×10^9	9.2
0.84	123.9	8.748×10^8	1317.5	1.419×10^9	13
0.86	126.8	1.116×10^9	1379.7	1.685×10^9	19
0.88	129.8	1.450×10^9	1456.0	2.048×10^9	28
0.90	132.7	1.927×10^9	1553.0	2.562×10^9	44
0.91	134.2	2.248×10^9	1613.0	2.905×10^9	57
0.92	135.7	2.649×10^9	1684.0	3.332×10^9	75
0.93	137.2	3.160×10^9	1768.0	3.875×10^9	100
0.94	138.6	3.831×10^9	1872.0	4.584×10^9	140
0.95	140.1	4.739×10^9	2004.0	5.541×10^9	210
0.96	141.6	6.019×10^9	2183.0	6.887×10^9	330
0.97	143.1	7.916×10^9	2444.0	8.879×10^9	540
0.98	144.5	1.089×10^{10}	2879.0	1.201×10^{10}	990
0.985	145.3	1.306×10^{10}	3244.0	1.431×10^{10}	1400
0.99	146.0	1.591×10^{10}	3897.0	1.738×10^{10}	2100

Table 1 (cont'd)

T/T_0	$T(\text{MeV})$	$\varepsilon^{(0)}(T) (\text{MeV}^4)$	$\bar{m}_B(T) (\text{MeV})$	$\varepsilon^{(1,B)}(T) (\text{MeV}^4)$	$p_0 (\text{GeV}/c)$
0.992	146.3	1.729×10^{10}	4250.0	1.891×10^{10}	2500
0.994	146.6	1.883×10^{10}	4826.0	2.066×10^{10}	2900
0.996	146.9	2.055×10^{10}	5793.0	2.273×10^{10}	3600
0.998	147.2	2.255×10^{10}	7974.0	2.551×10^{10}	4500
0.999	147.4	2.365×10^{10}	11058.0	2.773×10^{10}	5300
1		∞	∞	∞	∞

Remark: By comparing this table with the corresponding one of the old, weak bootstrap version [paper II, Appendix II] one sees that now T_0 is approached much faster See Eq. (2.10) for the explanation.

Table 2

Contributions considered in the present fit to
secondary particle spectra from p-p collisions
(Notation as in Ref. 17)

Particle	Processes considered
π^-	$Q_{\pi}^- \bigcirc \rightarrow \pi + C_{\pi}^- \sum_{m^*} A(m^*) \text{---} \bigcirc \xrightarrow{m^*} \begin{matrix} \nearrow \pi \\ \searrow N \end{matrix}$
K^-	$Q_K \text{---} \bigcirc \begin{matrix} \nearrow \bar{K} \\ \text{---} \searrow K \end{matrix}$
$\bar{p}$	$Q_{\bar{p}} \text{---} \bigcirc \begin{matrix} \nearrow \bar{p} \\ \text{---} \searrow N \end{matrix}$
π^+	$Q_{\pi}^- \bigcirc \rightarrow \pi + \sum_{m^*} A(m^*) \text{---} \bigcirc \xrightarrow{m^*} \begin{matrix} \nearrow \pi \\ \searrow N \end{matrix}$
$p^*)$	$\text{---} \bigcirc \rightarrow p + C_p \sum_{m^*} A(m^*) \text{---} \bigcirc \xrightarrow{m^*} \begin{matrix} \nearrow p \\ \searrow \pi \end{matrix}$

*) Note that the contribution $\bigcirc \begin{matrix} \nearrow p \\ \text{---} \searrow \bar{N} \end{matrix}$ has been neglected. If a calculation is done where this contribution is important (e.g. at 90° in the centre-of-momentum frame) it must be added; this is simple because it is exactly equal to the antiproton spectrum (in Figs. 10 and 11 this contribution is contained).

Table 3

Parameters determined in the present fit

Parameter	Value
a [Eq. (5.2)]	0.755
b [Eq. (5.3)]	4.58
θ [Eq. (5.5)]	1.05
v [Eq. (5.6)]	1.16
C_{π}^{-}	0.123
C_p	1.35
$A(1236)$	8.77×10^1
$A(1518)$	2.22×10^3
$A(2190)$	2.04×10^5
Q_{π}^{-}	2.60 (19.2 GeV/c), 1.65 (30 GeV/c), 1.87 (70 GeV/c)
Q_K	13.3 (19.2 GeV/c), 10.0 (70 GeV/c)
Q_p^{-}	39 (19.2 GeV/c), 19.5 (70 GeV/c)

Notes: 1) We expect $Q_p \lesssim Q_K \lesssim Q_{\pi}^{-} \approx 1$ at ISR energies, but we do not know how this regime will be approached.

2) All parameters from C_{π}^{-} downward are defined by the graphs in Table 2.

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a)

b)

Fig. 1 The decay of a heavy fireball (schematically);
 1a: in the weak bootstrap solution;
 1b: in the strong bootstrap solution

Fig. 2 Temperature T as function of the collective velocity in terms of λ . For increasing collision energy the curves tend rapidly to a limiting curve

Fig. 3 The integrand of Eq. (4.35) for $T = 150$, $\mu = 140$.
 For $y^* \ll 1$ the effect of $|\tanh \eta|$ is important,
 for $y^* > 4$ it becomes negligible

Fig. 4 The integral (Eq. 4.35) as a function of y^* for $T = 150$ and $\eta_0 = 10$ (corresponding to a collision energy $E_{\text{cm}} \approx 10^5 \text{ GeV}$). The two lower curves illustrate the effect of ± 1 in the denominator: the fictitious "fermions with pion mass" have the -1 replaced by $+1$

Fig. 5 Longitudinal rapidity distribution in c.m. for various collision energies and transverse momenta. Note the dip of width 2 and depth increasing with $p_{\perp}$

Fig. 6 The $\bar{p}/\pi^-$ ratio at various ISR energies [Bertin et al.²²⁾] compared to the prediction. Normalization parameters assumed to be equal: $Q_{\bar{p}} = Q_{\pi^-} = 1.65$. If one takes into account cooling along the decay chain, then $Q_{\bar{p}} < Q_{\pi^-}$ and the curve shifts down. The ratio is highly dependent on $p_{\perp}$

Fig. 7 The π^+/π^- ratio at various ISR energies [Bertin et al.²²⁾, Ratner et al.²³⁾] as function of x compared to the predictions. The theoretical curves for 500 to 1500 GeV/c coincide almost within drawing accuracy

Fig. 8 π^+ spectra as function of x with $p_{\perp}^2$ as parameter for 300 and 1500 GeV/c. For $x \gtrsim 0.2$, scaling is well approached, at low x it is not yet. Experimental data from Ratner et al.²³⁾

Fig. 9 π^- spectra as function of x with various $p_{\perp}^2$. The data grouped together belong to the same $p_{\perp}^2$ as the nearest curves. Data from Bertin et al.²²⁾

Fig. 10 Proton spectra as a function of x for various $p_{\perp}^2$ compared to the data of Ratner et al.²³⁾. Starting at about $x \approx 0.2$, the number of protons increases for falling x ; this is due to $p\bar{N}$ pair creation, which is probably overestimated here by a factor two (this contribution has not been included in the calculation of the 300 GeV curve).

Fig. 11 Proton spectra as a function of p_T^2 for different x compared to the data of Ratner et al.^{2,3}). Note that the slope is almost constant in contradistinction to the corresponding π^+ curves (Fig. 12)

Fig. 12 π^+ spectra as a function of $p_{\perp}^2$ compared to the data of Ratner et al.²³). Note the considerable change of shape between $p_{\perp}^2 \approx 0.3$ and 0.6 which is absent in the p spectra (Fig. 11). The disagreement between data and prediction would look much less dramatic if all curves were shifted down by 40%. Such a readjustment of normalization (Q_{π}^-) would make all curves in all figures agree better with the data