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CURRENTS AND GREEN'S FUNCTIONS FOR DUAL MODELS - II

OFF-SHELL DUAL AMPLITUDES

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A B S T R A C T

Off-shell dual amplitudes are constructed in a way consistent with factorization in the space of harmonic oscillators of the conventional model. They are shown to have only correctly located singularities. The use of domain variational theory on Riemann surfaces is of crucial importance to prove this result. The case of currents with quantum numbers is also discussed.

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1. INTRODUCTION

In a preceding paper ¹⁾ (hereafter referred to as I), Neveu and one of us (J.S.) proposed explicit rules to compute current amplitudes in the dual resonance model. The properties of the one-current amplitude were studied extensively, and proposals for scalar isoscalar currents, scalar currents with quantum numbers, vector currents were made. The possibility of constructing through this approach multi-current amplitudes was mentioned, but their construction was deferred to a subsequent paper. We show here explicitly how the construction can be carried, concentrating ourselves mostly on the case of scalar isoscalar current, and exhibit the resulting amplitudes. They are found to be free of the unphysical singularities which have plagued former models of currents ^{2),3)}. Other models are free of those unphysical singularities but have bad factorization properties, even on the leading trajectory ^{4),5)}.

The central idea of this approach is the picture that on-shell amplitudes correspond to points on the boundary of the Koba-Nielsen circle while off-shell amplitudes correspond to emit the momenta inside the disc. However, this picture is fully realized only after a change of variables, called "Jacobi transformation" which has a simple interpretation in terms of harmonic differentials on a surface. Therefore, although the end-result is an integrand containing factors of the type $|(z_i - z_j)(z_i - z_j^*)|^{-p_i p_j / 2}$ where the z_i are complex variables, because of the change of variable, additional "self-energy" terms $a_i^{-p_i^2 / 8}$ appear in the integrand, and those terms are responsible for transforming "badly" located singularities in the variable $p_i \cdot p_j$ into singularities located in $(p_i + p_j)^2$. Furthermore, these singularities receive a natural explanation in this model, as being associated with the Pomeron appearing in non-planar loop diagrams ⁶⁾.

All the techniques we use here have already been developed in studying single or multiloop amplitudes ⁷⁾⁻¹⁰⁾ in the dual model, or are refinements of those. For instance, the computation of multicurrents in their first form parallels exactly computing multiloops from the operator formalism ¹¹⁾, and the Jacobi transformation which helps putting the currents in a simple form was first met in the study of one-loop graphs ¹²⁾. Thanks to those techniques, it has been possible to obtain currents in a simple form while starting from operators which are exponentials of bilinear forms in the oscillators. A similar problem where exponentials of bilinear forms are met to define Born terms is in the multifermion amplitude ¹³⁾, but it has not yet been solved.

The multicurrent amplitudes obtained satisfy the constraints of factorization in the space of the conventional dual model, duality, absence of unphysical singularity. They have poles in the external masses. For fixed external masses, Regge behaviour is obtained, dominated both by Regge poles and Pomeron singularities, in a two-current isoscalar channel. For amplitude with quantum numbers, the Pomeron is decoupled, and we obtain a natural procedure to introduce an order in the complex plane so as to be able to attach Chan-Paton factors to the currents ¹⁴⁾.

The problem of eliminating spurious states for $\alpha(0) = 1$ ¹⁵⁾ from multiloop diagrams has still remained unsolved, and, as a consequence it still remains to be solved for multicurrent amplitudes. If one does not attempt to eliminate them, there is still some flexibility in the model, reflected in the "self energy" terms which allow to incorporate scaling behaviour in the Bjorken limit. Whether this is still possible when projection over physical states is required seems to put a constraint on the number of oscillators present in the dual model, in the same way as requiring the Pomeron to be a pole leads to Lovelace's conjecture ($D=26$) ⁵⁾. The two conditions may coincide eventually. Since those considerations are very speculative, one must be content to have achieved a reasonable model of currents for fixed masses, free of unphysical singularities.

In Section 2 we compute a two-current operator starting from the one-current operator of I. This Section uses the techniques of computing multiloop operators from one-loop operators ¹¹⁾. In Section 3, we obtain a first expression of a two-current amplitude by applying the two-current operator on a tree, and generalize to multicurrent amplitudes. Section 4 contains the core of this paper and devotes itself to the study of the Jacobi transformation. This transformation was first used in the dual theory to investigate the properties of one-loop graphs and renormalize them ^{6),12)}. Here it is generalized to a surface with N handles in the limit where they all vanish, which is the case studied here. As a result, the complicated expression obtained in Section 3 becomes expressible in terms of third Abelian integrals on the sphere, that is in terms of ordinary cross-ratios. There is still, however, a part remaining from the above mathematical manipulations, in the "self-energy" term, which have a complicated expression. In Section 5, we express the multicurrent amplitudes in that new form, study the integration region and the singularities of the amplitude. One notes that the "self-energy" terms convert unphysical into physical singularities. Section 6 essentially studies these selfenergy terms to prove that assertion, which is one of the main achievements of this paper.

In Section 7, we make some comments about the elimination of spurious states. We quote in Section 8 some results about the asymptotic behaviour of isoscalar currents, the proof of which is deferred to a forthcoming publication. In Section 9, we look at the current amplitudes with quantum numbers and explain how they are free from Pomeron singularity, and how Chan-Paton factors can be attached in the complex plane. The reader who does not feel inclined to study the mathematics of Sections 4 and 6 can skip them easily.

2. TWO-CURRENT OPERATOR

a. One-current operator

We start from the one-current operator as defined in I

$$\langle 0 | t(p) = \pi^{-\alpha(p^2)} \langle 0 | \exp \left[\frac{1}{2} (a | \tilde{\Delta} | a) + p (1 | \tilde{F}_T | a) \right] \quad (2.1)$$

where

$$\tilde{\Delta} = \sum_{n \neq 0} \left[\frac{1}{n} \right] M_+ \left[-\frac{1}{n} \right] \quad (2.2)$$

$$\tilde{F}_T = - \sum_{n \neq 0} \left[\frac{1}{n} \right] \quad (2.3)$$

By noting that

$$\left[\frac{1}{n} \right] M_+ \left[-\frac{1}{n} \right] = \Gamma[n] M_- \left[-\frac{1}{n} \right] \equiv \Gamma P_n \quad (2.4)$$

using the 2×2 representation it is seen that P_n is associated with the homographic transformation

$$\hat{P}_n(z) = z + n = \hat{P}^n(z) \quad (2.5)$$

where

$$\hat{P}(z) = z + 1 \quad (2.6)$$

is a parabolic transformation. Then $P_n = P^n$ with $P = M_- \begin{bmatrix} & -1 \\ -1 & \end{bmatrix}$. Then $t(p)$ takes the following form which exhibits its projective properties

$$\langle 0 | t(p) = \pi^{-\alpha(p^2)} \langle 0 | \exp \left\{ \frac{1}{2} \langle a | \Gamma \sum_{n \neq 0} P^n | a \rangle + p \langle a | \Gamma \sum_{n \neq 0} P^n | \infty \rangle \right\} \quad (2.7)$$

(note that ∞ is the invariant point of P). For notations and operator formalism we refer the reader to Refs. 16) and 17).

b. Two-current operator

We built it by attaching two one-current operators on a symmetric three-Reggeon vertex via a twisted projected propagator (Fig. 1)

$$\langle 0_a | t_2(p_1, p_2) = \langle 0_{a,b} | t(p_2) \mathcal{D}(-p_2) V(a, b, b^+) \mathcal{D}(p_1) t^+(p_1) | 0_b \rangle \quad (2.8)$$

Using the same techniques as to compute a two-loop tadpole from a one-loop tadpole as explained in Ref. 11), we obtain easily

$$\langle 0_a | t_2(p_1, p_2) = \pi^{-\alpha(p_1^2) - \alpha(p_2^2)} \int_0^1 dx_1 \int_0^1 dx_2 x_1^{-\alpha(p_1^2)-2} (1-x_1)^{-\alpha(p_1^2)-3} x_2^{-\alpha(p_2^2)-2} (1-x_2)^{-\alpha(p_2^2)-3}$$

$$(\det [I - (P_1 - I)(P_2 - I)])^{-2} \left(\frac{\bar{y}_2 - 1}{\bar{y}_2} \right)^{-P_2^2} \left(\frac{\bar{y}_1 - \bar{y}_2}{\bar{y}_1 \bar{y}_2} \right)^{-P_1 \cdot P_2}$$

$$\exp \left\{ \frac{1}{2} P_1^2 \langle \bar{y}_1 | \Gamma (P_{12} - P_1) | \bar{y}_1 \rangle + \frac{1}{2} P_2^2 \langle \bar{y}_2 | \Gamma (P_{12} - P_2) | \bar{y}_2 \rangle + P_1 \cdot P_2 \langle \bar{y}_2 | \Gamma (P_{12} - I) | \bar{y}_1 \rangle \right\}$$

$$\langle 0_a | \exp \left\{ \frac{1}{2} \langle a | \Gamma (P_{12} - I) | a \rangle + P_1 \langle a | \Gamma [(P_{12} - I) | \bar{y}_1 \rangle + | \bar{y}_1 \rangle - | \infty \rangle] + P_2 \langle a | \Gamma [(P_{12} - I) | \bar{y}_2 \rangle + | \bar{y}_2 \rangle - | \infty \rangle] \right\}$$

(2.9)

where

$$\mathcal{P}_1 = \sum_{n=-\infty}^{+\infty} P_1^n \quad ; \quad \mathcal{P}_2 = \sum_{n=-\infty}^{+\infty} P_2^n \quad (2.10)$$

P_1 and P_2 are associated with the parabolic homographic transformation

$$\frac{1}{\hat{P}_1(z) - \bar{y}_1} = \frac{1}{z - \bar{y}_1} + \frac{1}{\bar{C}_1} \quad (2.11)$$

with

$$\left\{ \begin{array}{l} \bar{y}_1 = \frac{1}{1-x_1} \quad ; \quad \bar{C}_1 = \frac{x_1}{x_1-1} \\ \bar{y}_2 = \frac{x_2-1}{x_2} \quad ; \quad \bar{C}_2 = \frac{x_2-1}{x_2} \end{array} \right. \quad (2.12)$$

$\mathcal{P}_{12}$ is the sum of all the elements of the group generated by P_1 and P_2 . In order to prove the result we have made use of the following formula which will be proved in Appendix A.

$$\langle \gamma | \Gamma \sum_{n \neq 0} P^n | \gamma \rangle = \frac{1}{2} \langle \gamma | \Gamma \sum_{n \neq 0} P^n | \gamma \rangle \quad (2.13)$$

where γ is the invariant point of $\hat{P}(z)$.

Let us note the apparent lack of symmetry in the integration measure. This is due to the fact that the operator is symmetric only up to gauge transformation (L_1). We shall see that the corresponding two-current amplitude is in fact symmetric.

As usual we have

$$\det[I - (\mathcal{P}_1 - I)(\mathcal{P}_2 - I)]^{-2} = \prod_{\alpha} f(w_{\alpha})^{-4} \quad (2.14)$$

where $f(w_{\alpha})$ is the usual partition function, w_{α} is the multiplier of P_{α} and the product is extended to all w_{α} which are different ¹⁰⁾ [we take into account only those which are not power of one another or are not related by a cyclic permutation since $w(P_{\alpha} P_{\beta} \dots P_{\gamma} P_{\gamma}) = w(P_{\gamma} P_{\alpha} P_{\beta} \dots P_{\gamma}) = \dots$].

3. MULTICURRENT AMPLITUDES: FIRST EXPRESSION

Let us first discuss in details the case of the two-current amplitudes. We built it by attaching the previous operator to a $N+1$ Reggeon vertex (with N on mass shell) via a usual non-twisted propagator (this is allowed since gauges are killed on the tree).

The $N+1$ Reggeon vertex ¹⁸⁾ is written as given by Yu ¹⁹⁾, for instance

$$\int \frac{\prod_{i=0}^N dy_i}{dV_{abc}} \prod_{i \neq j} |y_i - y_j|^{-\frac{1}{2} k_i \cdot k_j} \left[\frac{(y_N - y_0)(y_{N-1} - y_N)}{y_{N-1} - y_0} \right]^{-1 - \frac{1}{2} k_N^2}$$

$$: e^{(\alpha^+ (R_N - I)) \alpha} : \exp \left\{ \sum_{i=0}^{N-1} k_i [\langle y_i | - \langle y_0 |] \alpha^+ \right\} |0_a\rangle$$

(3.1)

where the y_i are ordered in the real axis (or the Koba-Nielsen circle)

$$\hat{R}_N(z) = \frac{y_N - y_{N-1}}{y_0 - y_{N-1}} \frac{y_0 - z}{y_N - z}$$

$$k_i^2 = -2 \quad \text{for } i = 0, \dots, N-1 \quad (3.2)$$

$$\text{and } k_N + \sum_{i=0}^{N-1} k_i = 0$$

Then the two-current amplitude is easily obtained. The group acting on the y_i differs from the previous one only by a global homographic transformation, more precisely the relation between them is

$$P = C^{-1} \bar{P} C \quad (3.3)$$

with

$$C = [x] R_N \quad (3.4)$$

where x is the Chan variable of the untwisted propagator between the tree and the operator.

The parameters of the new group are defined by

$$\begin{aligned} \gamma_{(2)}^{(1)} &= \widehat{C^{-1}}(\bar{\gamma}_{(2)}^{(1)}) \\ C_{(2)}^{(1)} &= \bar{C}_{(2)}^{(1)} \frac{\partial}{\partial z} \widehat{C^{-1}}(z) \Big|_{z=\bar{\gamma}_{(2)}^{(1)}} \end{aligned} \quad (3.5)$$

Noting that

$$\begin{cases} \gamma_N = \widehat{C^{-1}}(\infty) \\ \gamma_0 = \widehat{C^{-1}}(0) \end{cases} \quad (3.6)$$

and defining

$$\gamma = \widehat{C^{-1}}(1) \quad (3.7)$$

we obtain the relations

$$\begin{aligned} \gamma_N &= \widehat{P}_2(\gamma_0) \\ \gamma &= \widehat{P}_1(\gamma_N) \end{aligned} \quad (3.8)$$

They allow us to do the change of variables

$$(x_1, x_2, x, \gamma_N) \rightsquigarrow (\gamma_1, \gamma_2, c_1, c_2)$$

and the Jacobian is

$$\frac{dx_1}{x_1^2} \frac{dx_2}{x_2^2} \frac{dx}{x^2} \frac{d\gamma_N(\gamma_{N-1}-\gamma_0)}{(\gamma_N-\gamma_0)(\gamma_{N-1}-\gamma_N)} = \frac{d\gamma_1 dc_1}{c_1^2} \frac{d\gamma_2 dc_2}{c_2^2} \quad (3.9)$$

The points are ordered in the following way (up to a cyclic permutation)

$$\dots \gamma_2 < \gamma_0 < \dots \gamma_i \dots < \gamma_{N-1} < \gamma < \gamma_1 < \gamma_N < \gamma_2 \dots \quad (3.10)$$

It is seen that the integration region is the usual one of the two-loop amplitude. The γ_i 's ($i=0, \dots, N-1$) vary in the fundamental region of P_1P_2 . If we fix γ_1, c_1, γ_2 and integrate at c_2 fixed, we can see that because the ordering $\gamma_0 \leq \gamma_i \leq P_1P_2(\gamma_0)$ the γ_i 's ($i \neq 0$) are integrated in a fundamental region of P_1P_2 . However, from the construction and the

integration region of the Chan variables, we can see that y_0 is in fact integrated over all the region $[S_{12}, \gamma_{12}]$ where S_{12} and γ_{12} are the invariant points of $P_1 P_2$, and, therefore, in order to avoid an infinite multiple counting, we must, as for the two-loop amplitude ^{9),11)}, restrict ourselves to only one fundamental region of $P_1 \cdot P_2$.

If we fix γ_1, C_1 (or γ) and y_N then γ_2 varies in the fundamental region of P_1 $[\gamma_2 \geq y_N$ or $\gamma_2 \leq \gamma = P_1(y_N)]$.

Also as in the two-loop amplitude we must take care of the partition function which can provide some cut-off. The only partition function which can diverge is the one associated with $P_3 = (P_1 P_2)^{-1}$ ^{9),20)}. Its multiplier w_3 satisfies

$$\sqrt{w_3} + \frac{1}{\sqrt{w_3}} = \left| 2 - \frac{(\gamma_1 - \gamma_2)^2}{C_1 C_2} \right| \quad (3.11)$$

then we must have

$$\frac{(\gamma_1 - \gamma_2)^2}{C_1 C_2} \geq 4 \quad (3.12)$$

(Note that $(\gamma_1 - \gamma_2)^2 / C_1 C_2$ is a projective invariant)

As already discussed for the two-loop amplitude, this does not provide any restriction on the integration domain in the old variables x_1, x_2 ²⁰⁾ or in the new one. Fixing γ_1, C_1 and y_N (therefore γ is also fixed) we can show that the inequality is always satisfied and that the equality is obtained when

$$\gamma_0 = \dots = \gamma_{N-1} = \hat{P}_1(y_N)$$

and

$$\gamma_2 = \hat{P}_1^{\gamma_2}(y_N)$$

Using the new variables the two-current amplitude is written

as

$$\prod_{i=0}^{N-1} \frac{d\gamma_i}{dV_{abc}} \frac{d\gamma_1 dC_1}{C_1^2} \frac{d\gamma_2 dC_2}{C_2^2} \prod_{\alpha \neq 1,2} [f(w_\alpha)]^{-4} \prod_{i \neq j} |\gamma_i - \gamma_j|^{-\frac{1}{2} k_i \cdot k_j} \prod_{i=0}^{N-1} |\gamma_i - \gamma_1|^{-P_1 \cdot k_i} |\gamma_i - \gamma_2|^{-P_2 \cdot k_i}$$

$$\begin{aligned}
 & \prod_{i,j} \prod_{\alpha \neq I} \left[\frac{y_i - P_\alpha(y_j)}{y_i - P_\alpha(y_0)} \frac{y_0 - P_\alpha(y_0)}{y_0 - P_\alpha(y_j)} \right]^{-\frac{1}{2} k_i \cdot k_j} \\
 & \prod_{i=0}^{N-1} \prod_{\alpha \neq I} \left\{ \left[\frac{y_i - P_\alpha(y_1)}{y_i - P_\alpha(y_0)} \frac{y_0 - P_\alpha(y_0)}{y_0 - P_\alpha(y_1)} \right]^{-P_1 \cdot k_i} \left[\frac{y_i - P_\alpha(y_2)}{y_i - P_\alpha(y_0)} \frac{y_0 - P_\alpha(y_0)}{y_0 - P_\alpha(y_2)} \right]^{-P_2 \cdot k_i} \right\} \\
 & \prod_{\alpha \notin \mathcal{P}_1} \left[\frac{y_1 - P_\alpha(y_1)}{y_1 - P_\alpha(y_0)} \frac{y_0 - P_\alpha(y_0)}{y_0 - P_\alpha(y_1)} \right]^{-\frac{P_1^2}{2}} \prod_{\alpha \notin \mathcal{P}_2} \left[\frac{y_2 - P_\alpha(y_2)}{y_2 - P_\alpha(y_0)} \frac{y_0 - P_\alpha(y_0)}{y_0 - P_\alpha(y_2)} \right]^{-\frac{P_2^2}{2}} \\
 & \prod_{\alpha \neq I} \left[\frac{y_2 - P_\alpha(y_1)}{y_2 - P_\alpha(y_0)} \frac{y_0 - P_\alpha(y_0)}{y_0 - P_\alpha(y_1)} \right]^{-P_1 \cdot P_2} \quad (3.14)
 \end{aligned}$$

- The apparent dependence on y_0 is in fact spurious and disappears if we use the energy momentum conservation.
- This form is very suggestive of its projective properties and of its duality properties.

The generalization to a M current amplitude is obvious from the previous formula. We can prove it by recurrence. Let us write it for completeness

$$\begin{aligned}
 & \prod_{k=1}^M \alpha(P_k^2) \int \frac{\prod_{i=0}^{N-1} dy_i}{dV_{abc}} \prod_{k=1}^M \frac{dy_k dC_k}{C_k^2} \prod_{\alpha \neq 1, 2, \dots, M} [f(w_\alpha)]^{-4} \\
 & \prod_{i \neq j} |y_i - y_j|^{-\frac{1}{2} k_i \cdot k_j} \prod_{i,k} |y_i - y_k|^{-P_k \cdot k_i} \\
 & \prod_{i,j} \prod_{\alpha \neq I} \left[\frac{y_i - P_\alpha(y_j)}{y_i - P_\alpha(y_0)} \frac{y_0 - P_\alpha(y_0)}{y_0 - P_\alpha(y_j)} \right]^{-\frac{1}{2} k_i \cdot k_j} \prod_{i,k} \prod_{\alpha \neq I} \left[\frac{y_i - P_\alpha(y_k)}{y_i - P_\alpha(y_0)} \frac{y_0 - P_\alpha(y_0)}{y_0 - P_\alpha(y_k)} \right]^{-P_k \cdot k_i} \\
 & \prod_k \prod_{\alpha \notin \mathcal{P}_k} \left[\frac{y_k - P_\alpha(y_k)}{y_k - P_\alpha(y_0)} \frac{y_0 - P_\alpha(y_0)}{y_0 - P_\alpha(y_k)} \right]^{-\frac{P_k^2}{2}} \prod_{k \neq \ell} \prod_{\alpha \neq I} \left[\frac{y_k - P_\alpha(y_\ell)}{y_k - P_\alpha(y_0)} \frac{y_0 - P_\alpha(y_0)}{y_0 - P_\alpha(y_\ell)} \right]^{-\frac{P_k \cdot P_\ell}{2}} \quad (3.15)
 \end{aligned}$$

Let us make some remarks about this formula.

- The integration is the same as before:
 - all w_a are ≤ 1 ²¹⁾;
 - the y_i 's are ordered in the real axis;
 - because of duality properties, the γ_k 's are not ordered with respect to each other.
- In the configuration in which all the "off-shell particles" are adjacent, the γ 's are ordered with respect to the y_i 's:

$$\dots \gamma_i \dots < \gamma_0 < \dots < \gamma_{N-1} < \dots \gamma_i \dots$$

The other configurations are obtained by doing some projective transformations of the group on some y_i 's, in this way we obtain amplitudes with non-adjacent currents. Let us remark that these duality properties of the amplitude do not worry us since in the case $\alpha_0 = 1$, the complete amplitude is simply obtained by integrating all the variables without taking care of their order.

- If we want to study other properties of this amplitude such as singularities, asymptotic behaviour it seems quite uneasy (the problem is the same as for multiloop amplitude), moreover, the comparison with other previously proposed amplitudes does not appear very clearly. But we shall see that by using Jacobi transformation and identity on the integrand we arrive to a very compact form of the amplitude which will be discussed in Section 5. Before that we give some definitions and results on generalized Jacobi transformations and identities. The hurried reader can therefore skip Section 4 and go directly to Section 5.

4. GENERALIZED JACOBI'S TRANSFORMATION IN THE PARABOLIC LIMIT

1) General definition ^{*)}

Let us consider a closed and orientable Riemann surface which has the same topological properties as a sphere with M handles. We denote by C_μ and D_μ ($\mu = 1, \dots, M$) a canonical basis of cycles such that

$$\begin{aligned} I(C_\mu, D_\nu) &= \delta_{\mu\nu} \\ I(C_\mu, C_\nu) &= I(D_\mu, D_\nu) = 0 \end{aligned} \tag{4.1}$$

^{*)} For more details on the definition of various objects used here we refer the reader to ³⁾ and mathematical books on Riemann surface (22), (23).

where I designs the Kronecker index. Let us call $\phi_\mu(z)$ the first Abelian integrals defined on $\mathcal{R}$. There are only M independent first Abelian differentials $\phi_\mu(z)$ uniquely defined by their period along C_μ or D_μ (or the real part of the period along C_μ and D_μ).

Let us define $\phi_\mu(z)$ by

$$\oint_{C_\nu} d\phi_\mu(z) = 2i\pi \delta_{\mu\nu} \quad (4.2)$$

Then the period matrix is defined by

$$\oint_{D_\nu} d\phi_\mu(z) = A_{\mu\nu} \quad (= \oint_{D_\mu} d\phi_\nu(z)) \quad (4.3)$$

z is a complex uniformizer of the surface. In the same way we can define $\phi'_\mu(z')$ by

$$\oint_{-D_\nu} d\phi'_\mu(z') = 2i\pi \delta_{\mu\nu} \quad (4.4)$$

which define the matrix A' by

$$\oint_{C_\nu} d\phi'_\mu(z') = A'_{\mu\nu} \quad (I(-D_\nu, C_\mu) = \delta_{\mu\nu}) \quad (4.5)$$

But we can also build directly such functions from $\phi_\mu(z)$ since the functions

$$\bar{\Phi}_\mu(z) = - \sum_{\rho=1}^M 2i\pi (A^{-1})_{\mu\rho} \phi_\rho(z) \quad (4.6)$$

satisfy

$$\oint_{-D_\nu} d\bar{\Phi}_\mu(z) = 2i\pi \delta_{\mu\nu} \quad (4.7)$$

and

$$\oint_{C_\nu} d\Phi_\mu(z) = 4\pi^2 (A^{-1})_{\mu\nu}$$

Then, from the unicity theorem, there exists some change of uniformizer

$$z' = f(z)$$

such that

$$\Phi'_\mu(z') - \Phi'_\mu(z'_0) = -2i\pi \sum_{\rho=1}^M (A^{-1})_{\mu\rho} (\Phi_\rho(z) - \Phi_\rho(z_0)) \quad \forall \mu=1, \dots, M$$

(4.8)

and, moreover, we have the relation

$$A A' = 4\pi^2$$

(4.9)

Meromorphic single valued functions on $\mathcal{R}$ are automorphic functions on the complex plane under the double discontinuous group $G \sim \{T, R\}$ where the transformations of T (resp. R) correspond to go along some combination of the cycles D_μ (resp. C_μ). If we cut $\mathcal{R}$ along the cycles C_μ (resp. D_μ) T (resp. R) can be represented by a group of projective transformations.

This provides two parametrizations of the surface and two families of first Abelian integrals which are related following the previous discussion by a change of variable called the Jacobi transformation (uniformizer and parameters).

If we consider the third Abelian differentials in real normalization defined by their periods

$$\begin{cases} \operatorname{Re} \oint_{C_\mu} d\Omega_{ab}(z) = 0 \\ \operatorname{Re} \oint_{D_\mu} d\Omega_{ab}(z) = 0 \end{cases}$$

(4.10)

Since Ω_{ab} is defined in a symmetrical way with respect to C_μ and D_μ , it must be invariant under the Jacobi transformation (a and b are also transformed).

Let us now precise these statements in the case of a sphere with one handle.

2) Torus - Parabolic limit

a) General case

The group of automorphisms has only one generator. If we cut the torus along C_1 , the group is generated by T

$$\frac{T(z) - \xi}{T(z) - \eta} = w \frac{z - \xi}{z - \eta} \quad (4.11)$$

Going from z to $T(z)$ corresponds to going around D_1 . Going around C_1 corresponds to the transformation

$$\frac{z - \xi}{z - \eta} \rightarrow e^{2i\pi} \frac{z - \xi}{z - \eta} \quad (4.12)$$

The first Abelian integral is

$$\phi(z) - \phi(z_0) = \text{Log} \left(\frac{z - \xi}{z - \eta} \frac{z_0 - \eta}{z_0 - \xi} \right) \quad (4.13)$$

with periods

$$\begin{cases} P_{C_1}(\phi) = 2i\pi \\ P_{D_1}(\phi) = \text{Log} w \end{cases} \quad (4.14)$$

In the same way we can describe the surface by cutting along D_1 corresponding to a group generated by T'

$$\frac{T'(z') - \xi'}{T'(z') - \eta'} = w' \frac{z' - \xi'}{z' - \eta'} \quad (4.15)$$

Going along $-D_1$ corresponds to

$$\frac{z' - \xi'}{z' - \eta'} \rightarrow e^{2i\pi} \frac{z' - \xi'}{z' - \eta'} \quad (4.16)$$

The associated first Abelian integral is

$$\phi'(z') - \phi'(z'_0) = \text{Log} \left(\frac{z' - \xi'}{z' - \eta'} \frac{z'_0 - \eta'}{z'_0 - \xi'} \right) \quad (4.17)$$

with

$$\begin{aligned} P_{D_n}(\phi') &= 2i\pi \\ P_{C_n}(\phi') &= \text{Log } w' \end{aligned} \quad (4.18)$$

From the previous paragraph the two parametrizations are related by

$$\text{Log} \left(\frac{z' - \xi'}{z' - \eta'} \frac{z'_0 - \eta'}{z'_0 - \xi'} \right) = - \frac{2i\pi}{\text{Log } w} \text{Log} \frac{z - \xi}{z - \eta} \frac{z_0 - \eta}{z_0 - \xi} \quad (4.19)$$

$$\text{Log } w' \text{Log } w = 4\pi^2 \quad (4.20)$$

There are no other constraints on $\xi' \eta' z'_0$. If we want for example that for z real $\notin [\xi, \eta]$ z' to be real then we must have

$$\eta' = \xi'^* \quad (4.21)$$

[a segment of the real axis $[z \rightarrow T(z)] \notin [\xi, \eta]$ is mapped on the whole real axis by this transformation: z and $T(z)$ have the same transformed.

The third Abelian integral is defined by

$$\begin{aligned} v(z_i, z_j, z_k, z_\ell) &\equiv \Omega_{z_k z_\ell}(z_i) - \Omega_{z_k z_\ell}(z_j) \\ &= \sum_{n=-\infty}^{+\infty} \text{Log} \left(\frac{z_i - T^n(z_k)}{z_i - T^n(z_\ell)} \frac{z_j - T^n(z_\ell)}{z_j - T^n(z_k)} \right) + \frac{1}{\text{Log } w} \text{Log} \left(\frac{z_i - \eta}{z_i - \xi} \frac{z_j - \xi}{z_j - \eta} \right) \text{Log} \left| \frac{z_k - \eta}{z_k - \xi} \frac{z_\ell - \xi}{z_\ell - \eta} \right| \end{aligned} \quad (4.22)$$

$$[\Omega_{ab}(z) - \Omega_{ab}(z_0) = \omega_{ab}(z) - \omega_{ab}(z_0) + (\phi(z) - \phi(z_0))(\operatorname{Re} A)^{-1} \operatorname{Re}(\phi(a) - \phi(b))]$$

$$v'(z'_i, z'_j, z'_k, z'_e) = \sum_{n=-\infty}^{+\infty} \log \left(\frac{z'_i - T'^n(z'_k)}{z'_i - T'^n(z'_e)} \frac{z'_j - T'^n(z'_e)}{z'_j - T'^n(z'_k)} \right)$$

$$+ \frac{1}{\log w'} \log \left(\frac{z'_i - \eta'}{z'_i - \xi'} \frac{z'_j - \xi'}{z'_j - \eta'} \right) \log \left| \frac{z'_k - \eta'}{z'_k - \xi'} \frac{z'_e - \xi'}{z'_e - \eta'} \right|$$

(4.23)

and we have the fundamental relation

$$v(z_i, z_j, z_k, z_e) = v'(z'_i, z'_j, z'_k, z'_e)$$

(4.24)

[Let us note that if z_k and z_e are real $\notin [\xi, \eta]$ then the second term in v' disappears since $\phi(z_k) - \phi(z_e)$ is pure imaginary.]

Let us note that if we choose $T(z) = wz$ et $T'(z') = w'z'$ there we have the usual Jacobi transformation for the θ functions ¹²⁾

$$z' = e^{-2i\pi \frac{\log z}{\log w}}$$

(4.25)

Let us consider now the parabolic limit in which the relation becomes much more simple.

b) Parabolic limit

Let us first briefly recall some results of domain variational theory ²²⁾. A sphere with one handle $\mathcal{M}^*$ is built by cutting two holes in the sphere $\mathcal{M}$ following Jordan curves C_1 and C'_1 , and identifying the boundaries of the holes. D_1 and D'_1 are defined by (Fig. 2)

$$\begin{cases} D'_1 & V'(z', z'_0, \xi', \eta') = \operatorname{Re} \Omega'_{\xi'\eta'}(z') - \operatorname{Re} \Omega'_{\xi'\eta'}(z'_0) = \frac{1}{2} \log w' \\ D_1 & V'(z', z'_0, \xi', \eta') = -\frac{1}{2} \log w' \end{cases}$$

(4.26)

Then it can be proved that when $w' \rightarrow 0$ the first Abelian integrals on $\mathcal{M}^*$ are

$$\phi^{*'}(z') - \phi^{*'}(z'_0) = v(z', z'_0, \xi', \eta') \quad (4.27)$$

with

$$\begin{aligned} P_{-D_1}(\phi'^*) &= 2i\pi \\ P_{C_1}(\phi'^*) &\simeq \text{Log } w' \end{aligned}$$

and V^* defined on M^* is related to V' defined on M by

$$V'^*(z'_i, z'_j, z'_k, z'_e) = V(z'_i, z'_j, z'_k, z'_e) - V(z'_i, z'_j, \xi', \eta') \frac{1}{\text{Log } w'} V(z'_k, z'_e, \xi', \eta') + O(w') \quad (4.28)$$

So when $w' = 0$ we must have

$$V^*(z'_i, z'_j, z'_k, z'_e) = V(z'_i, z'_j, z'_k, z'_e) \quad (4.29)$$

if none of the z 's are on the boundaries D_1 or D'_1 .

The parabolic limit is defined by

$$w \rightarrow 1, \quad \xi \rightarrow \eta, \quad \frac{\xi - \eta}{1 - w} \rightarrow c \quad (4.30)$$

and

$$\frac{1}{T(z) - \eta} = \frac{1}{z - \eta} + \frac{1}{c}$$

then $P_{D_1}(\emptyset) \rightarrow 0$, if we want to use the previous results we must work in the other parametrization

$$\begin{cases} P_{-D_1}(\phi') = 2i\pi \\ P_{C_1}(\phi') = \text{Log } w' \rightarrow \infty \end{cases} \quad (4.31)$$

We obtain the relation for $w' = 0$

$$V^*(z_i, z_j, z_k, z_e) = V'^*(z'_i, z'_j, z'_k, z'_e) = V'(z'_i, z'_j, z'_k, z'_e)$$

with

$$V^*(z_i, z_j, z_k, z_e) = \sum_{n=-\infty}^{+\infty} \log \left(\frac{z_i - T^n(z_k)}{z_i - T^n(z_e)} \frac{z_j - T^n(z_e)}{z_j - T^n(z_k)} \right) \quad (4.32)$$

(The second part of V^* disappears in the parabolic limit.) Remembering that V on the sphere is just the anharmonic ratio:

$$V'(z'_i, z'_j, z'_k, z'_e) = \log \left(\frac{z'_i - z'_k}{z'_i - z'_e} \frac{z'_j - z'_e}{z'_j - z'_k} \right) \quad (4.33)$$

the Jacobi transformation becomes

$$\log \frac{z'_i - \xi'}{z'_i - \eta'} \frac{z'_j - \eta'}{z'_j - \xi'} = -2i\pi c \left(\frac{1}{z - \eta} - \frac{1}{z - \xi} \right) \quad (4.34)$$

and the Jacobi identity is

$$\sum_{n=-\infty}^{+\infty} \log \left(\frac{z_i - T^n(z_k)}{z_i - T^n(z_e)} \frac{z_j - T^n(z_e)}{z_j - T^n(z_k)} \right) = \log \left(\frac{z'_i - z'_k}{z'_i - z'_e} \frac{z'_j - z'_e}{z'_j - z'_k} \right) \quad (4.35)$$

If we use explicitly the Jacobi transformation, the dependence on ξ' η' z'_0 disappears and we have the more familiar form

$$\sum_{n=-\infty}^{+\infty} \log \left(\frac{z_i - T^n(z_k)}{z_j - T^n(z_e)} \frac{z_j - T^n(z_e)}{z_i - T^n(z_k)} \right) = \log \left\{ \frac{\sin \left[\pi c \left(\frac{1}{\eta - z_i} - \frac{1}{\eta - z_k} \right) \right] \sin \left[\pi c \left(\frac{1}{\eta - z_j} - \frac{1}{\eta - z_e} \right) \right]}{\sin \left[\pi c \left(\frac{1}{\eta - z_i} - \frac{1}{\eta - z_e} \right) \right] \sin \left[\pi c \left(\frac{1}{\eta - z_j} - \frac{1}{\eta - z_k} \right) \right]} \right\} \quad (4.36)$$

This can be proved directly for $T(z) = z-1$ and making the projective transformation $V(z) = 1 - c/(z - \eta)$, by using the well-known formula

$$\log \left(\frac{\sin \pi z}{\pi z} \right) = \sum_{n \neq 0} \log \left(1 - \frac{z}{n} \right) = \sum_{n \neq 0} \log \left(\frac{z-n}{-n} \right)$$

$$\log\left(\frac{\sin \pi z}{\pi z}\right) = \sum_{n \neq 0} \log \frac{T^n(z)}{T^n(0)} \quad ; \quad T(z) = z-1$$

c) Further remarks

When $z_i = z_k$ by a limiting process we obtain easily

$$\sum_{n \neq 0} \log\left(\frac{z_i - T^n(z_i)}{z_i - T^n(z_e)} \frac{z_j - T^n(z_e)}{z_j - T^n(z_i)}\right) = \log\left(\frac{dz'_i}{dz_i} \frac{z_i - z_e}{z'_i - z'_e} \frac{z'_j - z'_e}{z_j - z_e} \frac{z_j - z_i}{z'_j - z'_i}\right) \quad (4.37)$$

We need also the expression of this infinite sum when some point is the invariant point. We can see that the infinite sum is finite if we take together the terms n and $-n$. We can define it by a limiting process by taking the point $z_k \rightarrow \gamma$, but the Jacobi transformation is singular at this point, in fact we see that

$$\begin{aligned} z_k &\rightarrow \gamma + i\varepsilon \quad ; \quad z'_k \rightarrow \xi' \\ z_k &\rightarrow \gamma - i\varepsilon \quad ; \quad z'_k \rightarrow \gamma' \end{aligned}$$

It seems then natural to define the infinite sum as the half sum of it when $z_k \rightarrow \gamma + i\varepsilon$ and $z_k \rightarrow \gamma - i\varepsilon$.

Let us justify this prescription by taking some point z'_j on the boundary D_1 , and its conjugate z'^{*}_j on D_1 . Then

$$\begin{aligned} V'(z'_k, z'_e, \xi', \gamma') &= -\frac{1}{2} \log w' \\ V'(z'^{*}_k, z'_e, \xi', \gamma') &= \frac{1}{2} \log w' \end{aligned}$$

the function $V'^{*}(z'_i, z'_j, z'_k, z'_e)$ has zero period along C_1 , then

$$V'^{*}(z'_i, z'_j, z'_k, z'_e) = V'^{*}(z'_i, z'_j, z'^{*}_k, z'_e)$$

so that by using the results of b)

$$V'^*(z'_i, z'_j, z'_k, z'_e) = \frac{1}{2} [V'(z'_i, z'_j, z'_k, z'_e) + V'(z'_i, z'_j, z'^*_k, z'_e)]$$

if $w' \rightarrow 0$, $z'_k \rightarrow \xi'$, $z'^*_k \rightarrow \eta'$ then we can define

$$V^*(z_i, z_j, \eta, z_e) = \frac{1}{2} [V'(z'_i, z'_j, \eta', z'_e) + V'(z'_i, z'_j, \xi', z'_e)] \quad (4.38)$$

The Jacobi identity is then written as

$$\begin{aligned} \sum_{n>0} \log \left(\frac{(z_i - \eta)^2}{(z_i - T^n(z_e))(z_i - T^{-n}(z_e))} \frac{(z_j - T^n(z_e))(z_j - T^{-n}(z_e))}{(z_j - \eta)^2} \right) \\ + \log \left(\frac{z_i - \eta}{z_i - z_e} \frac{z_j - z_e}{z_j - \eta} \right) = \frac{1}{2} \log \left(\frac{(z'_j - z'_e)^2}{(z'_i - z'_e)^2} \frac{z'_i - \xi'}{z'_k - \xi'} \frac{z'_i - \eta'}{z'_k - \eta'} \right) \\ = \log \left\{ \frac{\sin \left[\pi C \left(\frac{1}{\eta - z_j} - \frac{1}{\eta - z_e} \right) \right]}{\sin \left[\pi C \left(\frac{1}{\eta - z_i} - \frac{1}{\eta - z_e} \right) \right]} \right\} \end{aligned} \quad (4.39)$$

(4.40)

3) Study of a sphere with N handles in the limit where all the handles shrink to zero

For simplicity of writing we limit ourselves to the case of two handles. The results are trivially extended to N handles. Let us consider a sphere in the two handles $\mathcal{M}^*$ which are built by cutting four holes in the sphere $\mathcal{M}$ following curves C_1 , C'_1 , C_2 and C'_2 and identifying $C_1 \sim C'_1$ and $C_2 \sim C'_2$ (Fig. 3). We are interested in the limit where the four holes shrink to zero.

The holes are defined by

$$\begin{aligned} C_1 \quad V(z', z'_0, \xi'_1, \eta'_1) &= \frac{1}{2} \log e_1 \\ C'_1 \quad V(z', z'_0, \xi'_1, \eta'_1) &= -\frac{1}{2} \log e_1 \end{aligned} \quad (4.41)$$

where, as usual

$$V(z', z_0, \xi_1, \eta_1) = \operatorname{Re} (\Omega_{\xi_1, \eta_1}(z') - \Omega_{\xi_1, \eta_1}(z_0)) [= \operatorname{Re} v] \quad (4.42)$$

z_0' is a fixed point. (In our case since $\mathcal{R}_6$ is a sphere $\Omega \equiv \omega$ since there are no cycles on the sphere.) Once we have chosen two points $z_1^{(0)} \in C_1$ and $z_1^{*(0)} \in C_1'$ that we wish to identify, we identify $z \in C_1$ and $z^* \in C_1'$ if they satisfy

$$v(z', z_0, \xi_1, \eta_1) - v(z_1^{(0)}, z_0, \xi_1, \eta_1) = v(z^*, z_0, \xi_1, \eta_1) - v(z_1^{*(0)}, z_0, \xi_1, \eta_1) \quad (4.43)$$

[Since we are interested by symmetric surface we can choose $z^{(0)}$ and $z^{(0)*}$ in such a way that $\operatorname{Im} v(z^{(0)}, \dots) = \operatorname{Im} v(z^{*(0)}, \dots)$.]

In the same way C_2 and C_2' are defined by

$$\begin{cases} C_2 & V(z', z_0, \xi_2, \eta_2) = \frac{1}{2} \operatorname{Log} p_2 \\ C_2' & V(z', z_0, \xi_2', \eta_2') = -\frac{1}{2} \operatorname{Log} p_2 \end{cases} \quad (4.44)$$

a) First Abelian integrals on $\mathcal{R}_6^*$ and period matrix

Let us consider the function

$$\phi_1(z') - \phi_1(z_0) = v(z', z_0, \xi_1, \eta_1) \quad (4.45)$$

ϕ_1 is analytic on $\mathcal{R}_6^*$ since

- ξ_1 and η_1 are not on $\mathcal{R}_6^*$;
- the relation (4.43) ensures the analyticity after identification of C_1 and C_1' (the discontinuity from C_1 to C_1' is constant along C_1);
- along C_2 and C_2' $\phi_1(z) - \phi_1(z_0)$ can be considered as constant up to order $\sqrt{g_2}$.

Moreover

$$\oint_{C_1} d\phi_1(z) = 2i\pi \quad \oint_{C_2} d\phi_2(z) = 0$$

Then $\phi_1(z') - \phi_1(z'_0)$ is a first Abelian integral on $\mathcal{M}^*$. The same result is true for $\phi_2(z')$ defined by

$$\phi_2(z') - \phi_2(z'_0) = \nu(z, z'_0, \xi'_2, \eta'_2) \quad (4.46)$$

with

$$\oint_{C_2} d\phi_2(z) = 2i\pi \quad \oint_{C_1} d\phi_2(z) = 0$$

So the two Abelian integrals are ϕ_1 and ϕ_2 . We can now compute the period matrix defined as the period along the cycles D_1 and D_2 along the handles.

$$\begin{aligned} \oint_{D_1} d\phi_1(z) &= \phi_1(C_1) - \phi_1(C_1^*) \\ &= \log e_1 + i \operatorname{Im}[\nu(z_1^{(0)}, z'_0, \xi'_1, \eta'_1) - \nu(z_1^{*(0)}, z'_0, \xi'_1, \eta'_1)] \\ &\equiv \log e_1 + i \alpha_1 \end{aligned} \quad (4.47)$$

$$\begin{aligned} \oint_{D_2} d\phi_1(z) &= \phi_1(C_2) - \phi_1(C_2^*) = \nu(\xi'_2, z'_0, \xi'_1, \eta'_1) - \nu(\eta'_2, z'_0, \xi'_1, \eta'_1) + \mathcal{O}(\sqrt{e_2}) \\ &= \nu(\xi'_2, \eta'_2, \xi'_1, \eta'_1) + \mathcal{O}(\sqrt{e_2}) \end{aligned} \quad (4.48)$$

$$\begin{aligned} \oint_{D_2} d\phi_2(z) &= \log e_2 + i \operatorname{Im}[\nu(z_2^{(0)}, z'_0, \xi'_2, \eta'_2) - \nu(z_2^{*(0)}, z'_0, \xi'_2, \eta'_2)] \\ &\equiv \log e_2 + i \alpha_2 \end{aligned} \quad (4.49)$$

$$\oint_{D_1} d\phi_2(z) = \nu(\xi'_1, \eta'_1, \xi'_2, \eta'_2) + \mathcal{O}(\sqrt{e_1}) \quad (4.50)$$

Using the property of V

$$V(z_1, z_2, z_3, z_4) = V(z_3, z_4, z_1, z_2) \quad (4.51)$$

We verify that the matrix period is symmetric (up to order $\sqrt{p_1}, \sqrt{p_2}$) and the real part is simply

$$\operatorname{Re} A = \begin{pmatrix} \log p_1 & V(\xi'_1, \eta'_1, \xi'_2, \eta'_2) \\ V(\xi'_1, \eta'_1, \xi'_2, \eta'_2) & \log p_2 \end{pmatrix} \quad (4.52)$$

b) Third Abelian integral in real normalization

In order to compute $V^*(z'_1 z'_j z'_k z'_l)$ we apply the following general theorem on variational problem, Ref. 22).

Theorem

Let M and M^* be two closed Riemann surfaces which have the domain M_0 in common. Suppose that there exists a set K of cycles K_v in M_0 which is canonical with respect to M and can be completed to a canonical set with respect to M^* . Then for every quadruple $p p_0 q q_0$ in M_0 we have

$$V^*(p, p_0, q, q_0) - V(p, p_0, q, q_0) = \operatorname{Re} \left\{ \frac{1}{2i\pi} \int_{\partial M_0} \Omega_{q q_0}(t) d\Omega_{p p_0}^*(t) \right\} \quad (4.53)$$

Let us now apply it to our problem. Taking care of the orientation we have

$$\partial M_0 = -c_1 + c'_1 - c_2 + c'_2$$

In order to compute these integrals we develop $\Omega_{z'_i z'_j}(t)$ around ξ'_j for the integration on C_1

$$\Omega_{z'_i z'_j}(t) = \Omega_{z'_i z'_j}(\xi'_1) + (t - \xi'_1) \Omega'_{z'_i z'_j}(\xi'_1) + \dots \quad (4.54)$$

then

$$\begin{aligned} \int_{C_1} d\Omega_{z'_k z'_l}^*(t) \Omega_{z'_i z'_j}(t) &= \Omega_{z'_i z'_j}(\xi'_1) \int_{C_1} d\Omega_{z'_k z'_l}^*(t) \\ &+ \Omega'_{z'_i z'_j}(\xi'_1) \int_{C_1} (t - \xi'_1) d\Omega_{z'_k z'_l}^*(t) + \dots \end{aligned} \quad (4.55)$$

The function

$$F_{\lambda}(z'_k) - F_{\lambda}(z'_e) = \frac{1}{2i\pi} \int_{C_1} d\Omega^*_{z'_k z'_e}(t) \quad (4.56)$$

is a real harmonic function of z'_k (z'_e is taken as a fixed parameter). It has zero period along $C_1 C_2 D_2$ (because Ω^* has only imaginary periods). It has a period 1 along D_1 , because $d\Omega^*_{z'_k z'_e}(t)$ is singular on C_1 when z'_k crosses C_1 . Then it is related to $\text{Re } \phi_1$ and $\text{Re } \phi_2$ by

$$F_{\lambda}(z'_k) - F_{\lambda}(z'_e) = (\text{Re } A)^{-1}_{11} \text{Re}[\phi_1(z'_k) - \phi_1(z'_e)] + (\text{Re } A)^{-1}_{12} \text{Re}[\phi_2(z'_k) - \phi_2(z'_e)] \quad (4.57)$$

We remark that $\text{Re } A^{-1} \rightarrow 0$ when ϱ_1 and $\varrho_2 \rightarrow 0$, then $V^* - V \rightarrow 0$. Let us now compute the second term. We can choose, to compute it, two uniformizers around ξ_1 and η_1 such that

$$\begin{aligned} t \in C_1 & \quad v(t, z'_0, \xi'_1, \eta'_1) = \text{Log}(t - \xi'_1) \\ t^* \in C_1 & \quad v(t^*, z'_0, \xi'_1, \eta'_1) = -\text{Log}(t^* - \eta'_1) \end{aligned} \quad (4.58)$$

if t and t^* are identified on $\mathcal{P}^*$ then

$$v(t, z'_0, \xi'_1, \eta'_1) - v(t^*, z'_0, \xi'_1, \eta'_1) = \text{Log } \varrho_1 + i\alpha_1 \quad (4.59)$$

and then we have

$$(t - \xi'_1)(t^* - \eta'_1) = \varrho_1 e^{i\alpha_1} \quad (4.60)$$

Now we can compute the integral on C'_1 instead of C_1

$$\int_{C_1} (t - \xi'_1) d\Omega^*_{z'_k z'_e}(t) = \varrho_1 e^{i\alpha_1} \int_{C'_1} \frac{d\Omega^*_{z'_k z'_e}(t)}{t^* - \eta'_1} \quad (4.61)$$

and since $\Omega^* - \Omega \rightarrow 0$ as $\rho_1 \rightarrow 0$ we replace Ω^* by Ω in the integral and apply the residue theorem, then

$$\frac{1}{2i\pi} \int_{c_1} (t - \xi'_1) d\Omega_{z'_k z'_e}^*(t) = \rho_1 e^{i\alpha_1} \Omega'_{z'_k z'_e}(\eta'_1) \quad (4.62)$$

Then collecting all the contributions and defining

$$[V(z'_i, z'_j)] = \begin{pmatrix} V(z'_i, z'_j, \xi'_1, \eta'_1) \\ V(z'_i, z'_j, \xi'_2, \eta'_2) \end{pmatrix} \quad (4.63)$$

we obtain

$$V^*(z'_i, z'_j, z'_k, z'_e) = V(z'_i, z'_j, z'_k, z'_e) - [V(z'_i, z'_j)] (\text{Re } A)^{-1} [V(z'_k, z'_e)] + O(\rho_1, \rho_2) \quad (4.64)$$

where $\sim$ denotes the transposition.

Let us note, however, that we know $\text{Re } A$ only up to order $\sqrt{\rho_1}, \sqrt{\rho_2}$. The generalization is obvious for N handles by defining $[V(z'_i, z'_j)]$ as a N dimensional vector and $\text{Re } A$ is now

$$\begin{cases} \text{Re } A_{ij} \simeq V(\xi'_i, \eta'_i, \xi'_j, \eta'_j) \\ \text{Re } A_{ii} \simeq \text{Log } \rho_i \end{cases} \quad (4.65)$$

4) Explicit expression of period matrix and first Abelian integral in the parabolic limit

a) Period matrix

In terms of the group the period matrix is given by ^{10), 11)}

$$A_{ij} = \delta_{ij} \text{Log } w_i + \sum_{\alpha \in Q_{ij}} \text{Log} \left(\frac{\xi_i - T_\alpha(\xi_j)}{\xi_i - T_\alpha(\eta_j)} \frac{\eta_i - T_\alpha(\eta_j)}{\eta_i - T_\alpha(\xi_j)} \right) \quad (4.66)$$

where Q_{ij} is the sum on all elements T_α which have not T_j^n at the right and T_i^m at the left.

A_{ij} is real. It is a general result when the surface is symmetric. Let us now take all the generators parabolic

$$\frac{1}{T_i(z) - \eta_i} = \frac{1}{z - \eta_i} + \frac{1}{c_i} \quad (4.67)$$

Then by developing the logarithms around 1 ($\xi_i \rightarrow \eta_i, \xi_j \rightarrow \eta_j$) we obtain

$$A_{ii} = \log w_i + c_i^2 (1-w_i)^2 \sum_{\alpha \in Q_{ii}} \frac{\partial T_\alpha}{\partial z} \Big|_{z=\eta_i} \frac{1}{(\eta_i - T_\alpha(\eta_i))^2} + O((1-w_i)^4)$$

$$A_{ii} = \log w_i + b_i (1-w_i)^2 + O((1-w_i)^4)$$

(4.68)

$$A_{ij} = c_i c_j (1-w_i)(1-w_j) \sum_{\beta \in Q_{ij}} \frac{\partial T_\beta}{\partial z} \Big|_{z=\eta_i} \frac{1}{(\eta_i - T_\beta(\eta_j))^2} + O((1-w_i)^2(1-w_j)^2)$$

$$A_{ij} = (1-w_i)(1-w_j) b_{ij} + O((1-w_i)^2(1-w_j)^2)$$

(4.69)

We can now compute the inverse retaining only terms which are infinite or finite

$$\begin{cases} (A^{-1})_{ij} \simeq -b_{ij} \\ (A^{-1})_{ii} \simeq \frac{1}{\log w_i} - b_i \end{cases}$$

(4.70)

b) First Abelian integrals

In term of the group they are defined by

$$\Phi_i(z) - \Phi_i(z_0) = \sum_{\alpha \in Q_i} \text{Log} \left(\frac{T_\alpha(z) - \xi_i}{T_\alpha(z) - \eta_i} \frac{T_\alpha(z_0) - \eta_i}{T_\alpha(z_0) - \xi_i} \right) \quad (4.71)$$

where Q_i is the sum of all elements T_α which have not T_i^n at the left. Then if $w_i \rightarrow 1$ ($T_\alpha(z) \notin [\xi_i, \eta_i]$ which is realized if z is in a fundamental region of $T_1 T_2 \dots T_n$)

$$\Phi_i(z) - \Phi_i(z_0) \simeq (1 - w_i) C_i \sum_{\alpha \in Q_i} \frac{\partial T_\alpha}{\partial y} \Big|_{y=\eta_i} \left(\frac{1}{z - T_\alpha(\eta_i)} - \frac{1}{z_0 - T_\alpha(\eta_i)} \right) \quad (4.72)$$

There is no further change when $w_j \rightarrow 1$ in $\emptyset_i$ (except in T_α).

5) Generalized Jacobi transformation in the parabolic limit

We now apply the results of 1), 3) and 4).

The fact that $A_{ij} \rightarrow 0$ in the limit means that we must do the Jacobi transformation in order to apply the result of the discussion in 3). Then

$$\Phi'_i(z') - \Phi'_i(z'_0) = \Omega_{\xi'_i \eta'_i}(z') - \Omega_{\xi'_i \eta'_i}(z'_0)$$

with Ω defined on the sphere

$$\Phi'_i(z') - \Phi'_i(z'_0) = \text{Log} \left(\frac{z' - \xi'_i}{z' - \eta'_i} \frac{z'_0 - \eta'_i}{z'_0 - \xi'_i} \right) \quad (4.73)$$

and the relation $A'A = 4\pi^2$ allows us to identify (introducing w'_i instead of ρ_i)

$$\text{Log } w'_i \sim 4\pi^2 \left(\frac{1}{\text{Log } w_i} - \theta_i \right) \quad (4.74)$$

$$\text{Log} \left(\frac{\xi'_i - \xi'_j}{\xi'_i - \eta'_j} \frac{\eta'_i - \eta'_j}{\eta'_i - \xi'_j} \right) = -4\pi^2 \theta_{ij}$$

(4.75)

Using the results of 4), the relations between z' and z are now written

$$-\frac{1}{2i\pi} \text{Log} \left(\frac{z' - \xi'_i}{z' - \eta'_i} \frac{z'_0 - \eta'_i}{z'_0 - \xi'_i} \right) = c_i \sum_{\alpha \in Q_i} \frac{\partial T_\alpha}{\partial y} \Big|_{y=\eta_i} \left(\frac{1}{z - T_\alpha(\eta_i)} - \frac{1}{z_0 - T_\alpha(\eta_i)} \right)$$

(4.76)

and this whatever $i=1, \dots, N$. The consistency of these relations is ensured by the relation between the ξ'_i, η'_i *).

We have no other constraints on these parameters. If we want that for z real z' be also real then we need

$$\xi'_i = \eta'_i^*$$

(4.77)

(If we want z' on the unit circle, then ξ'_i and η'_i must be conjugate with respect this circle $\xi'_i \eta'_i = 1$.)

Let us note that z and $T_\alpha(z)$ where T_α is a member of the group have the same transform z' .

6) Generalized Jacobi identity

a) From 3), when $w \rightarrow 1$ (or $w' \rightarrow 0$) we have

$$V^*(z_i, z_j, z_k, z_l) = V^{*'}(z'_i, z'_j, z'_k, z'_l) = V(z'_i, z'_j, z'_k, z'_l)$$

(4.78)

then, writing them explicitly

*) Let us note that contrary to the case of one generator, once z_0 is chosen, z'_0 is fixed and cannot be chosen arbitrarily.

$$\sum_{\alpha} \log \left(\frac{z_i - T_{\alpha}(z_k)}{z_i - T_{\alpha}(z_e)} \frac{z_j - T_{\alpha}(z_e)}{z_j - T_{\alpha}(z_k)} \right) = \log \left(\frac{z'_i - z'_k}{z'_i - z'_e} \frac{z'_j - z'_e}{z'_j - z'_k} \right) \quad (4.79)$$

where z' is defined by the Jacobi transformation.

b) As in the case of one generator we can extend this identity in the cases $z_k \rightarrow z_i$ and $z_k \rightarrow \gamma_{\mu}$

$$\sum_{d \neq i} \log \left(\frac{z_i - T_d(z_i)}{z_i - T_d(z_e)} \frac{z_j - T_d(z_e)}{z_j - T_d(z_i)} \right) = \log \left(\frac{dz'_i}{dz_i} \frac{z_i - z_e}{z'_i - z'_e} \frac{z'_k - z'_e}{z_k - z_e} \frac{z_k - z_i}{z'_k - z'_i} \right) \quad (4.80)$$

When $z_k \rightarrow \gamma_{\mu}$ we must take together the term in T_{α} and $(T_{\alpha})^{-1}$ in order for the series to be convergent. Taking a point on C_{μ} and its conjugate on C'_{μ} and making the limit $w'_{\mu} \rightarrow 0$ we obtain

$$V^*(z_i, z_j, \gamma_{\mu}, z_e) = \frac{1}{2} \left[V'(z'_i, z'_j, \gamma'_{\mu}, z'_e) + V'(z'_i, z'_j, \xi'_{\mu}, z'_e) \right] \quad (4.81)$$

Then

$$\sum_{\alpha} \log \left(\frac{z_i - T_{\alpha}(\gamma_{\mu})}{z_i - T_{\alpha}(z_e)} \frac{z_j - T_{\alpha}(z_e)}{z_j - T_{\alpha}(\gamma_{\mu})} \right) = \frac{1}{2} \log \left(\frac{(z'_j - z'_e)^2}{(z'_i - z'_e)^2} \frac{z'_i - \xi'_{\mu}}{z'_j - \xi'_{\mu}} \frac{z'_i - \gamma'_{\mu}}{z'_j - \gamma'_{\mu}} \right) \quad (4.82)$$

c) We need also to compute these infinite sums when z_i and $z_k \rightarrow \gamma_{\mu}$ and the sums exclude the terms T_{α} of the type T_{μ}^n . We limit here ourselves to the case of a group with two generators. The result will also be true in the general case.

We then take two points z'_i and z'_j on the cycle C_1 (their conjugates are on C'_1). If the sum is extended to all the group the sum is just

$$V^{1 \times *}(z'_i, z'_j, z'_k, z'_e) = V^{* *}(z_i, z_j, z_k, z_e) = V^{1 \times *}(z'_i, z'_j, z'^{*}_k, z'_e) \quad (4.83)$$

$$V'^{**}(z'_i, z'_j, z'^*_k, z'_e) = V'(z'_i, z'_j, z'^*_k, z'_e) - \widetilde{[V'(z'_i, z'_j)]} (\text{Re } A')^{-1} [V'(z'^*_k, z'_e)] \quad (4.84)$$

with

$$\begin{aligned} V'(z'_i, z'_j, \xi'_1, \eta'_1) &= V'(z'_i, z'_0, \xi'_1, \eta'_1) + V'(z'_0, z'_j, \xi'_1, \eta'_1) \\ &= \frac{1}{2} \text{Log } \varepsilon_1 + V'(z'_0, z'_j, \xi'_1, \eta'_1) \end{aligned} \quad (4.85)$$

$$V'(z'^*_k, z'_e, \xi'_1, \eta'_1) = -\frac{1}{2} \text{Log } \varepsilon_1 + V'(z'_0, z'_e, \xi'_1, \eta'_1) \quad (4.86)$$

From the previous paragraph we have

$$(\text{Re } A')^{-1} = \frac{A}{4\pi^2} \sim \begin{bmatrix} \text{Log } w_1 + \theta_1(1-w_1)^2 & \theta_{12}(1-w_1)(1-w_2) \\ \theta_{12}(1-w_1)(1-w_2) & \text{Log } w_2 + \theta_2(1-w_2)^2 \end{bmatrix} \quad (4.87)$$

with

$$\text{Log } \varepsilon_1 = 4\pi^2 \left[\frac{1}{\text{Log } w_1} - \theta_1 \right] \quad (4.88)$$

Then, keeping only the non-zero term in the limit $w_1 \rightarrow 1$ (we can put $w_2 = 1$)

$$\begin{aligned} V'^{**}(z'_i, z'_j, z'^*_k, z'_e) &= V'(\xi'_1, z'_j, \eta'_1, z'_e) + \frac{\pi^2}{\text{Log } w_1} - \pi^2 b_1 \\ &\quad - \frac{1}{2} V'(z'_j, z'_e, \xi'_1, \eta'_1) + o(\sqrt{\varepsilon_1}) \end{aligned} \quad (4.89)$$

We must now subtract the term which comes from the sum of the terms generated only by T_1 . We can still do the Jacobi transformation but we must be careful because it is not the same as before, but the one described in 2).

$$V^*(z_i, z_j, z_k, z_l) = V^{*''}(z_i'', z_j'', z_k'', z_l'') = V^{*''}(z_i'', z_j'', z_k^{*''}, z_l'') \quad (4.90)$$

We can choose ξ_1'' , and η_1'' identical to ξ_1' , η_1' as well as $z_0'' = z_0'$. Then

$$V^{*''}(z_i'', z_j'', z_k^{*''}, z_l'') = V''(z_i'', z_j'', z_k^{*''}, z_l'') - V''(z_i'', z_j'', \xi_1', \eta_1') \frac{1}{\log \varepsilon_1'} V''(z_k^{*''}, z_l'', \xi_1', \eta_1') \quad (4.91)$$

with now

$$V''(z_i'', z_j'', \xi_1', \eta_1') = \frac{1}{2} \log \varepsilon_1' + V''(z_0', z_1'', \xi_1', \eta_1') \quad (4.92)$$

$$V''(z_k^{*''}, z_l'', \xi_1', \eta_1') = -\frac{1}{2} \log \varepsilon_1' + V''(z_0', z_l'', \xi_1', \eta_1') \quad (4.93)$$

and

$$\log \varepsilon_1' \log w_1 = 4\pi^2 \quad (4.94)$$

Then,

$$V^{*''}(z_i'', z_j'', z_k^{*''}, z_l'') = V''(\xi_1', z_j'', \eta_1', z_l'') + \frac{\pi^2}{\log w_1} - \frac{1}{2} V''(z_j'', z_l'', \xi_1', \eta_1') + O(\sqrt{\varepsilon_1'}) \quad (4.95)$$

Then we obtain (note that only the difference is defined in the limit $w_1 \rightarrow 0$)

$$\begin{aligned} V^{**}(\gamma_1, z_j, \gamma_1, z_e) - V^*(\gamma_1, z_j, \gamma_1, z_e) \\ = -\pi^2 b_1 + V'(\xi'_1, z'_j, \gamma'_1, z'_e) - V''(\xi'_1, z''_j, \gamma'_1, z''_e) \\ - \frac{1}{2} [V'(z'_j, z'_e, \xi'_1, \gamma'_1) - V''(z''_j, z''_e, \xi'_1, \gamma'_1)] \end{aligned} \quad (4.96)$$

or

$$\begin{aligned} \sum_{T_\alpha \in \mathcal{P}_1} \text{Log} \frac{\gamma_1 - T_\alpha(\gamma_1)}{\gamma_1 - T_\alpha(z_e)} \frac{z_j - T_\alpha(z_e)}{z_j - T_\alpha(\gamma_1)} = -\pi^2 b_1 \\ + \text{Log} \left(\frac{\xi'_1 - z''_e}{\xi'_1 - z'_e} \frac{z'_j - \gamma'_1}{z''_j - \gamma'_1} \frac{z''_j - z''_e}{z'_j - z'_e} \right) \\ - \frac{1}{2} \text{Log} \left(\frac{z'_j - \xi'_1}{z'_j - \gamma'_1} \frac{z'_e - \gamma'_1}{z'_e - \xi'_1} \frac{z''_j - \gamma'_1}{z''_j - \xi'_1} \frac{z''_e - \xi'_1}{z''_e - \gamma'_1} \right) \end{aligned} \quad (4.97)$$

where z'_1 is defined by (4.76) and z''_1 by (4.34).

We defined a_{11} by

$$\text{Log} a_{11} = -4\pi^2 b_1 \quad (4.98)$$

When $z_j \rightarrow z_e = y_0$ we obtain (we take $\gamma' = \xi'^*$)

$$\sum_{T_\alpha \in \mathcal{P}_1} \text{Log} \left(\frac{\gamma_1 - T_\alpha(\gamma_1)}{\gamma_1 - T_\alpha(y_0)} \frac{y_0 - T_\alpha(y_0)}{y_0 - T_\alpha(\gamma_1)} \right) = \text{Log} \left(a_{11}^{\frac{1}{4}} \frac{\xi'^* - \xi'_1}{2i\pi c_1} \frac{(y_0 - \gamma_1)^2}{|y'_0 - \xi'_1|^2} \frac{dy'_0}{dy_0} \right) \quad (4.99)$$

We can also compute this quantity by noting that if z_i and z_k are on C_1 we have, using (4.84) and (4.91)

$$\begin{aligned} V'^{**}(z'_i, z'_j, z'_k, z'_e) - V''^*(z''_i, z''_j, z''_k, z''_e) \\ = \frac{1}{4} \left\{ V' (z'_i, z'_j, z'_k, z'_e) - V''(z''_i, z''_j, z''_k, z''_e) \right. \\ + V' (z'^*_i, z'_j, z'_k, z'_e) - V''(z''^*_i, z''_j, z''_k, z''_e) \\ + V' (z'_i, z'_j, z'^*_k, z'_e) - V''(z''_i, z''_j, z''^*_k, z''_e) \\ \left. + V' (z'^*_i, z'_j, z'^*_k, z'_e) - V''(z''^*_i, z''_j, z''^*_k, z''_e) \right\} \end{aligned} \quad (4.100)$$

and we take in the second member

$$z''_i, z'_i \rightarrow \xi'_1 \quad ; \quad z''_k, z'_k \rightarrow \xi'_1$$

$$z''^{*}_i, z'^{*}_i \rightarrow \gamma'_1 \quad ; \quad z''^{*}_k, z'^{*}_k \rightarrow \gamma'_1$$

It is shown in the Appendix how we obtain also the result in this way.

5. MULTICURRENT AMPLITUDE - SECOND EXPRESSION

1) Second expression of the multicurrent amplitude

Let us now use the results of Section 4 to simplify the multicurrent amplitude. First, we shall list for the hurried reader the formulas of the previous sections that we need explicitly.

$$\prod_{\alpha \neq I} \frac{y_i - P_{\alpha}(y_j)}{y_i - P_{\alpha}(y_0)} \frac{y_0 - P_{\alpha}(y_0)}{y_0 - P_{\alpha}(y_j)} = \frac{y'_i - y'_j}{y_i - y_j} \frac{y_i - y_0}{y'_i - y'_0} \frac{y_j - y_0}{y'_j - y'_0} \frac{dy'_0}{dy_0} \quad (5.1)$$

$$\prod_{\alpha \neq I} \frac{y_i - P_{\alpha}(y_i)}{y_i - P_{\alpha}(y_0)} \frac{y_0 - P_{\alpha}(y_0)}{y_0 - P_{\alpha}(y_i)} = \frac{dy'_i}{dy_i} \left(\frac{y_i - y_0}{y'_i - y'_0} \right)^2 \frac{dy'_0}{dy_0} \quad (5.2)$$

$$\prod_{\alpha \neq I} \frac{y_i - P_{\alpha}(y_k)}{y_i - P_{\alpha}(y_0)} \frac{y_0 - P_{\alpha}(y_0)}{y_0 - P_{\alpha}(y_k)} = \frac{|y'_i - y'_k|}{y_i - y_k} \frac{y_i - y_0}{y'_i - y'_0} \frac{y_0 - y_k}{|y'_0 - y'_k|} \frac{dy'_0}{dy_0} \quad (5.3)$$

$$\prod_{\alpha \neq I} \frac{y_k - P_{\alpha}(y_e)}{y_k - P_{\alpha}(y_0)} \frac{y_0 - P_{\alpha}(y_0)}{y_0 - P_{\alpha}(y_e)} = \frac{|(y'_k - y'_e)(y'_k - y'_e)^*|^{\frac{1}{2}}}{|y_k - y_e|} \frac{(y_k - y_0)(y_e - y_0)}{|y'_k - y'_0| |y'_e - y'_0|} \frac{dy'_0}{dy_0} \quad (5.4)$$

$$\prod_{\alpha \neq J_k} \frac{y_k - P_{\alpha}(y_k)}{y_k - P_{\alpha}(y_0)} \frac{y_0 - P_{\alpha}(y_0)}{y_0 - P_{\alpha}(y_k)} = a_{kk}^{\frac{1}{4}} \frac{(y_k - y_0)^2}{|y'_k - y'_0|^2} \frac{y'_k - y'_k}{2i\pi C_k} \frac{dy'_0}{dy_0} \quad (5.5)$$

where the y'_k 's are defined up to a homographic transformation by

$$\log a_{ke} \equiv \log \frac{(y'_k - y'_e)(y'_k - y'_e^*)}{(y'_k - y'_e^*)(y'_k - y'_e)} = -4\pi^2 c_k c_e \sum_{\alpha \in Q_{ke}} \frac{\partial P_\alpha}{\partial z} \Big|_{z=y_k} \frac{1}{(y_e - P_\alpha(y_k))^2} \quad (5.6)$$

and

$$\log a_{kk} = -4\pi^2 c_k^2 \sum_{\beta \in Q_{kk}} \frac{\partial P_\beta}{\partial z} \Big|_{z=y_k} \frac{1}{(y_k - P_\beta(y_k))^2} \quad (5.7)$$

where Q_{kl} is the sum of all the elements of the group which have not P_k^n ($n \neq 0$) at the right and P_l^m at the left.

Since the projective transformations P_α satisfy $a_\alpha d_\alpha - b_\alpha c_\alpha = 1$ then $\log a_{kl} < 0$. y'_i is only function of y_i and of the parameters of the group. We note that the terms dy'_i/dy_i occur with the power $-P_i^2/2=1$ and that because of the energy momentum conservation all the dependence on y_0 cancels.

Then, collecting everything together we obtain the desired result for the M currents amplitude doing the change of variable $y_i \rightarrow y'_i$

$$\begin{aligned} & \frac{1}{\pi^M} \int \frac{\prod_{i=0}^{N-1} dy'_i}{dV_{abc}} \prod_{k=1}^M \frac{dy_k dc_k}{c_k^2} \prod_{\alpha \neq 1,2,\dots,M} [f(w_\alpha)]^{-4} \prod_{k,e} a_{ke}^{-\frac{P_k \cdot P_e}{2}} \\ & \prod_{i \neq j} |y'_i - y'_j|^{-\frac{1}{2} k_i \cdot k_j} \prod_{i,k} |y'_i - y'_k|^{-P_k \cdot k_i} \prod_k \left| \frac{y'_k - y'_k}{2} \right|^{-P_k^2/2} \prod_{k \neq l} |y'_k - y'_l|^{-\frac{P_k \cdot P_l}{2}} \end{aligned} \quad (5.8)$$

2) Integration domain

From the expression of the Jacobi transformation and by noting that two points y which are related by a transformation of the group have the same transform y' we deduce that the y' must be integrated on the whole real axis. Their order is the same as for the y .

The γ' are expressed only via some projective invariant quantities. There are $2M-3$ such quantities. However, using the projective invariance of the integrand as usual we can integrate over $2M$ parameters if we keep fixed three points y_i . So, in principle, we can pass from the (γ_k, c_k) variables to the γ'_l complex variables.

The integration on the γ' 's is made on all the half complex plane with the ordering condition on the imaginary parts of γ'

$$\text{Im } \gamma'_1 \leq \text{Im } \gamma'_2 \leq \dots \leq \text{Im } \gamma'_M$$

This restriction is easily interpreted since our current amplitude possesses some duality properties (as a M loop planar amplitude) which allow in particular to interchange the order of the currents. Therefore, it is sufficient to integrate in this domain in order to have the full amplitude. If we want to integrate on all the half complex plane we must divide the amplitude by $M!$ In this explicit symmetric form, it is more easy to exhibit the various singularities, and all a_{kl} (≤ 1) can reach 0 directly (without use of a non-trivial change of variables in order to exhibit the singularities associated with this part of the integration domain). We shall discuss this more precisely in the case of two-current amplitude.

In this new variable γ' , the natural boundary due to the partition function does not provide further restrictions. This point will be discussed later.

Now, if we want the complete amplitude, it is sufficient to integrate all the variables (y_i or γ') without taking care of their order.

If the particles have quantum numbers, introduced by means of the Chan-Paton factor, we need some ordering condition and we shall see later how this problem can be solved. We note that all the $a_{k,l}$ ($k \neq l$) are not independent since there are only $2M-3$ independent cross-ratios.

3) Comparison with other approaches

Let us discuss briefly the main difference with other approaches. Some of the models contain only the part of the integrand in

$$|z_i - z_j^*|^{-\frac{p_i \cdot p_j}{2}} \quad \text{and} \quad |z_i - z_i^*|^{-p_i^2/2}$$

These models have good singularities but the spectrum is larger than the original one (especially they used two kinds of operators in order to factorize the amplitude) ⁴⁾. Some attempts have led to amplitude with the terms $a_{k\ell}$ ($k \neq \ell$) (and $a_{kk}=1$) but have been rejected because they have abnormal singularities in the variables $p_k \cdot p_\ell$ ²⁾. Let us note that this model corresponds to the same one-current amplitude as other approaches but the multicurrent amplitudes are built using a different factorization (essentially starting with exponential of linear form in the annihilation operators).

The present model has moreover terms a_{kk} which, as we shall see later, change the abnormal singularities in normal one.

We also note that the integration measure is quite different from other models by the partition function and the Jacobian of the transformation: $\gamma_i, C_i \rightarrow \gamma'_j, \gamma_j^{*'}.$

4) Singularities

We use the fully symmetric amplitude to exhibit them.

a) Poles

We have the usual resonance poles in the channel of on shell particles when $y_k' \rightarrow y_j'$ ($i \leq k < j$) in $(k_i + \dots k_j)^2$.

When $\gamma_k' \rightarrow$ real axis we have the pole in the external variables p_k^2 . It corresponds in the old variables to $C_k \rightarrow 0$. In this case the corresponding transformation P_k becomes

$$P_k(z) = \gamma_k$$

Then, the contribution of the terms containing P_k in all the various products on the whole group disappears. The group is now generated only by $M-1$ homographic transformations, and the Jacobi transformation is the one associated with this new group.

The γ_k' are not ordered on the real axis with respect to the y_i' . This means that, as expected, we obtain the sum of amplitude with $M-1$ off-shell particles and the one on-shell associated in the γ_k' in all various possible configurations.

- When $y_j' \rightarrow y_k'$ on real axis we exhibit the poles in the variable $(k_j + p_k)^2$.
- When $y_k' \rightarrow y_j'^*$ on the real axis we exhibit the poles in variable $(p_k + p_j)^2$.

b) Other singularities

Other singularities are also present because the partition function can become zero if one of the multipliers is equal to 1²¹⁾ and if the $a_{k\ell}$ can reach 0. In fact all these limits are obtained simultaneously. We shall prove it explicitly for the two-current amplitude in the next section.

When some y_i' ($i \in I$ subset of $1, \dots, M$) collapse together then $a_{ij} \rightarrow 0$ if i and $j \in I$, $a_{ii} \rightarrow 0$ if $i \in I$ and one $w(\prod_{i \in I} P_i)$ is equal to 1 so that the partition function diverges. (There is only one order of the P_i 's up to a cyclic permutation such that the multiplier of the product can be one.)

In this limit, we have

$$f(w_\alpha)^{-4} \simeq \frac{\text{Log}^2 w_\alpha}{4\pi^2} e^{\frac{2\pi^2}{3(1-w_\alpha)}} \quad (5.9)$$

$$\begin{cases} \text{Log } a_{ij} \simeq -\frac{4\pi^2}{1-w_\alpha} \\ \text{Log } a_{ii} \simeq -\frac{4\pi^2}{1-w_\alpha} \end{cases} \quad (5.10)$$

Then the integrand behaves like

$$\exp\left\{\frac{\pi^2}{1-w_\alpha} \left[\frac{1}{2}(\sum_{i \in I} p_i)^2 + \frac{2}{3}\right]\right\} \equiv \varepsilon_\alpha^{\frac{1}{4}S_\alpha + \frac{1}{3}} \quad (5.11)$$

This corresponds to the so-called Pomeranchuk singularity in the non-planar loop amplitude⁶⁾. Then the amplitude has a cut (because of the presence of logarithms) with branch point at $s_\alpha = -4/3$. This is, of course,

a bad property for the current amplitude, however, one may hope that in a more realistic model, the Pomeron singularity may become a pole, as required by unitarity. It is obvious that some of the bad properties of the conventional model (zero mass ρ , tachyon, cut structure for the Pomeron) are reflected in our current model, and we do not attempt to solve these. One can also escape this problem by considering currents amplitudes with quantum numbers (see Section 9).

5) Two-current amplitude

In order to precise a little bit more our statements, let us write down the two-current amplitude with two on-shell particles (we replace ξ' by z), Fig. 4

$$A(s, t, p_1^2, p_2^2) = \frac{1}{\pi^2} \int_{y_1 < y_2} \frac{dy_1 dy_2}{dV_{abc}} \frac{dy_1 dc_1}{c_1^2} \frac{dy_2 dc_2}{c_2^2} \prod_{\alpha \neq 1, 2} [F(w_\alpha)]^{-4} \\ |y_1 - y_2|^{-k_1 \cdot k_2} |y_1 - z_2|^{-p_2 \cdot k_1} |y_2 - z_1|^{-p_1 \cdot k_2} |y_1 - z_1|^{-p_1 \cdot k_1} |y_2 - z_2|^{-p_2 \cdot k_2} \\ \left(\frac{z_1 - z_1^*}{2i}\right)^{-p_1^2/2} \left(\frac{z_2 - z_2^*}{2i}\right)^{-p_2^2/2} |z_2 - z_1^*|^{-p_1 \cdot p_2} a_{11}^{-p_1^2/4} a_{22}^{-p_2^2/4} a_{12}^{-p_1 \cdot p_2/4} \quad (5.12)$$

with

$$\log a_{12} = \log \frac{(z_1 - z_2)(z_1^* - z_2^*)}{(z_1 - z_2^*)(z_1^* - z_2)} = -4\pi^2 c_1 c_2 \sum_{\alpha \in Q_{12}} \frac{\partial p_\alpha / \partial z |_{z=y_1}}{(y_2 - p_\alpha(y_1))^2} \quad (5.13)$$

$$\log a_{11} = -4\pi^2 c_1^2 \sum_{\alpha \in Q_{11}} \frac{\partial p_\alpha / \partial z |_{z=y_1}}{(y_1 - p_\alpha(y_1))^2} \quad (5.14)$$

$$\log a_{22} = -4\pi^2 c_2^2 \sum_{\alpha \in Q_{22}} \frac{\partial p_\alpha / \partial z |_{z=y_2}}{(y_2 - p_\alpha(y_2))^2} \quad (5.15)$$

The group generated by P_1 and P_2 is now defined (up to a global homographic transformation) by only one invariant parameter

$$\rho = \frac{c_1 c_2}{(\gamma_1 - \gamma_2)^2} > 0 \quad (5.16)$$

Then $\log a_{ij}$ is only function of ρ . This implies in particular since ρ is symmetric with respect to $(\gamma_1, c_1) \leftrightarrow (\gamma_2, c_2)$ that

$$a_{11} = a_{22} \equiv a(\rho) \quad (5.17)$$

For completeness we shall give in the Appendix C, the series expansion of $\log a$ and $\log a_{12}$ in ρ .

As for the two-loop amplitude only one multiplier can reach one

$$w_3 \equiv w(P_2^{-1} P_1^{-1}) [= w(P_1 P_2) = w(P_2 P_1) = w(P_1^{-1} P_2^{-1})] \quad (5.18)$$

it satisfies the equation

$$\sqrt{w_3} + \frac{1}{\sqrt{w_3}} = |2 - \frac{1}{\rho}| \geq 2 \quad (5.19)$$

then since $\rho > 0$ we must have

$$\rho \leq \frac{1}{4} \quad (5.20)$$

and $w_3 = 1$ if $\rho = 1/4$

$$\sqrt{w_3} = \frac{1 - \sqrt{1 - 4\rho}}{1 + \sqrt{1 - 4\rho}} \quad (5.21)$$

If we fix $\gamma_1 c_1 \gamma_2$ we can write

$$dV_{abc} = \frac{dc_1}{c_1} \frac{d\gamma_1 d\gamma_2}{(\gamma_1 - \gamma_2)^2} \quad (5.22)$$

then

$$\frac{1}{dV_{abc}} \frac{dy_1 dc_1 dy_2 dc_2}{c_1^2 c_2^2} = \frac{dc_2}{c_2} \frac{(y_1 - y_2)^2}{c_1 c_2} = \frac{d\rho}{\rho^2} \quad (5.23)$$

ρ is integrated from 0 to $1/4$.

We shall show in the next section that when $\rho \rightarrow 1/4$, then $\log a$ and $\log a_{12} \rightarrow \infty$ as

$$\log a \sim \log a_{12} \underset{\rho \rightarrow \frac{1}{4}}{\sim} -\frac{4\pi^2}{1-w_3} \sim -\frac{\pi^2}{\sqrt{1-4\rho}} \quad (5.24)$$

The complex variables z_1 and z_2 are defined up to a real homographic transformation. Therefore, we can fix three real parameters, for example

$$\begin{cases} z_1 = i x & (x > 0) \\ z_2 = i \end{cases} \quad (5.25)$$

Then we have

$$a_{12} = \left(\frac{1-x}{1+x} \right)^2 \quad (5.26)$$

It is easily seen that when $c_1 = 0$ ($\rho = 0$) then $a_{12} = 1$ so that when ρ goes from 0 to $1/4$ a_{12} goes from 1 to 0, and then x from 0 to 1. This corresponds to the ordering condition $\text{Im } z_1 \leq \text{Im } z_2$. We shall show below that despite this fact the amplitude is symmetric in p_1^2 and p_2^2 as expected. Let us now write the amplitude in terms of s, t, p_1^2, p_2^2 .

$$\begin{aligned}
 A(s, t, p_1^2, p_2^2) = & \int_0^1 dx \mu(x^2) \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{y_1} \frac{dy_2}{(y_1 - y_2)^2} \\
 & \left[\frac{(y_1^2 + x^2)(y_2^2 + 1)}{(y_1^2 + 1)(y_2^2 + x^2)} \right]^{-\frac{s}{4} - \frac{1}{2}} \left[\frac{(y_1 - y_2)^2 (1 - x^2)}{(y_2^2 + x^2)(y_1^2 + 1)} \right]^{-t/4} \\
 & \left(\frac{(y_1^2 + 1)}{(y_2^2 + x^2)} \frac{x^2}{1 - x^2} a^{\frac{1}{2}} \right)^{-p_1^2/4} \left(\frac{(y_2^2 + x^2)}{(y_1^2 + 1)} \frac{1}{1 - x^2} a^{\frac{1}{2}} \right)^{-p_2^2/4}
 \end{aligned}
 \tag{5.27}$$

with

$$\mu(x^2) = \frac{1}{\pi^2} \frac{1}{x^2} \frac{d\varphi}{dx} \left[\prod_{d \neq 1, 2} f(w_d) \right]^{-4}
 \tag{5.28}$$

We can see the symmetry in p_1^2 , p_2^2 by making the change of variables

$$y_1' = \frac{x}{y_2} \quad ; \quad y_2' = \frac{x}{y_1}
 \tag{5.29}$$

6) Propagator for external particles

The propagator of the external particles is obtained from the previous expression (5.12) by dropping the variables of external on-shell particles and choosing the same configuration for off-shell particle variables as in (5.27). We then obtain

$$\mathcal{P}(p^2) = \int_0^1 dx \mu(x^2) \left[\frac{x^2 a(x^2)}{(1 - x^2)^2} \right]^{-p^2/4}
 \tag{5.30}$$

It has poles at $\alpha(p^2) = 0, 2, \dots, 2n$ coming from the singularity of the integrand at $x=0$ (the integrand is only function of x^2).

Let us now study what happens in the neighbourhood of $x = 1$

$$\mu(x^2) \sim C \frac{(\log(1-x))^{-4}}{(1-x)^{4/3}} \quad (5.31)$$

$$\frac{a(x^2)}{(1-x^2)^2} \sim A \text{ finite} \quad (5.32)$$

Then, there is no singularity in p^2 but the propagator $\mathcal{P}(p^2)$ is infinite. Therefore, in order to give a meaning to this propagator we must do some regularization.

- A first way to do it is to subtract some infinite counter-term which is an entire function in p^2 , for example, we can define

$$\bar{\mathcal{P}}(p^2) = \int_0^1 dx \mu(x^2) \left\{ \left[\frac{x^2 a(x^2)}{(1-x^2)^2} \right]^{-D/4} - A^{-D/4} \right\} \quad (5.33)$$

but it is not clear that such a procedure does not affect the construction of multicurrent amplitude via factorization. Moreover, we have a great arbitrariness for the choice of this counter-term.

- A second way is to start from the two-point amplitude, but formally we keep p_1^2 and p_2^2 independent. Then the point $x=1$ is associated with the "Pomeranchuk singularity" in the variable $t = (p_1 + p_2)^2$ and we do an analytic continuation in this variable up to a region including $t=0$.
- Finally we can use a regularization and renormalization procedure which has been developed recently by 't Hooft and Veltman ²⁴⁾ using analytic continuation in the space-time dimension. In this amplitude, if we had worked in a space time of dimension D , the only change should be in the power of the partition function which passes from -4 to $-D$. Then we now have

$$\mu(x^2) \underset{x \rightarrow 1}{\sim} C \frac{(\log(1-x))^{-D/2-2}}{(1-x)^{D/2+1}} \quad (5.34)$$

(Let us remark that this power could be also changed by elimination of spurious states.) Then, if $D < 0$ the propagator as given by (5.30) is finite. By analytic continuation in D up to $D=4$ we then obtain a finite result. We note that such a procedure can also be used in order to regularize the dual multiloop amplitudes.

- Let us finally remark that by using these procedures of analytic continuation, the propagator acquires probably some imaginary part because of the presence of logarithm, but this does not affect the analytic structure in p^2 especially the residues of the poles.

6. PERIOD MATRIX IN THE LIMIT ASSOCIATED WITH THE "POMERANCHUK" SINGULARITY

We want to show in this section that when the multiplier of some elements of the group $P_i P_j \dots P_n$ goes to 1 then the corresponding a_{ij} 's and a_{kk} 's go to zero in a definite way. This will be done using the methods of domain variational theory on Riemann surfaces. In order to understand what is the corresponding problem on the surface, let us study some properties of the period matrix under the change of generators.

1) Symmetry properties of the period matrix

a) Most of the properties of the Abelian integrals depend only on the group and not on the basis of generators. It has already been seen that in the planar multiloop amplitude, different configurations of a same planar diagram can be described using different basis of generators ^{9),11)} In particular, the divergences of the partition function happens when some element of the group which can always be chosen as a generator, becomes parabolic. The study of such a case is then simplified if we choose that element as one of the generators.

In terms of cycles, this change of basis of generators corresponds to a change of basis of cycles D_μ . We must also change the basis of cycles C_ν because we must preserve the definition relation

$$I(C_\mu, D_\nu) = \delta_{\mu\nu} \quad (6.1)$$

If when we go around D_μ , it means that we do the transformation P_μ , then if we take the element $(P_{\mu_1} P_{\mu_2} \dots P_{\mu_\ell})^{-1}$ as a generator, the associated cycle D is homotopic to the curve

$$-D_{\mu_1} - D_{\mu_2} - D_{\mu_3} \dots - D_{\mu_\ell}$$

Then, from their general definitions, we see immediately that the third Abelian integrals are invariant under this change. For the one in complex normalization it is obvious also from the expression in terms of the elements of the group

$$\omega_{ab}(z) - \omega_{ab}(z_0) = \sum_{\alpha} \text{Log} \left(\frac{P_{\alpha}(z) - a}{P_{\alpha}(z) - b} \frac{P_{\alpha}(z_0) - b}{P_{\alpha}(z_0) - a} \right) \quad (6.2)$$

where the sum over α is extended to all the elements of the group.

It is less obvious for the one in real normalization because it is related to the properties of the first Abelian integral and of the period matrix.

b) we shall use here the original expression of first Abelian integrals in terms of the Poincaré θ series ⁸⁾

$$\theta(z, a) = \sum_{\alpha} \frac{1}{(c_{\alpha} z + d_{\alpha})^2} \frac{1}{P_{\alpha}(z) - a} \quad (6.3)$$

where

$$P_{\alpha}(z) = \frac{a_{\alpha} z + b_{\alpha}}{c_{\alpha} z + d_{\alpha}} \quad ; \quad a_{\alpha} d_{\alpha} - b_{\alpha} c_{\alpha} = 1 \quad (6.4)$$

$\phi_{\mu}(z)$ is then defined by

$$d\phi_{\mu}(z) = \theta(z, P_{\mu}(\infty)) dz \quad (6.5)$$

$$\phi_{\mu}(z) - \phi_{\mu}(z_0) = \sum_{\alpha} \text{Log} \left(\frac{P_{\alpha}(z) - P_{\mu}(\infty)}{P_{\alpha}(z_0) - P_{\mu}(\infty)} \right) \quad (6.6)$$

The crucial property of $\theta(z, a)$ that we need is

$$\theta(z, a) - \theta(z, P_\alpha(a)) = \theta(z, P_\alpha^{-1}(\infty)) \quad (6.7)$$

whatever a , then we deduce in particular by choosing

$$\begin{aligned} a' = P_\alpha^{-1}(a) \quad \theta(z, P_\alpha^{-1}(a)) - \theta(z, a) &= \theta(z, P_\alpha^{-1}(\infty)) \\ &= -\theta(z, P_\alpha(\infty)) \end{aligned}$$

$$\theta(z, P_\alpha^{-1}(\infty)) = -\theta(z, P_\alpha(\infty)) \quad (6.8)$$

and $a = P_\beta(\infty)$

$$\theta(z, P_\alpha P_\beta(\infty)) = \theta(z, P_\alpha(\infty)) + \theta(z, P_\beta(\infty)) \quad (6.9)$$

Then, the first Abelian integrals have the following properties.

- If ϕ_μ is associated with the generator P_μ and $\phi_{\mu^{-1}}$ with P_μ^{-1} we have

$$d\phi_\mu(z) = -d\phi_{\mu^{-1}}(z) \quad (6.10)$$

- If $\phi_{\mu_1 \dots \mu_N}$ is associated with $P_{\mu_1 \dots \mu_N} = P_{\mu_1} P_{\mu_2} \dots P_{\mu_N}$ then

$$d\phi_{\mu_1 \mu_2 \dots \mu_N}(z) = d\phi_{\mu_1}(z) + d\phi_{\mu_2}(z) + \dots + d\phi_{\mu_N}(z) \quad (6.11)$$

Now, if we represent the set of M independent first Abelian integrals by a column vector, ϕ , we see that a change of basis is represented by a $M \times M$ matrix S independent of z and its elements are integer number

$$\phi' = S\phi \quad (6.12)$$

We can now compute the period matrix A' in this new basis, we see immediately that it is related to A by

$$A' = S A S^t \quad (6.13)$$

These relations allow us to study the period matrix when one element of the group which is not in the original basis becomes parabolic.

c) Let us apply the previous results to the case of two generators P_1 and P_2 . We define

$$P_3 = (P_1 P_2)^{-1} \quad (6.14)$$

Since there are only three projective invariant parameters to describe the group, we choose them as the multipliers of P_1 , P_2 and P_3 : w_1, w_2, w_3 .

We define

$$P_1(\phi_1) = F(w_1, w_2, w_3) \quad (6.15)$$

since ϕ_1 does not depend on the fact that we have chosen P_2 or P_3 as second generators, $F(w_1, w_2, w_3)$ must be symmetric in w_2, w_3 .

The three transformations are related by

$$P_1 P_2 P_3 = I \quad (6.16)$$

and play therefore a very similar role and then we can write

$$P_2(\phi_2) = F(w_2, w_3, w_1) \quad (6.17)$$

We define also

$$P_1(\phi_2) = P_2(\phi_1) = G(w_1, w_2, w_3) \quad (6.18)$$

which is obviously symmetric in w_1 and w_2 . From the previous section we now have the relations

$$\begin{cases} P_3(\phi) = -P_1(\phi) - P_2(\phi) \\ d(\phi_1 + \phi_2 + \phi_3) = 0 \end{cases}$$

(6.19)

in particular

$$P_3(\phi_3) = F(w_3, w_1, w_2) = P_1(\phi_1) + P_1(\phi_2) + P_2(\phi_1) + P_2(\phi_2)$$

(6.20)

so that G is defined in terms of F by

$$G(w_1, w_2, w_3) = \frac{1}{2} [F(w_3, w_1, w_2) - F(w_1, w_2, w_3) - F(w_2, w_3, w_1)]$$

(6.21)

Therefore, the period matrix is defined only by one function of three variables. This will be useful in the next section, in order to check that the surface has been described in the correct way.

2) Parametrization of the period matrix in the parabolic limit

Let us now study the limit in which we are interested. For simplicity of writing we limit ourselves to the case of two generators, but the proof is easily extended to more general cases, we shall give during the proof the differences occurring in the general case.

We consider a sphere with two handles (or a torus) $\mathcal{M}^*$. We choose as a canonical basis of cycles the curves $C_1 C_2 D_1^0 D_2^0$ or $C_1 C_2 D_1 D_2$ since D_1 (resp. D_2) is homotopic to D_1^0 (resp. D_2^0) (Fig. 5).

The cycle D_3 is not an independent one since $D_1 + D_2 + D_3$ is homotopic to a point. (This corresponds in the language of projective transformation to $P_1 P_2 P_3 = I$.) We are interested in the limit in which the three cycles D_1, D_2, D_3 shrink to a point at the same time.

Let us now apply the general methods of domain variational theory in order to compute the period matrix. We cut the surface $\mathcal{M}^*$ following the three curves $D_1 D_2 D_3$, that leads to a surface $\mathcal{M}$ homotopic to a sphere. In $\mathcal{M}$ the three cycles can be defined in the limit by Jordan curves in the following way (Fig. 6)

$$\begin{cases} D_1 & V(p, p_0, q_1, q_2) = \text{Log } \epsilon_1 \\ D'_1 & V(p, p_0, q_1, q_3) = \text{Log } \epsilon_1 \end{cases}$$

$$\begin{cases} D_2 & V(p, p_0, q_2, q_3) = \text{Log } \epsilon_2 \\ D'_2 & V(p, p_0, q_2, q_1) = \text{Log } \epsilon_2 \end{cases} \quad (6.22)$$

(6.23)

$$\begin{cases} D_3 & V(p, p_0, q_3, q_1) = \text{Log } \epsilon_3 \\ D'_3 & V(p, p_0, q_3, q_2) = \text{Log } \epsilon_3 \end{cases}$$

(6.24)

A_i and B_i denote the intersection points of D_i and D'_i . M^* is built by identifying $A_1 D_1 B_1$ and $A_2 D'_2 B_2$; $A_2 D_2 B_2$ and $A_3 D'_3 B_3$, $A_3 D_3 B_3$ and $A_1 D'_1 B_1$ using the imaginary part of $v(p, p_0, q_i, q_j)$ for example $p \in D_1$ and $p' \in D'_2$ are identified if

$$\begin{cases} v(p, p_0, q_1, q_2) = \text{Log } \epsilon_1 + i \alpha_1(p) \\ v(p', p_0, q_1, q_2) = -\text{Log } \epsilon_2 + i \alpha_1(p') + i \alpha_{12} \end{cases}$$

(6.25)

with α_{12} fixed.

There must be some condition in order that if A_1 and A_2 are identified then B_1 and B_2 are also identified. In general it is wrong but it is true in the limit $\epsilon_i \rightarrow 0$ (up to order ϵ_i).

The points A_i satisfy by definition

$$V(A_i, p_0, q_j, q_k) = V(A_i, p_0, q_i, q_k) - V(A_i, p_0, q_i, q_j) = 0$$

(6.26)

and then in the limit $\epsilon_i \rightarrow 0$ since $A_i \rightarrow q_i$

$$V(q_i, p_0, q_j, q_k) = 0 \quad (6.27)$$

(In the general case we still have some relations but only for consecutive points q_i, q_{i-1}, q_{i+1} .)

Now let us consider the function defined on $\mathcal{W}_6$

$$\Phi_1^*(p) - \Phi_1^*(p_0) = v(p, p_0, q_1, q_3) \quad (6.28)$$

This function is analytic on $\mathcal{W}_6^*$ (since q_1 and q_3 are not on $\mathcal{W}_6^*$) and has zero period along D_2 and $2i\pi$ along D_1 . Therefore it is a first Abelian integral on $\mathcal{W}_6^*$ (up to order ϵ_i). In the same way

$$\Phi_2^*(p) - \Phi_2^*(p_0) = v(p, p_0, q_1, q_2) \quad (6.29)$$

is the first Abelian integral defined by zero period along D_1 and $2i\pi$ along D_2 .

We can now compute their periods along C_1 and C_2 by taking their discontinuity from D_i to D_i' . Since the original surface is symmetric the period matrix is real and we do not take care of the imaginary part

$$\begin{aligned} \Phi_1^*(A_1) - \Phi_1^*(A_3) &= V(A_1, p_0, q_1, q_3) - V(A_3, p_0, q_1, q_3) \\ &= \text{Log } \epsilon_1 + \text{Log } \epsilon_3 \end{aligned}$$

$$\begin{aligned} \Phi_1^*(A_2) - \Phi_1^*(A_3) &= V(A_2, p_0, q_1, q_3) - V(A_3, p_0, q_1, q_3) \\ &= V(q_2, p_0, q_1, q_3) + o(\epsilon_2) + \text{Log } \epsilon_3 \\ &= \text{Log } \epsilon_3 + o(\epsilon_2) \end{aligned} \quad (6.30)$$

(6.31)

$$\begin{aligned}
 \Phi_2^*(A_1) - \Phi_2^*(A_3) &= V(A_1, p_0, q_2, q_3) - V(A_3, p_0, q_2, q_3) \\
 &= V(q_1, p_0, q_2, q_3) + \mathcal{O}(\epsilon_1) + \text{Log } \epsilon_3 \\
 &= \text{Log } \epsilon_3 + \mathcal{O}(\epsilon_1)
 \end{aligned}$$

$$\begin{aligned}
 \Phi_2^*(A_2) - \Phi_2^*(A_3) &= V(A_2, p_0, q_2, q_3) - V(A_3, p_0, q_2, q_3) \quad (6.32) \\
 &= \text{Log } \epsilon_2 + \text{Log } \epsilon_3
 \end{aligned}$$

(6.33)

Then the period matrix is written as

$$A' \simeq \begin{pmatrix} \text{Log } \epsilon_1 + \text{Log } \epsilon_3 & \text{Log } \epsilon_3 \\ \text{Log } \epsilon_3 & \text{Log } \epsilon_2 + \text{Log } \epsilon_3 \end{pmatrix} \quad (6.34)$$

[In the case of N handles we would have (defining $P_{N+1} = (P_1 \dots P_N)^{-1}$)

$$A'_{ii} \simeq \text{Log } \epsilon_i + \text{Log } \epsilon_{N+1} + C_{ii} \quad ; \quad A'_{ij} \simeq \text{Log } \epsilon_{N+1} + C_{ij}$$

where C_{ii} and C_{ij} are finite in the limit $\epsilon_i \rightarrow 0$.]

In order to apply that to the problem in which P_1 , P_2 and P_3 become parabolic in the same time we must do the Jacobi transformation. Then A' is related to the period matrix A defined by periods along D_i , by

$$A' = 4\pi^2 A^{-1} \quad (6.35)$$

recalling that

$$A^{-1} \sim \begin{pmatrix} \frac{1}{\log w_1} - b_{11}(w_3) & -b_{12}(w_3) \\ -b_{12}(w_3) & \frac{1}{\log w_2} - b_{11}(w_3) \end{pmatrix} \quad (6.36)$$

one can now identify

$$\log \varepsilon_3 \sim -4\pi^2 b_{12}(w_3) \quad (6.37)$$

That shows that in the limit ε_i is function only of w_i since all three w 's play the same role from the previous section. Then we deduce

$$\log \varepsilon_i \sim 4\pi^2 \left(\frac{1}{\log w_i} + C \right) \quad (6.38)$$

where C is an unknown constant so that we obtain

$$b_{11}(w_3) \underset{w_3 \rightarrow 1}{\sim} -\frac{1}{\log w_3} - 2C \sim \frac{1}{1-w_3} - 2C - \frac{1}{2} \quad (6.39)$$

$$b_{12}(w_3) \underset{w_3 \rightarrow 1}{\sim} -\frac{1}{\log w_3} - C \sim \frac{1}{1-w_3} - C - \frac{1}{2} \quad (6.40)$$

This proves the statements used for the singularities occurring when

$\eta_i \rightarrow \eta_j$, in particular that this happens when $P_i P_j$ becomes parabolic. Defining ε'_i by

$$\log \varepsilon'_i \log \varepsilon_i = 4\pi^2 \quad (6.41)$$

we find the expression of A

$$A \simeq \begin{pmatrix} \frac{\log \varepsilon'_1 \log \varepsilon'_2 \varepsilon'_3}{\log \varepsilon'_1 \varepsilon'_2 \varepsilon'_3} & -\frac{\log \varepsilon'_1 \log \varepsilon'_2}{\log \varepsilon'_1 \varepsilon'_2 \varepsilon'_3} \\ -\frac{\log \varepsilon'_1 \log \varepsilon'_2}{\log \varepsilon'_1 \varepsilon'_2 \varepsilon'_3} & \frac{\log \varepsilon'_2 \log \varepsilon'_1 \varepsilon'_3}{\log \varepsilon'_1 \varepsilon'_2 \varepsilon'_3} \end{pmatrix} \quad (6.42)$$

we see immediately that it satisfies all the symmetry requirements of the previous section. ε_i' is related to w_i by

$$\text{Log } \varepsilon_i' \approx \frac{\text{Log } w_i}{1 + C \text{Log } w_i} \quad (6.43)$$

In the general case (N) we still have the same kind of behaviour when $w_{N+1} \rightarrow 1$, namely

$$b_{ij} \sim \frac{1}{1 - w_{N+1}} + c_{ij} \quad (6.44)$$

but we have not a complete parametrization because there are now $3N - 3$ parameters.

7. SOME COMMENTS ON THE ELIMINATION OF SPURIOUS STATES

The current amplitudes we have constructed are not free of spurious states. Indeed, by expanding the state $t^+(p_1)|0\rangle$ in eigenstates of L_0 , one can show that at $N=2$ already a spurious state is introduced. However, the construction of multicurrent amplitudes using our method rests on the same assumption as the construction of multiloop amplitudes: one implements first the duality properties of the desired amplitudes, and obtains an integral expression. Then one assumes that the structure of this integral expression remains the same when spurious states are projected out: for instance, in the one-loop case, it has been shown that the terms raised to the power $p_i \cdot p_j$ are unaffected, while the integration measure is modified¹⁵⁾. In that way one finds that the correction to the one-loop planar amplitude is $(1-w)f(w)$. However, because of the technical difficulty of the problem, the analogous problem has not yet been solved for multi-loop amplitudes. It is, however, a likely conjecture that the $p_i \cdot p_j$ ($i \neq j$) terms, which have an interpretation in terms of the analogue model are unaffected by the projection over physical states, while the integration measure has to be changed.

Similarly for the current amplitude, one has to find how it is modified when it is constructed with an operator like

$$\prod_{i \geq 1} \exp(\alpha_i A_i^+) \mathcal{D}(p) t^+(p) |0\rangle \quad (7.1)$$

when the A_i^+ are Virasoro gauges ²⁵⁾: $A_i^+ = L_0 - L_{-i} + i - 1$ instead of being constructed with the operator $\mathcal{D}(p) t^+(p) |0\rangle$ which we have considered in this paper. The one-current amplitudes are identical since the A_i^+ annihilate a tree, however, multicurrent amplitude will be different from the one we have been studying.

It was noticed in I that the same one-current amplitudes are obtained starting from the operator $\mathcal{D}(p) t^+(p) |0\rangle$ or from the operator

$$\int_0^\pi d\theta \sin \theta^{-1-\alpha(p^2)} \exp\left(\frac{p}{2} [\varphi^+(z) + \varphi^+(z^*)]\right) |0\rangle = J^+(p) |0\rangle \quad (7.2)$$

where

$$\varphi^+(z) = \sum_{n=1}^{\infty} \frac{a_n^+}{\sqrt{n}} z^n \quad ; \quad z = 1 - e^{-i\theta} \quad (7.3)$$

$\mathcal{D}(p) t^+(p) |0\rangle$ and $J^+(p) |0\rangle$ are therefore related by Virasoro gauges

$$J^+(p) |0\rangle = \prod_i \exp \alpha_i A_i^+ \mathcal{D}(p) t^+(p) |0\rangle$$

and we can use $J^+(p) |0\rangle$ to see what happens to multicurrent amplitudes. One then obtains Drummond's model (in unrevised version) ²⁾, which is a special case of ours if the coefficients a_{kk} are made equal to one, and the integration measure changed in a suitable way.

So one suspects that using an operator of the type (7.1) or another will merely change the integration measure and the dependence of the "self-energy" terms a_{kk} on the cross ratios

$$\frac{(\gamma'_i - \gamma'_j)(\gamma'^*_i - \gamma'^*_j)}{(\gamma'_i - \gamma'^*_j)(\gamma'^*_i - \gamma'_j)}$$

It is indeed possible to show that this is the case; we outline here the proof. Since the operators $\mathcal{D}(p) t^+(p) |0\rangle$ and $J^+(p) |0\rangle$ are equivalent up to gauges, we can compute an expression of the type

$$\langle 0 | J(p_1) \prod_{i \geq 1} \exp(\alpha_i A_i) V(p_2) \frac{1}{L_0 - 1} \dots V(p_{N-1}) \prod_{j \geq 1} \exp(\beta_j A_j^\dagger) J^\dagger(p_N) | 0 \rangle \quad (7.4)$$

One can shift the A_i to the right and the $A_j^\dagger$ to the left by using their commutation relations with the vertices, propagators and between themselves so that finally one is left with the problem of evaluating the action of the L_{-N} generators on the current $J^\dagger(p) | 0 \rangle$ which we have considered in because $Q^\dagger(Z)$ has simple transformation properties under the L_{-N}

$$[L_{-N}, Q^{\mu\dagger}(z)] = -z^N p^\mu + z^{N+1} \frac{d}{dz} Q^{\mu\dagger}(z) - \sum_{n=1}^{N-1} \sqrt{n} \alpha^{\mu\nu} z^{N-n} \quad (7.5)$$

From this commutation relation one can deduce that

$$(L_{-N} - z^{N+1} \frac{\partial}{\partial z} - z^{*N+1} \frac{\partial}{\partial z^*} - \frac{p^2}{8} g(z, z^*)) \exp\left[\frac{p}{2} (Q^\dagger(z) + Q^\dagger(z^*))\right] | 0 \rangle = 0 \quad (7.6)$$

where g is a function of Z, Z^* . Integrating this differential equation one obtains that a gauge operator $\exp \frac{\alpha}{N} L_{-N}$, acting on $J^\dagger(p) | 0 \rangle$ replaces it by

$$J^\dagger(p) | 0 \rangle = \int_0^\pi d\theta \sin \theta^{-1-\alpha(p^2)} [f(z, z^*, \alpha)]^{-\frac{p^2}{8}} \exp\left[\frac{p}{2} (Q^\dagger(z') + Q^\dagger(z'^*))\right] | 0 \rangle \quad (7.7)$$

where Z' is related to Z by the transformation:

$$z' = [z^{-N} - \alpha]^{-N} \quad (7.8)$$

Therefore, the net effect of including those gauge factors will be to change the self-energy terms and the integration measure: the part of the integrand depending on $p_i \cdot p_j$ ($i \neq j$) will always be obtained from the expression:

$$\langle 0 | \exp\left[\frac{p_1}{2} (Q^-(z_1) + Q^-(z_1^*))\right] V(p_2) X_2^R V(p_3) \dots V(p_{N-1}) \exp\left[\frac{p_N}{2} (Q^+(z_N) + Q^+(z_N^*))\right] | \quad (7.9)$$

and will therefore keep the same expression in terms of cross ratios.

So, one is free to introduce gauge operators to modify the N current amplitudes obtained above. However, at the same time, one would like to keep the property discussed in Section 5 that

$$\lim_{a_{ij} \rightarrow 0} \frac{a_{ii}}{a_{ij}} < \infty$$

and

$$\lim_{z_i \rightarrow z_i^*} a_{ii} = 1$$

It is not known at present if some other scheme satisfies those conditions, although it is rather plausible. If one limits oneself to two-current amplitude, one can certainly fix the gauge so as to obtain a particular function $a(x)$ and this will be important for scaling.

To project out spurious states in, for instance, the Compton amplitude, we consider now that in expression (5.27) $\mu(x^2)$ and $a(x^2)$ are unknown and given by an expansion in x^2

$$x^2 \mu = 1 + \mu_1 x^2 + \dots$$

$$a = 1 - \beta_1 x^2 + \dots$$

To determine the unknown coefficients in the expansion, we expand the integrand in power of x^2 and obtain an expansion of $A(s, t, p_1^2, p_2^2)$

$$A(s, t, p_1^2, p_2^2) \simeq \frac{1}{\alpha(p_1^2) - 2N} R_{2N}(s, t, p_2^2) + \dots \quad (7.10)$$

At the level $\alpha(p_1^2) = 2N$ the operator

$$\mathcal{O}(p_1) t^+(p_1) |0\rangle \simeq \frac{1}{\alpha(p_1^2) - 2N} [|P_{2N}(p_1)\rangle + |S_{2N}(p_1)\rangle] \quad (7.11)$$

where $|P_{2N}(p_1)\rangle$ is a physical state while $|S_{2N}(p_1)\rangle$ is a spurious state. $|P_{2N}\rangle$ and $|S_{2N}\rangle$ can be explicitly computed (at least in principle) for each level. Then one obtains an equation for R_{2N} by identifying with

$$R_{2N}(s, t, p_2^2) = \langle 0 | t(p_2) \mathcal{O}(p_3) \mathcal{O}(p_4) | P_{2N}(p_1) \rangle \quad (7.12)$$

The first non-trivial result using this method is obtained for $N=1$ where there is only one physical scalar state $|P_2(p_1)\rangle$. After some algebra, one obtains

$$\mu_1 = \frac{D+4}{26-D} \quad ; \quad \beta_1 = \frac{40}{26-D} \quad (7.13)$$

Where D is the dimension of space-time, which we can keep arbitrary. In the model presented in this paper, including spurious states, we replace -4 by $-D$ in the exponent of the partition function and the coefficients μ_1^0 , a_1^0 can be computed directly, and one gets:

$$\mu_1^0 = \frac{D+4}{18} \quad ; \quad \beta_1^0 = \frac{4}{3} \quad (7.14)$$

which coincide with the above projected value in the case $D=-4$. Although this result cannot be easily generalized to higher level providing determination of μ and a , it calls for two remarks.

Firstly, if one wants to project out spurious states, both $\mu(x^2)$ and $a(x^2)$ are different from the explicit function given in this paper. One may, however, hope that the properties at $x=1$ will remain the same for projected functions so that the absence of unphysical singularities is kept as a feature of the model.

Secondly, the projected functions μ and a will depend on the number of dimension of space-time. In the one-loop case, this does not affect the asymptotic properties of the amplitude (Regge behaviour). However, here $a(x^2)$ controls the asymptotic behaviour in p^2 and the behaviour in the Bjorken limit so that the presence or absence of scaling will, in a spurious-state-free model of current, depend on the number of dimension of space-time.

Finally, one notices that $D=26$ is a singular case for projecting out spurious states. The above calculation in fact fails at $D=26$ since the physical state $|P_2\rangle$ becomes a zero norm state. This zero norm state decouples from the tree graphs, and, therefore, we have to decouple it from the current also. The difference with the case $D<26$ is that one has firstly to decouple it even from one current amplitude

imposing that the one-current amplitude has zero residue at $\alpha(p^2) = 2$. This can be done by subtracting a pole term. For two-current amplitude, one can thus both change μ , a , and subtract pole terms. When this is done, one obtains $\beta_1 = 2$, $\mu_1 = 0$ for $D = 26$.

Since it seems difficult to obtain a complete solution of the problem of eliminating spurious states, we will leave this problem; however, we will keep in mind that $\mu(x^2)$ and $a(x^2)$ are model dependent if one does not impose elimination of spurious states, and that they still depend on the dimension D of space-time if this elimination is imposed. So we will mention in the next section the asymptotic properties of the current amplitude obtained by keeping μ and $a(x^2)$ arbitrary. A full discussion and derivation of these asymptotic behaviours will be reserved for a forthcoming paper.

8. ASYMPTOTIC PROPERTIES OF THE TWO-CURRENT AMPLITUDE

We consider the Compton amplitude for two scalar-isoscalar currents as given by Eq. (5.27) with $\mu(x^2)$, $a(x^2)$ satisfying the boundary conditions

$$\lim_{x \rightarrow 0} x^2 \mu(x^2) = a(0) = 1 \quad (8.1)$$

$$\lim_{x \rightarrow 1} \frac{a(x^2)}{(1-x^2)^2} = A < \infty \quad (8.2)$$

Let us mention first the results obtained at fixed p_1^2, p_2^2 . One can show that $A(s, t, p_1^2, p_2^2)$ is Regge behaved, both in the t and s channels. One obtains for the large t behaviour:

$$A(s, t, p_1^2, p_2^2) \underset{t \rightarrow \infty}{\simeq} \left[\left(\frac{t}{2} \right)^{\alpha(s)} + \left(-\frac{t}{2} \right)^{\alpha(s)} \right] \Gamma(-\alpha(s)) f(s, p_1^2) f(s, p_2^2) \quad (8.3)$$

where

$$f(s, p^2) = B \left(-\frac{1+\alpha(s)}{2}; \frac{\alpha(p^2)}{2} \right) \quad (8.4)$$

i.e., the large t behaviour is controlled by s channel exchange of Regge poles whose coupling to the current is the same as in the one-current amplitude as was to be expected.

For the large s behaviour, one obtains

$$A(s, t, p_1^2, p_2^2) \underset{s \rightarrow \infty}{\simeq} \left[\left(\frac{s}{2}\right)^{\alpha(t)} + \left(-\frac{s}{2}\right)^{\alpha(t)} \right] \Gamma(-\alpha(t)) f_R(t, p_1^2, p_2^2) \\ + \frac{\left[s^{\alpha_P(t)} + (-s)^{\alpha_P(t)} \right]}{\text{Log}^n |s|} \Gamma(-\alpha_P(t)) A^{-\frac{p_1^2 + p_2^2}{8}} f_P(t)$$

(8.5)

So, as expected, we get a Regge behaviour controlled by both Regge and Pomeron exchange in the t channel, as is the case in the non-planar one-loop diagram. In our model $n=5$ so that the Pomeron appears as a cut and not as a pole, but this depends on the details of the integration measure $\mu(x^2)$. Also, A is a certain number bigger than one so that we have some exponentially behaved form factors in p_1^2, p_2^2 in this model.

The large p_1^2, p_2^2 behaviour is much more model dependent since it depends crucially on the function $a(x^2)$. One can show that unless

$$a(x^2) = (1-x^2)^2$$

(8.6)

one obtains an exponential behaviour in the p_1^2 complex plane, at s, t, p_2^2 fixed. Of course, by redefining the form factors in the one-current amplitude, one can get rid of this exponential behaviour, but then our power-behaved form factors become exponentially behaved, so we will not consider this possibility.

As mentioned in the above section, it seems likely that one can construct a factorizable model with spurious states so as to obtain the desired form of a . It is also possible that it is obtained automatically by eliminating spurious states for $D=26$; indeed $a=(1-x^2)^2$ implies $\beta_1=2$, and the Pomeron is likely to be a pole for that dimension⁵⁾. Since this exponential behaviour seems to be a general feature, one may be tempted to attribute it to a basic non-locality underlying dual resonance model.

In the special case $a(x^2) = (1-x^2)^2$, one obtains

$$A(s, t, p_1^2, p_2^2) \underset{p_1^2 \rightarrow \infty}{\simeq} (-p_1^2)^{-\frac{1}{2}} G(s, t, p_2^2) + (-p_1^2)^{\alpha_P(t)} h(t)$$

(8.7)

and the amplitude is polynomially bounded in the p_1^2 plane. An extra bonus of this particular form of $a(x^2)$ is that one obtains a scaling behaviour in the Bjorken limit:

$$A(s, t, p_1^2, p_2^2) \underset{s \rightarrow \infty}{\simeq} (-p^2)^{\alpha_p(t)} F(t, \omega) \quad (8.8)$$

where $p_1^2 = p_2^2 = p^2$

$$\frac{s}{p^2} = 1 - \omega \quad (8.9)$$

we note, however, that we obtain an anomalous dimension for scaling, and that the exponent of p^2 depends on the momentum transfer. The scaling function $\text{Im} F(0, \omega)$ has an explicit form, behaves like a power of ω^{-1} near threshold and like

$$A \omega^{\alpha(0)} + \mu \omega^{\alpha_p(0)} \quad (8.10)$$

for large values of ω , so that we have consistency with Regge behaviour.

It is also possible that the large p^2 properties of those amplitudes do not have to be taken too seriously, since we have mentioned in I the possibility of constructing a large class of different scalar currents, which can also be excited for large p^2 . Also, we have discussed here only the properties of isoscalar currents while, because of their decoupling from the Pomeron singularity, scalar currents with quantum number will exhibit a different asymptotic behaviour.

9. AND WHAT ABOUT CURRENTS WITH QUANTUM NUMBERS?

It has already been noted that the amplitude studied up to now has more duality properties than required. If the particles have no quantum numbers it is all right since the total amplitude has such properties and can be represented if $\alpha_0 = 1$ by a single integral representation. However, such properties are incompatible with the introduction of quantum numbers especially via Chan-Paton factors¹⁴⁾ and we need amplitudes with only planar duality.

In I, it has been shown that for a one-current amplitude this problem can be solved by cutting the integration region in such a way that asymptotic behaviour in the mass² of the off-shell particle is not

exponential. Furthermore, it has been shown that such an amplitude can be constructed in a factorizable scheme starting from the current operator

$$\langle 0 | t'(p) = \left(\frac{\pi}{2} \right)^{-\alpha(p^2)} \langle 0 | \exp \left[\frac{1}{2} c a | \tilde{\Delta}' | a \rangle + p c a | \tilde{F}'_T | a \rangle \right] \quad (9.1)$$

with

$$\tilde{\Delta}' = \sum_{n \neq 0} \left[\frac{1}{2n} \right] M_+ \left[-\frac{1}{2n} \right] \quad (9.2)$$

$$\tilde{F}'_T = - \sum_{n \neq 0} \left[\frac{1}{2n} \right] \quad (9.3)$$

As previously, it can be written as

$$\langle 0 | t'(p) = \left(\frac{\pi}{2} \right)^{-\alpha(p^2)} \langle 0 | \exp \left\{ \frac{1}{2} c a | \Gamma \left(\sum_{n \neq 0} p^{2n} \right) | a \rangle + p c a | \Gamma \left(\sum_{n \neq 0} p^{2n} \right) | \infty \rangle \right\} \quad (9.4)$$

where P is the same as for the previous current operator without quantum numbers

$$\hat{P}(z) = z + 1 \quad (9.5)$$

Then, starting with this operator, we can, as previously, construct multicurrent amplitudes. All the algebraic manipulations are exactly the same. All that we must keep in mind is that the integration variables are defined with respect to the projective transformation P_1 , P_2 while the group is generated by P_1^2 and P_2^2 for a two-current amplitude for example. In particular, this tells us that the integration region for the external variables is

$$\dots y_2 \leq \xi_{12} \leq y_0 \leq \dots y_i \dots \leq P_1 P_2(y_0) \leq y_{12} \leq y_1 \leq P_2(y_0) \leq y_2 \dots$$

$$(\xi_{12} \text{ and } \eta_{12} \text{ are the invariant points of } P_1 P_2). \quad (9.6)$$

But, now, since we have lost voluntarily some unrequired duality properties, we have no reason, as before, to restrict ourselves to a fundamental region of $P_1 P_2$ for y_0 and, therefore, we must only respect this ordering (and, of course, keep three points fixed).

Let us remember also the presence of the partition function which is defined by the group generated by P_1^2 and P_2^2 . In principle, that provides a restriction of the integration domain different from those associated with the currents without quantum number. However, we have seen that, in fact, this condition was automatically fulfilled by construction, then the integration region for the internal variable (defined by P_1, P_2) is the same. Moreover, we can see that in this limit ($w_{12} \rightarrow 1$) $\xi_{12} \rightarrow \eta_{12}$ and then the integration region of the external variables goes to zero.

If we want an amplitude where the two currents are not adjacent, it is sufficient to integrate some external variables between ξ_{12}, η_{12} and the others between ξ_{21} and η_{21} (which are the invariant points of $P_2 P_1$) related to ξ_{12} and η_{12} by

$$\begin{cases} \xi_{21} = P_1^{-1}(\xi_{12}) \\ \eta_{21} = P_1^{-1}(\eta_{12}) \end{cases} \quad (9.7)$$

This can be generalized for a n current amplitude described by means of n projective transformation $P_1, \dots, P_n$. Let us denote

$$P_{i,i-1} = P_i P_{i+1} \dots P_n P_1 P_2 \dots P_{i-1} \quad (9.8)$$

with invariant points $\xi_{i,i-1}$ and $\eta_{i,i-1}$.

The external variables are now integrated in the following ordering

$$\begin{aligned} \dots \leq \xi_{1n} \leq \dots \eta_{1n} \leq \eta_1 \leq \xi_{21} \leq \eta_{21} \leq \eta_2 \leq \dots \\ \dots \leq \xi_{i,i-1} \leq \eta_{i,i-1} \leq \eta_i \leq \xi_{i+1,i} \leq \eta_{i+1,i} \leq \eta_{i+1} \leq \xi_{i+2,i+1} \dots \end{aligned}$$

(9.9)

The internal variables are integrated in the same region as previously.

Now we can do the Jacobi transformation on the external variables, but we must remember that it is associated now to the group generated by $P_1^2, \dots, P_n^2$, so that the points $\xi_{i-1,i}$ and $\eta_{i,i-1}$ are ordinary points for it. Therefore, the integration region of the new variables y_i' 's is just the transformed of that of the y_i 's.

$$\dots \leq \xi'_{1n} \leq \eta'_{1n} \leq \xi'_{2n} \leq \eta'_{2n} \leq \dots \leq \xi'_{i,i-1} \leq \eta'_{i,i-1} \leq \xi'_{i+1,i} \leq \dots \quad (9.10)$$

The invariant points η_i are mapped onto two complex points z_i' and $z_i'^*$. Hence, that provides some order on the variables of the on-shell particles as well as those of the off-shell particles (we associate the i th off-shell particle with the two points $\eta'_{i,i-1}$ and $\xi'_{i+1,i}$ as well as with z_i') (Fig. 7).

Now the complex variables z_i' are not integrated in all the half complex plane. Indeed, since the integration domain of the variable (η_{ic_i}) is the same as for the amplitude without quantum numbers and that the function a_{ij} and a_{ii} as well as the partition function are defined by $P_1^2, \dots, P_n^2$. These quantities cannot reach 0 and, in particular

$$a_{ij} \geq a_{ij}^0 \neq 0 \quad (9.11)$$

then it means that z_i' cannot collapse with z_j' and, therefore, there are no singularities of the previous type in the multicurrent channels.

However, let us note that it seems very hard to precise more the integration domain of the z_i' 's.

Finally, let us remark that this scheme provides some natural way of ordering points in the complex plane. A solution of this problem has recently been proposed by Dorren ²⁶⁾, but it is not clear that it is compatible with factorization, let us note that this solution cannot coincide with the one described here since, already at the level of one-current amplitude, the two proposed amplitudes are different.

Before leaving the problem of quantum numbers, let us say that we can use these amplitudes even if there are no particles with quantum numbers. Of course, this new extrapolation off-mass shell is different from the one considered up to now since there is no "Pomeranchuk singularity". This does not mean that the "Pomeron" does not couple, but only to higher order (by including non-planar loop). The choice between the two amplitudes must be done on physical arguments like the asymptotic behaviour on the mass variable. We hope to answer this question in a near future.

10. CONCLUDING REMARKS

The main advantage of our model, despite of its mathematical sophistication, is its internal consistency. We start building multicurrent amplitudes from a current operator which has been derived from the self-energy non-planar operator. Hence, we ensure factorization in the original spectrum, duality properties, and absence of unphysical singularities from the start. This last property has been explicitly checked in this paper. The model also provides a natural way to introduce Chan-Paton factors for off-shell amplitude. We note also that it is possible to extend the scheme for $\alpha(0) \neq 1$. The above properties remain valid, the drawback being that the integration measure now contains additional terms, mixing "old" y_i and "new" y_i' variables, which are rather inelegant.

We note that this construction of off-shell amplitude should generalize to other dual models ²⁷⁾. Hopefully in a hypothetical dual model where the Pomeron singularity is a pole it should also be obtained as a pole for isoscalar scalar off-shell amplitudes. In general, the unphysical properties of our off-shell amplitudes seem to be directly related from those of the dual model from which they stem rather than from their construction. If off-shell dual amplitudes can be obtained for vector currents satisfying the constraints of current algebra and, eventually also of scaling light-cone behaviour, they could be identified with physical currents. However, we expect that it is possible to obtain such amplitudes only from a realistic model of hadrons. The present off-shell amplitudes can be used internally in the dual model for solving the problem met in the loop diagrams of computing wave-function renormalization effects.

It had been conjectured previously that the dual resonance model was based on a non-local field theory because of the presence of the dual Pomeron ²⁸⁾. An independent indication of this is obtained here since

we obtain amplitudes with momentum growths in p^2 of index one, while strictly localizable theories require the index to be smaller than $\frac{1}{2}$ ²⁹⁾. (However, this theorem requires the absence of unstable particles, an assumption obviously violated by the D.R.M.) We hope, therefore, that the construction of off-shell amplitudes for the dual resonance model may help also to clarify its relationship with field theory.

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APPENDIX A - A USEFUL IDENTITY

If P is a general projective transformation of invariant points ξ, γ and multiplier W then we have the following formula

$$\langle x | \prod_{n \neq 0} P^n | x \rangle = \langle x | \prod_{n \neq 0} P^n (|\gamma\rangle + |\xi\rangle) \quad (\text{A.1})$$

Proof

Let us compute explicitly the two members

$$A \equiv \langle x | \prod_{n \neq 0} P^n | x \rangle = - \sum_{n \neq 0} \text{Log} \left(\frac{1 - \frac{x}{\hat{P}^n(x)}}{1 - \frac{x}{\hat{P}^n(0)}} \right) \quad (\text{A.2})$$

using

$$\hat{P}(x) = \frac{(\xi - W\gamma)x - \xi\gamma(1-W)}{(1-W)x - \gamma + W\xi} \quad (\text{A.3})$$

we obtain

$$\begin{aligned} A &= - \sum_{n \neq 0} \text{Log} \left(\frac{-(x-\gamma)(x-\xi)\xi\gamma(1-W^n)^2}{(-(\gamma - W^n \xi)x + \xi\gamma(1-W^n))((\xi - W^n \gamma)x - \xi\gamma(1-W^n))} \right) \\ &= - \sum_{n \neq 0} \text{Log} \left(\frac{(x-\gamma)(x-\xi)\xi\gamma(1-W^n)(1-W^{-n})}{[(\xi - W^n \gamma)x - \xi\gamma(1-W^n)][(\gamma - W^{-n} \xi)x - \xi\gamma(1-W^{-n})]} \right) \\ &= -2 \sum_{n > 0} \text{Log} (C F(n) F(-n)) \end{aligned} \quad (\text{A.4})$$

with

$$C = (x-\gamma)(x-\xi)\xi\gamma \quad (\text{A.5})$$

$$F(n) = \frac{1 - w^n}{(\xi - w^n \eta) x - \xi \eta (1 - w^n)} \quad (A.6)$$

$$B \equiv \langle x | \Gamma \sum_{n \neq 0} P^n (|y\rangle + |\xi\rangle) = - \sum_{n \neq 0} \text{Log} \left(\frac{(1 - \frac{x}{\xi})(1 - \frac{x}{\eta})}{(1 - \frac{x}{\xi \eta})^2} \right) \quad (A.7)$$

$$\begin{aligned} B &= - \sum_{n \neq 0} \text{Log} \left(\frac{(x - \eta)(x - \xi) \xi \eta (1 - w^{-n})^2}{[(\xi - w^{-n} \eta) x - \xi \eta (1 - w^{-n})]^2} \right) \\ &= - \sum_{n \neq 0} \text{Log} (C F^2(-n)) \\ &= - \sum_{n > 0} \text{Log} (C^2 F^2(-n) F^2(n)) \end{aligned}$$

$$B = - 2 \sum_{n > 0} \text{Log} (C F(-n) F(n))$$

(A.8)

then $B \equiv A$.

When P becomes parabolic, we must take together the terms n and $-n$ in order for the expression A and B be defined. The formula is then written

$$\langle x | \Gamma \sum_{n > 0} (P^n + P^{-n}) | x \rangle = 2 \langle x | \Gamma \sum_{n > 0} (P^n + P^{-n}) | y \rangle \quad (A.9)$$

Let us note that this formula, however, does not mean that we can linearize every quadratic form in annihilation operator because A is not linear in $\langle x |$

$$(\langle x | + \langle y |) \sum_{n \neq 0} P^n (|x\rangle + |y\rangle) \neq \langle x | \sum_{n \neq 0} P^n |x\rangle + \langle y | \sum_{n \neq 0} P^n |y\rangle$$

APPENDIX B - DIRECT COMPUTATION OF (4.99)

We shall compute

$$D = \sum_{\alpha \notin \mathcal{P}_1} \text{Log} \left(\frac{y_1 - P_\alpha(y_1)}{y_1 - P_\alpha(y_0)} \frac{y_0 - P_\alpha(y_0)}{y_0 - P_\alpha(y_1)} \right) \quad (\text{B.1})$$

by an average limiting process. More precisely, we write

$$D = \frac{1}{4} \left\{ \lim_{\substack{x \rightarrow y_1 + i\varepsilon \\ y \rightarrow y_1 + i\varepsilon}} + \lim_{\substack{x \rightarrow y_1 + i\varepsilon \\ y \rightarrow y_1 - i\varepsilon}} + \lim_{\substack{x \rightarrow y_1 - i\varepsilon \\ y \rightarrow y_1 + i\varepsilon}} + \lim_{\substack{x \rightarrow y_1 - i\varepsilon \\ y \rightarrow y_1 - i\varepsilon}} \right\} \\ \left\{ \sum_{\alpha \neq I} \text{Log} \left(\frac{x - P_\alpha(y)}{x - P_\alpha(y_0)} \frac{y_0 - P_\alpha(y_0)}{y_0 - P_\alpha(y)} \right) - \sum_{n \neq 0} \text{Log} \left(\frac{x - P_1^n(y)}{x - P_1^n(y_0)} \frac{y_0 - P_1^n(y_0)}{y_0 - P_1^n(y)} \right) \right\} \quad (\text{B.2})$$

$$D = \frac{1}{4} \left\{ \lim_{\substack{x', x'' \rightarrow y_1' \\ y', y'' \rightarrow y_1'}} + \lim_{\substack{x', x'' \rightarrow y_1' \\ y', y'' \rightarrow y_1'^*}} + \lim_{\substack{x', x'' \rightarrow y_1'^* \\ y', y'' \rightarrow y_1'}} + \lim_{\substack{x', x'' \rightarrow y_1'^* \\ y', y'' \rightarrow y_1'^*}} \right\} \\ \left\{ \text{Log} \left(\frac{x' - y'}{x - y} \frac{x - y_0}{x' - y_0'} \frac{y_0 - y}{y_0' - y'} \frac{dy_0'}{dy_0} \right) \right. \\ \left. - \text{Log} \left(\frac{x'' - y''}{x - y} \frac{x - y_0}{x'' - y_0'} \frac{y_0 - y}{y_0' - y''} \frac{dy_0''}{dy_0} \right) \right\} \quad (\text{B.3})$$

where x' is the Jacobi transform of x for the whole group and x'' is the one only for the group generated by P_1

$$D = \frac{1}{4} \lim_{\substack{x' \rightarrow y_1', y' \rightarrow y_1'^* \\ x'' \rightarrow y_1', y'' \rightarrow y_1'^*}} \text{Log} \left(\frac{x' - y_1'}{x'' - y_1'} \frac{y_1' - y_1'^*}{y_1'' - y_1'^*} \right) \\ + \text{Log} \left(\frac{(y_1 - y_0)^2}{(y_1' - y_0)(y_1'^* - y_0)} \frac{y_1' - y_1'^*}{2i\pi C_1} \frac{dy_0'}{dy_0} \right) \quad (\text{B.4})$$

In order to evaluate the first part we shall use explicitly the Jacobi transformation

$$-\frac{1}{2i\pi} \operatorname{Log} \left(\frac{y'' - y_1'}{y'' - y_1'^*} \frac{y_0' - y_1'^*}{y_0' - y_1'} \right) = C_1 \left(\frac{1}{y - y_1} - \frac{1}{y_0 - y_1} \right) \quad (\text{B.5})$$

$$-\frac{1}{2i\pi} \operatorname{Log} \left(\frac{y' - y_1'}{y' - y_1'^*} \frac{y_0' - y_1'^*}{y_0' - y_1'} \right) = C_1 \sum_{\alpha \in Q_1} \left(\frac{1}{y - P_\alpha(y_1)} - \frac{1}{y_0 - P_\alpha(y_1)} \right) \frac{\partial P_\alpha}{\partial z} \Big|_{z=y_1} \quad (\text{B.6})$$

Q_1 is the sum of all elements not of form $T_\alpha T_1^r$. It can be written as

$$Q_1 = Q_{11} + (P_1 - I)Q_{11} + I \quad (\text{B.7})$$

We have

$$\lim_{x \rightarrow y_1 + i\epsilon} \operatorname{Log} \frac{x' - y_1'}{x'' - y_1'} = -2i\pi C_1 \lim_{x \rightarrow y_1 + i\epsilon} \sum_{\substack{\alpha \in Q_1 \\ \alpha \neq I}} \frac{\partial P_\alpha}{\partial z} \Big|_{z=y_1} \left(\frac{1}{y - P_\alpha(y_1)} - \frac{1}{y_0 - P_\alpha(y_1)} \right) \quad (\text{B.8})$$

$$\lim_{y \rightarrow y_1 - i\epsilon} \operatorname{Log} \frac{y' - y_1'}{y'' - y_1'^*} = 2i\pi C_1 \lim_{y \rightarrow y_1 - i\epsilon} \sum_{\substack{\alpha \in Q_1 \\ \alpha \neq I}} \frac{\partial P_\alpha}{\partial z} \Big|_{z=y_1} \left(\frac{1}{y - P_\alpha(y_1)} - \frac{1}{y_0 - P_\alpha(y_1)} \right) \quad (\text{B.9})$$

in the sum Q_1 , the part Q_{11} gives a function regular when $x \rightarrow y_1$ and, therefore, this part cancels when we sum the two expressions. Then,

$$\begin{aligned} & \lim_{\substack{x \rightarrow y_1 + i\epsilon \\ y \rightarrow y_1 - i\epsilon}} \operatorname{Log} \left(\frac{x' - y_1'}{x'' - y_1'} \frac{y' - y_1'}{y'' - y_1'^*} \right) = \\ & -2i\pi C_1 \lim_{\substack{x \rightarrow y_1 + i\epsilon \\ y \rightarrow y_1 - i\epsilon}} \sum_{r \neq 0} \sum_{\beta \in Q_{11}} \frac{\partial (P_1^r P_\beta(z))}{\partial z} \Big|_{z=y_1} \left(\frac{1}{x - P_1^r P_\beta(y_1)} - \frac{1}{y - P_1^r P_\beta(y_1)} \right) \end{aligned} \quad (\text{B.10})$$

Taking together the terms r and $-r$ we can sum over r using

$$\sum_{n>0} \frac{1}{z^2 - n^2} = \frac{\pi \cotg \pi z}{2z} - \frac{1}{2z^2}$$

(B.11)

Then, taking the limit we see that the two parts are purely imaginary and, therefore, do not cancel. We obtain finally

$$\begin{aligned} \lim_{\substack{x \rightarrow y_n + i\varepsilon \\ y \rightarrow y_n - i\varepsilon}} \log \left(\frac{x' - y_n}{x'' - y_n} \frac{y' - y_n^*}{y'' - y_n^*} \right) &= -4\pi^2 c_1^2 \sum_{\beta \in \mathcal{Q}_{11}} \frac{\partial T_\beta}{\partial z} \Big|_{z=y_n} \frac{1}{(y_n - T_\beta(y_n))^2} \\ &= -4\pi^2 b_n \end{aligned} \quad (\text{B.12})$$

We then prove the result (4.99) in this way.

APPENDIX C - EXPANSION OF $\log a_{12}$ AND $\log a_{11}$

Let us remember that a_{12} and a_{11} are defined by

$$\log a_{12} = -4\pi^2 c_1 c_2 \sum_{\alpha \in Q_{12}} \frac{\partial P_\alpha / \partial z |_{z=y_1}}{(y_2 - P_\alpha(y_1))^2} \quad (C.1)$$

$$\log a_{11} = -4\pi^2 c_1^2 \sum_{\beta \in Q_{11}} \frac{\partial P_\beta / \partial z |_{z=y_1}}{(y_1 - P_\beta(y_1))^2} \quad (C.2)$$

They are only functions of the invariant parameter

$$\rho = \frac{c_1 c_2}{(y_1 - y_2)^2} \quad (C.3)$$

Then we can choose a particular representation of the group and take

$$P_2(z) = z - 1 \quad (C.4)$$

$$P_1(z) = \frac{\rho z}{z + \rho} \quad (C.5)$$

Noting

$$P_\alpha(z) = \frac{a_\alpha z + b_\alpha}{c_\alpha z + d_\alpha} \quad (C.6)$$

we obtain

$$\log a_{12} = -4\pi^2 \rho \sum_{\alpha \in Q_{12}} \frac{a_\alpha d_\alpha - b_\alpha c_\alpha}{d_\alpha^2} \quad (C.7)$$

$$\log a_{\alpha} = -4\pi^2 p^2 \sum_{\beta \in \mathcal{Q}_{\alpha}} \frac{a_{\beta} d_{\beta} - b_{\beta} c_{\beta}}{b_{\beta}^2} \quad (C.8)$$

We compute b_{α} and c_{α} following a recursive way. P_{α} is of the type

$$P_{\alpha_n} = P_1^{s_1} P_2^{r_1} \dots P_1^{s_n} P_2^{r_n} = P_1^{s_n} P_2^{r_n} P_{\alpha_{n-1}} \quad (C.9)$$

then, we can show that

$$a_{\alpha_n} d_{\alpha_n} - b_{\alpha_n} c_{\alpha_n} = \frac{p^2}{r_n^2 s_n^2} (a_{\alpha_{n-1}} d_{\alpha_{n-1}} - b_{\alpha_{n-1}} c_{\alpha_{n-1}}) \quad (C.10)$$

and

$$d_{\alpha_n} = d_{\alpha_{n-1}} \left(1 - \frac{p}{r_n s_n}\right) - \frac{p}{r_{n-1} s_{n-1}} \left(d_{\alpha_{n-1}} + p \frac{d_{\alpha_{n-2}}}{r_{n-1} s_{n-1}}\right) \quad (C.11)$$

with

$$d_{\alpha_0} = 1, \quad d_{\alpha_1} = 1 - \frac{p}{r_1 s_1} \quad (C.12)$$

therefore, d_{α_n} is a polynomial of degree n in p

$$d_{\alpha_n} = \sum_{p=0}^n A_p^{\alpha_n} p^p \quad (C.13)$$

Then, we can solve these relations

$$a_{\alpha_n} d_{\alpha_n} - b_{\alpha_n} c_{\alpha_n} = \frac{p^{2n}}{(r_n s_n \dots r_1 s_1)^2} \quad (C.14)$$

$$A_p^{d_n} = (-1)^p \sum_{\left\{ \begin{array}{l} 1 \leq i_1 < \dots < i_p \leq n \\ \varepsilon_{i_k} = 0, -1 \\ i_k + \varepsilon_{i_k} + i_{k+1} + \varepsilon_{i_{k+1}} \end{array} \right\}} \frac{1}{r_{i_1} r_{i_2} \dots r_{i_p} s_{i_1 + \varepsilon_{i_1}} s_{i_2 + \varepsilon_{i_2}} \dots s_{i_p + \varepsilon_{i_p}}} \quad (C.15)$$

P_β is of the type

$$P_{\beta_n} = P_2^e P_{\alpha_n} \quad (C.16)$$

then

$$a_{\beta_n} d_{\beta_n} - b_{\beta_n} c_{\beta_n} = \frac{1}{\ell^2} (a_{d_n} d_{d_n} - b_{d_n} c_{d_n}) \quad (C.17)$$

$$b_{\beta_n} = d_{\alpha_n} - \frac{\rho}{\ell s_n} \left(d_{\alpha_n} + \rho \frac{d_{d_{n-1}}}{r_n s_n} \right) \quad (C.18)$$

Then, expanding $d_{\alpha_n}^{-2}$ and $d_{\beta_n}^{-2}$ in powers of ρ , summing over the r_n , s_n ($\neq 0$) and using the relation

$$\sum_{\gamma=0}^{\inf(\alpha, \beta)} \frac{(N-\gamma+1)! (-1)^\gamma}{(\alpha-\gamma)! (\beta-\gamma)! \gamma!} = \frac{(N-\alpha+1)! (N-\beta+1)!}{\alpha! \beta! (N-\alpha-\beta+1)!} \quad (C.19)$$

we obtain

$$= \sum_{n=0}^{\infty} (2\rho)^{2n} \sum_{N=0}^{\infty} \rho^{2N} \sum_{k_1 + \dots + k_{2n-1} = N} \prod_{i=0}^{2n-1} \frac{\mathfrak{S}(2k_i + 2k_{i+1} + 2) (2k_i + 2k_{i+1} + 1)!}{(2k_i + 1)! (2k_{i+1})!} \quad (C.20)$$

where

$$\mathfrak{S}(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \quad (\alpha \geq 2) \quad (C.21)$$

and we have put $k_0 = k_n = 0$.

$$= \sum_{n=0}^{\infty} (2\rho)^{2n+1} \sum_{N=0}^{\infty} \rho^{2N} \sum_{k_1+\dots+k_{2n}=N} \prod_{i=0}^{2n} \frac{\Gamma(2k_i+2k_{i+1}+2) (2k_i+2k_{i+1}+1)!}{(2k_i+1)! (2k_{i+1})!} \quad (C.22)$$

where we have put $k_0 = k_{n+1} = 0$.

Unfortunately, we have not been able up to now to prove directly using these series that they diverge at $\rho = \frac{1}{4}$. However, they can be used for a numerical study, this will be useful for the discussion of asymptotic properties.

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FIG.1

FIG.2

FIG.3

FIG.4

FIG.5

FIG.6

FIG.7