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IMPROVEMENT OF THE FROISSART BOUND : CROSS-SECTIONS
ESSENTIALLY CONSTANT UP TO ISR ENERGIES

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ABSTRACT

By simultaneous use of unitarity in all three channels we are able to show that total $\pi\pi$ cross-sections are bounded by a constant $\simeq 130$ mb, inelastic ones by 31 mb, up to energies of ~ 300 GeV, and that it is only above that than the logarithmic increase permitted by the Froissart bound could begin to show up.

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In recent papers ¹⁾⁻⁴⁾ it has been shown that knowledge of the t channel D wave scattering length a_2^t enables one to put absolute bounds on averaged total and inelastic cross-sections in the s channel. These bounds are valid at all energies and can therefore be tested against experiment. At high energies, such bounds imply the Froissart bounds ⁵⁾

$$\sigma_{tot}(s) \lesssim \frac{\pi}{\mu^2} \log^2(s/s_0); \quad \sigma_{inel}(s) \lesssim \frac{\pi}{4\mu^2} \log^2(s/s_0). \quad (1)$$

We consider $\pi^0 \pi^+$ scattering for definiteness and will henceforth take units where mass of pion $= \mu = 1$; later we will extend the analysis to other processes. Even in their most improved versions ⁴⁾ these absolute bounds are still one or two orders of magnitude larger than "experimental" cross-sections in the 10 GeV region, and hopelessly large at hundred GeV.

In this note we want to show that it is possible to improve greatly upon the bounds of Refs. 1) to 4) by using some more t channel information. The point is, to obtain the bounds of Refs. 1) to 4), one uses the fact that the quantity

$$a_2^t = \lim_{t \rightarrow 4} f_2(t)/(t-4)^2$$

is finite (f_2 is the t channel D wave). However, we have more information available: due to elastic unitarity in the t channel, not only a_2^t but also the quantities

$$h_2' \equiv \lim_{t \rightarrow 4} \frac{d}{dt} \left[\frac{f_2(t)}{(t-4)^2} \right], \quad h_2'' \equiv \lim_{t \rightarrow 4} \frac{d^2}{dt^2} \left[\frac{f_2(t)}{(t-4)^2} \right]$$

are finite. Although this does not improve on the Froissart bound for $s \rightarrow \infty$, this new information implies that the scale factor s_0 in (1) increases with the energy, roughly as $s_1 \log^7(s/s_2)$, s_1, s_2 constants. Therefore the logarithmic increase in the Froissart bound will not become apparent until the multi-GeV region, where $s \gg s_1 \log^7(s/s_2)$. Detailed results will be found at the end of this note; roughly speaking, what we have is that total cross-sections are effectively bounded by a constant, $\sigma_{tot} < 130$ mb, and inelastic ones by $\sigma_{inel} < 32$ mb, up to values of $p_{lab} \simeq 300$ GeV/c. The relevance of these results is enhanced by recent theoretical arguments ⁶⁾ and experimental hints [both from ISR and cosmic rays ⁷⁾] of a $\log^2$ increase in total and inelastic cross-sections.

The details of the argument are as follows : taking $\pi^0 \pi^+$ scattering as stated before, we have, for $0 \leq t \leq 4$ the Froissart-Gribov representation

$$f_2(t) = \frac{4}{\pi(4-t)} \int_4^\infty ds F_s(s,t) Q_2\left(\frac{2s+t-4}{4-t}\right), \quad (2)$$

where F_s is the s channel absorptive part that can be expanded as

$$F_s(s,t) = \sqrt{\frac{s}{s-4}} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell\left(1 + \frac{2t}{s-4}\right) \alpha_\ell(s), \quad (3)$$

$$\alpha_\ell(s) \equiv [(s-4)/s]^{1/2} \text{Im} f_\ell(s),$$

and, due to s channel unitarity, the α_ℓ satisfy

$$0 \leq \alpha_\ell(s) \leq 1. \quad (4)$$

On the other hand, t channel elastic unitarity implies that we can write

$$f_2^{(I)}(t) = \frac{(t-4)^2 t^{1/2}}{-i(t-4)^{5/2} + \psi^{(I)}(t) t^{1/2}}, \quad (5)$$

$\psi^{(I)}$ analytic around $t=4$. I is the isospin index, $I=0,2$ and

$$f_2(t) \equiv f_2^{\pi^0 \pi^0 \rightarrow \pi^+ \pi^-}(t) = \frac{2}{3} [f_2^{(0)}(t) - f_2^{(2)}(t)] \quad (6)$$

From (5) and (6), it is then clear that one can differentiate $h_2(t) \equiv f_2(t)/(t-4)^2$ one or two times and still get a finite result as $t \rightarrow 4$.

Note, however, that $h_2'''(t=4) = \infty$, so in this sense our results are the best that can be obtained by these methods. Equating the values of $h_2^{(n)}(t=4)$, $n=0,1,2$ obtained from (5), (6) and from (2) and (3), we get the relations

$$\gamma_0 = \int_4^\infty ds \frac{s^{1/2}}{s^3 (s-4)^{1/2}} \sum_0^\infty (2\ell+1) P_\ell\left(\frac{2}{3}\right) \alpha_\ell(s), \quad (7.a)$$

$$\begin{aligned} \gamma_1 = & \int_4^\infty ds \frac{s^{1/2}}{s^3 (s-4)^{3/2}} \sum_1^\infty (2\ell+1) P_\ell'(z) \alpha_\ell'(s) \\ & - \frac{3}{4} \int_4^\infty ds \frac{s^{1/2}}{s^4 (s-4)^{1/2}} \sum_0^\infty (2\ell+1) P_\ell(z) \alpha_\ell(s), \end{aligned} \quad (7.b)$$

$$\begin{aligned} \gamma_2 = & \int_4^\infty ds \frac{s^{1/2}}{s^3 (s-4)^{5/2}} \sum_2^\infty (2\ell+1) P_\ell''(z) \alpha_\ell(s) \\ & - \frac{3}{4} \int_4^\infty ds \frac{s^{1/2}}{s^4 (s-4)^{3/2}} \sum_1^\infty (2\ell+1) P_\ell'(z) \alpha_\ell(s) \\ & + \frac{10}{7} \int_4^\infty ds \frac{s^{1/2}}{s^5 (s-4)^{1/2}} \sum_0^\infty (2\ell+1) P_\ell(z) \alpha_\ell(s), \end{aligned} \quad (7.c)$$

where $z \equiv (s+4)/(s-4)$ and

$$\begin{aligned} \gamma_0 \equiv & 10\pi \{ h_2^{(0)}(4) - h_2^{(2)}(4) \}, \quad \gamma_1 \equiv 5\pi \{ h_2^{(0)'}(4) - h_2^{(2)'}(4) \}, \\ \gamma_2 \equiv & \frac{5\pi}{2} \{ h_2^{(0)''}(4) - h_2^{(2)''}(4) \}, \end{aligned} \quad (8.a)$$

and

$$\begin{aligned} h_2^{(1)}(4) = & 1/\psi^{(1)}(4), \quad h_2^{(1)'}(4) = -\psi^{(1)'}(4)/[\psi^{(1)}(4)]^2, \\ h_2^{(1)''}(4) = & -\psi^{(1)''}(4)/[\psi^{(1)}(4)]^2 + 2[\psi^{(1)'}(4)]^2/[\psi^{(1)}(4)]^3. \end{aligned} \quad (8.b)$$

We have used positivity to differentiate Eq. (2) under the integration sign. In Refs. 1) to 4) only the contents of Eq. (7.a) were used; our improvement comes from using also (7.b) and (7.c). One can see this intuitively as follows : Eq. (7.c) implies that $\sum (2\ell+1) P_\ell'' \alpha_\ell/s^5$ is integrable, so we may write

$$\sum_2^\infty (2\ell+1) P_\ell''(z) \alpha_\ell(s) \lesssim C s^4 / \log s.$$

The ~~Lukaszyk~~-Martin argument ⁵⁾ then gives $\sigma_{\text{tot}}(s) \lesssim 16 \pi L^2/s$, where L is such that

$$P_L'''(z) + P_{L+1}'''(z) = \sum_2^L (2\ell+1) P_\ell''(z) = C s^4 / \log s.$$

Using then the asymptotic properties of the P_L''' we can verify that this gives $L = (s^{\frac{1}{2}}/4) \log(s/s_1 \log^7(s/s_2))$, hence the value of s_0 in (1), quoted at the beginning. Of course this is only qualitative; to obtain quantitative results, we have to begin by defining averaged cross-sections. Following Ref. 1), we set

$$\begin{aligned} \overline{\sigma}_{\text{tot}}^{(n)}(s) &\equiv \frac{n+1}{(s-4)^{n+1}} \int_4^s ds' (s'-4)^n \sigma_{\text{tot}}(s') \\ &= 16\pi \frac{n+1}{(s-4)^{n+1}} \int_4^s ds' (s'-4)^{n-1} \sum_0^\infty (2\ell+1) \alpha_\ell(s'), \end{aligned} \quad (9)$$

and the same for σ_{inel} . We choose these averages because one can show that if σ does not oscillate too much, then

$$\overline{\sigma}^{(n)}(s) = \sigma(s) + O\left(\frac{1}{ns}\right).$$

Hence, at large s or n , $\overline{\sigma}$ is equivalent to σ . Since we are interested in large s ; we will take $n=1$. The problem now is: find the maximum of (9) compatible with the constraints (7) and (4). This is solved quite easily using the methods of Lagrange multipliers as developed in Ref. 8). The answer is as follows: consider the equation

$$\begin{aligned} (s'-4)^{n-1} &= D_0(s) \frac{s'^{1/2}}{s'^3 (s'-4)^{1/2}} C_L^{1/2}(z') \\ &+ D_1(s) \frac{s'^{1/2}}{s'^4 (s'-4)^{3/2}} \left\{ s' C_{L-1}^{3/2}(z') - \frac{3(s'-4)}{4} C_L^{1/2}(z') \right\} \\ &+ D_2(s) \frac{s'^{1/2}}{s'^4 (s'-4)^{5/2}} \left\{ 3s' C_{L-2}^{5/2}(z') - \frac{3(s'-4)}{4} C_{L-1}^{3/2}(z') \right\}. \end{aligned} \quad (10)$$

Given the D , this determines L as function of s, s' (the C_ℓ^λ are the Gegenbauer polynomials). One then cuts the sums in the right-hand sides of Eqs. (7) at $\ell = [L] \equiv \text{maximum integer} \leq L$. This makes the right-hand side

of Eqs. (7) functionals of D_0, D_1, D_2 . These D 's are then varied until Eqs. (7) are as nearly satisfied as possible, that is, we let D_i^{eff} be the minimum values of the D_i such that the right-hand sides of Eqs. (7) are still smaller than the left-hand sides. Through Eq. (10), this determines $L_{\text{eff}}(s, s')$. The best possible bound on $\bar{\sigma}_{\text{tot}}$ is then ⁸⁾

$$\bar{\sigma}_{\text{tot}}^{(n)}(s) \leq \frac{16\pi(n+1)}{(s-4)^{n+1}} \int_4^s ds' (s'-4)^{n-1} \left\{ [L_{\text{eff}}(s, s')]^2 + 2L_{\text{eff}}(s, s') + 1 \right\}. \quad (11.a)$$

For σ_{inel} we use the expansion

$$\sigma_{\text{inel}}(s) = \frac{16\pi}{s-4} \sum_0^\infty (2\ell+1) \beta_\ell(s),$$

$$\beta_\ell(s) \equiv \text{Im} \hat{f}_\ell(s) - |\hat{f}_\ell(s)|^2, \quad 0 \leq \beta_\ell(s) \leq 1/4,$$

and the corresponding arguments give the bound

$$\bar{\sigma}_{\text{inel}}^{(n)}(s) \leq \frac{4\pi(n+1)}{(s-4)^{n+1}} \int_4^s ds' (s'-4)^{n-1} \left\{ [L_{\text{eff}}(s, s')]^2 + 2L_{\text{eff}}(s, s') + 1 \right\}. \quad (11.b)$$

Of course, since Eq. (10) is transcendental, it is impossible to write down explicitly the value of L_{eff} in terms of h_2, h_2', h_2'' ; but, given numerical values for these, the computation of (11) offers no difficulty.

This brings us to the problem of finding the values of the h 's. We have parametrized

$$\psi^{(0)}(t) = \frac{1}{G^2} (m_f^2 - t)(1 + R^2(t-4)), \quad \psi^{(2)}(t) = b^{(2)}$$

The values of $G^2, R^2, b^{(2)}$ we have found by requiring the f^0 resonance to have its correct width (m_f = mass of f^0) and the formulae to reproduce the best values ⁹⁾ of the scattering lengths, $a_2^{(0)} = 1.4 \times 10^{-3}$, $a_2^{(2)} = -1.5 \times 10^{-4}$. This gives $\gamma_0 = 3.2 \times 10^{-3}$, $\gamma_1 = 1 \times 10^{-5}$, $\gamma_2 = 2.1 \times 10^{-7}$. To test the dependence of our bounds on the parametrization used, we have tried other parametrizations for ψ ; it turns out that the values of the h, h', h'' do not change very much, thus leading support to our calculation. We have computed two bounds for the $\bar{\sigma}$, using two different values for the parameters γ : the "best" values, and upper bounds, obtained by taking the largest values of the γ compatible with the largest reasonable values for the $a_2^{(I)}$ and the width of the f^0 . The physical reason for the smallness of the γ_1, γ_2 as

compared to γ_0 is, obviously, the large mass of the f^0 and the absence of $I=2$ resonances. The ensuing bounds, $\bar{\sigma}^{(B)}$ and $\bar{\sigma}^{(U)}$ are displayed in the Figure. The small variation from $\bar{\sigma}^{(B)}$ to $\bar{\sigma}^{(U)}$ is remarkable; it is due to the fact that the effective dependence of the bounds on γ_0 , γ_1, γ_2 is only logarithmic [cf., Refs. 2), 8), for example], and this reinforces the credibility of our results.

DISCUSSION

Further details of this work, and a more complete discussion will be published elsewhere, but we want to draw here the attention to a few points.

First of all, it is not difficult to extend the analysis to other $\pi\pi$ cross-sections. It is also possible to obtain bounds on $\pi N \rightarrow \text{all}$. All we need is the property that $(d^n/dt^n)[g_2(t)/(t-4)^2]$ be finite at $t=4$ for $n=0,1,2$; g_2 is the $\pi\pi \rightarrow \bar{N}N$ D wave. But this is obvious as, due to generalized unitarity ¹⁰⁾, one can write $g_2(t) = \varphi(t)f_2(t)$ with φ analytical around $t=4$, so the analysis can be directly extended [spinology and the problem of the pole term can be dealt with as in Ref. 3]. However, estimates of φ and, mainly, of its derivatives, will not be as reliable as one would wish. Perhaps the best here would be to invert the argument and use the known $\sigma_{\pi N \rightarrow \text{all}}$ to bound ¹¹⁾ $\varphi, \varphi', \varphi''$. Also bounds on $\bar{p}p \rightarrow \pi^+\pi^-$ cross-sections can be improved with the present method.

Then comes the point of comparison with experiment. Since $\pi\pi$ cross-sections are not directly measurable, we have to use some theory. We write

$$\sigma_{\text{tot, exp}}(s) = \frac{48\pi}{s-4} \cdot \frac{g_F^2 (s-4)^3}{(m_F^2 - s)^2 + g_F^2 (s-4)^3} + \sigma(\infty) \lambda(s) e^{-(a/s)^2}, \quad (12.a)$$

$$\lambda(s) = \begin{cases} 1 & \text{if } s < s_F \\ 1 + \log^2(s/s_F) & \text{if } s \geq s_F. \end{cases} \quad (12.b)$$

a gives the energy at which the Pomeranchuk begins to show up; we have taken $a=0.6(\text{GeV})^2$. For $\sigma(\infty)$ we consider two possibilities: $\sigma(\infty) = 18 \text{ mb}$ (implied by factorization and the quark model) which gives us $\sigma_{\text{tot, exp}}^{(FQ)}$, and $\sigma(\infty) = 40 \text{ mb}$, as suggested by the model of Ref. 12); this gives $\sigma_{\text{tot, exp}}^{(CMN)}$. The parameter s_F is the value at which

the logarithmic increase permitted by the Froissart bound could begin to appear. $s_F = \infty$ means constant total cross-sections at infinity. For inelastic cross-sections we take $\sigma_{\text{inel,exp}} = \sigma_{\text{tot,exp}} - \sigma_{\text{elastic,exp}}$, $\sigma_{\text{elastic,exp}}$ calculated with the same models as above.

CONCLUSIONS

First of all, it is clear that we have enormously improved previous bounds : for example, the best previous bounds [Ref. 4] give $\bar{\sigma}_{\text{tot}} < 540$ mb, as compared with our result $\bar{\sigma}_{\text{tot}} < 150-170$ mb at 17.4 GeV/c. (The improvement is more dramatic at higher energies.) Our bounds are very tight, especially for σ_{inel} where they almost rule out the CMN model. The most important result, in our opinion, is the fact that the bounds are practically flat at accelerator energies, so logarithmic increases are only allowed at ISR or cosmic energies. For example, the parametrization (12) with $\sigma(\infty) = 18$ mb violates the bound if $s_F < 500 \mu^2$ (see Figure). If we take as a criterion for possible detection of a logarithmic increase that it be at least 20% of the constant term, this means that there is no hope of detecting logarithmic increases at energies smaller than 60 GeV/c.

Finally, the tightness of the bound makes it very interesting to try to improve it, even slightly (it is clear that drastic improvements are not likely as the bounds for σ_{inel} are quite close to the expected values). There is some chance of doing so by bringing into the game s-t crossing symmetry (only s-u crossing has been fully used here) through Roy's equations ¹³⁾, as suggested by Common and Pidcock ¹⁴⁾. This will be left for future work.

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FIGURE CAPTION

- (a) Bounds and "experimental" values for total cross-sections.
- (b) Bounds and "experimental" values for inelastic cross-sections.

Solid lines : bounds. Broken lines : "experimental" values.