CM-P00058297

Ref.TH.1477-CERN

Archives

SOLVABLE UNITARY MODEL WITH CLUSTER DECOMPOSITION
FOR MULTIPRODUCTION OF DISTINGUISHABLE PARTICLES

L. B. Rédei *)

CERN - Geneva

A B S T R A C T

A solvable model is presented for multiproduction of distinguishable particles. The scattering amplitude satisfies many - particle unitarity, cluster decomposition with respect to the produced particles, and Lorentz invariance. The analytic properties and the high-energy limit of the model are discussed.

*) Permanent address : Department of Physics,
Umeå University, Umeå,
Sweden.

1. INTRODUCTION

In a previous article ¹⁾ (which we refer to as I), we have described a model or a class of models for multiproduction of particles which was solvable, relativistic and which gave rise to an S matrix satisfying many-particle unitarity exactly at all energies ²⁾. Also, the scattering amplitude had reasonable analytic properties, in so far as it was the boundary value of a single analytic function with branch points at production thresholds. The model was obtained by making a simple Hermitian ansatz for the K matrix and solving the resulting Heitler equations ³⁾ for the scattering amplitude on the energy shell. We found that there was a very large number of models of this kind, in fact the model contained a large number of essentially arbitrary input functions. Therefore the predictive power of models with only these properties is very small. The S matrix obtained in I, apart from not being crossing symmetric, had one very important physical property missing, that of cluster decomposition. Cluster decomposition is very important both from the point of view of physical interpretation of the S matrix ⁴⁾ and from the point of view of mathematics, since it restricts severely the structure of the scattering amplitude.

In the present article we are going to describe a model which, in addition to unitarity and relativistic invariance, also has the right cluster decomposition property with respect to the emitted particles. However, in order to be able to obtain an explicit solution, we now have to assume that the produced particles are distinguishable. Physically, this corresponds to neglecting exchange effects. Under certain circumstances, this need not be too bad an approximation. In fact, distinguishable models for indistinguishable particles have a long history, starting from Hartree's method and the Thomas-Fermi model in atomic physics ⁵⁾ to the multiperipheral model ^{6),7)} for particle production, where exchange effects are usually neglected.

The model is again obtained by making an ansatz for the K matrix. It will have the following properties.

Solvability

Yes. The scattering amplitude can be given in closed, explicit form.

Relativistic invariance

Yes.

Multiproduction

The number of produced particles can be arbitrarily large (for large enough energy).

Unitarity

Yes. The S matrix satisfies many-particle unitarity exactly at all energies.

Cluster decomposition

Yes.

Analyticity

Yes and no. The requirement that the elastic amplitude should be the boundary value of a single analytic function can be formulated as a non-linear relation for the input function but which cannot be solved explicitly.

Crossing symmetry

No.

Statistics

The amplitude is not symmetric in the momenta of the produced particles.

There is only S wave scattering in the elastic channel and there are no internal quantum numbers. This is an unrealistic feature of the model which precludes the possibility of making a meaningful comparison, at this stage, with the experimental situation. However, the model to be presented is just the simplest possible example of a class of solvable models and it is therefore hoped that more sophisticated versions of it might eventually allow a comparison with experiment.

In the next Section we shall present the model, solve it and discuss its properties. This will be followed by Section 3, where the high energy limit will be discussed and where it will be shown that certain two-particle distributions exhibit limiting behaviour.

2. SOLVABLE UNITARY MODEL FOR MULTIPARTICLE PRODUCTION

We consider two kinds of scalar particles with masses m and μ . Let $m > \mu$ and we refer to the two particles as nucleons and mesons, respectively. We shall consider the scattering of two nucleons with four-momenta p_1 and p_2 . We denote the four-momenta of the outgoing nucleons by p_1' and p_2' (the number of nucleons will be conserved in the model) and the momenta of the produced mesons by k_j . We shall use the notation $P = p_1 + p_2$ and $P^2 = s$.

We first discuss the well-known connection ³⁾ between the K matrix and the scattering amplitude. We write the scattering operator S as

$$S = (1 + iK)(1 - iK)^{-1}, \quad (1)$$

where $\langle b|K|a \rangle$ is the K matrix. Unitarity and Poincaré invariance of S requires K to be Hermitian and Poincaré invariant.

From the above equation, it follows that

$$T = 2iK + iK \cdot T, \quad (2)$$

where we have put

$$T = S - 1. \quad (3)$$

We take the matrix element of Eq. (2) and insert a complete set of orthonormal states in the right-hand side to obtain that

$$\langle b|M|a \rangle = 2i \langle b|W|a \rangle + i \sum_c \langle b|W|c \rangle \langle c|M|a \rangle \delta(P_b - P_c) \quad (4)$$

for $P_b = P_a$, where $\langle b|M|a \rangle$ is the scattering amplitude on the energy shell

$$\langle b|T|a \rangle = \delta(P_b - P_a) \langle b|M|a \rangle \quad (5)$$

and where we have put

$$\langle b|K|a \rangle = \delta(P_b - P_a) \langle b|W|a \rangle. \quad (6)$$

Equation (4) is a coupled channel linear integral equation for the scattering amplitude on the energy shell and our model consists of solving this equation for a specific W .

The ansatz we make for W is the following :

$$\langle \underline{k}'_1, \dots, \underline{k}'_n; \underline{p}'_1, \underline{p}'_2 | W | \underline{k}_1, \dots, \underline{k}_n; \underline{p}_1, \underline{p}_2 \rangle = \frac{1}{2} (2\pi)^{-2} \beta(P^2) \frac{1}{\sqrt{E'_2 E'_1 E_2 E_1}} \quad (7)$$

$$\delta(\underline{k}'_1 - \underline{k}_1) \dots \delta(\underline{k}'_n - \underline{k}_n),$$

$$\langle \underline{k}'_1, \dots, \underline{k}'_{n+1}; \underline{p}'_1, \underline{p}'_2 | W | \underline{k}_1, \dots, \underline{k}_n; \underline{p}_1, \underline{p}_2 \rangle = \langle \underline{k}_1, \dots, \underline{k}_n; \underline{p}_1, \underline{p}_2 | W |$$

$$\underline{k}'_1, \dots, \underline{k}'_{n+1}; \underline{p}'_1, \underline{p}'_2 \rangle^* = \frac{1}{2} (2\pi)^{-7/2} \alpha(P^2) \frac{1}{\sqrt{2\omega_{n+1}}} \frac{1}{\sqrt{E'_2 E'_1 E_2 E_1}} \quad (8)$$

$$\delta(\underline{k}'_1 - \underline{k}_1) \dots \delta(\underline{k}'_n - \underline{k}_n),$$

where $P = \underline{p}_1 + \underline{p}_2$, $E = (\underline{p}^2 + m^2)^{\frac{1}{2}}$, $\omega = (\underline{k}^2 + \mu^2)$ and where $\alpha(s)$ and $\beta(s)$ are arbitrary functions, except that $\beta(s)$ is real. The other matrix elements of W we put equal to zero.

The W as defined above is Hermitian and Lorentz invariant, giving rise to a unitary and Poincaré invariance S matrix. Since the disconnected part of K is given by the connected part multiplied by a momentum δ function for each particle which is going straight through and since the scattering amplitude can be written as

$$\langle b | T | a \rangle = 2 \sum_{n=1}^{\infty} \langle b | (iK)^n | a \rangle \quad (9)$$

the S matrix will have the correct cluster decomposition properties ⁴⁾ with respect to an arbitrary number of mesons going straight through. On the other hand, the matrix elements of W are not properly symmetrized in the meson variables and therefore our model is a model for producing distinguishable particles: in fact, W numbers the produced mesons according to the sequence of their emission. The W can also annihilate one meson at a time, in such a way that it is always the last emitted meson that gets absorbed first. This implies that in the diagrammatic representation of the power series expansion given by Eq. (9), there will be no diagrams with crossed meson lines. Therefore, our model can be considered to be an approximation to a model for producing indistinguishable particles, the approximation consisting of neglecting exchange effects. A schematic representation of W is given in the Figure.

We introduce the invariant amplitudes $\overline{M}^{(n)}$ for the production of n mesons $[\overline{M}^{(0)}$ is the elastic amplitude] by

$$\langle k_1, \dots, k_n; p'_1, p'_2 | M | p_1, p_2 \rangle = \frac{1}{2} (2\pi)^{-2} (2\pi)^{-3n/2} \frac{1}{\sqrt{E'_2 E'_1 E_2 E_1}} \quad (10)$$

$$\left(\prod_{j=1}^n \frac{1}{\sqrt{2\omega_j}} \right) \bar{M}^{(n)}(k_1, \dots, k_n; p'_1, p'_2; p_1, p_2).$$

Just as in paper I, for $P = p_1 + p_2 = p'_1 + p'_2 + \sum_{j=1}^n k_j$ and for the simple W used here, we may write

$$\bar{M}^{(n)} = \bar{M}^{(n)}(k_1, \dots, k_n; P). \quad (11)$$

Taking now in Eq. (4) for $|a\rangle$ a two-nucleon state and for $|b\rangle$ a state with two nucleons and $n \geq 2$ mesons, we obtain, after using Eqs. (7), (8) and (10) and integrating over the intermediate nucleon momenta, that

$$\begin{aligned} \bar{M}^{(n)}(k_1, \dots, k_n; P) = & i (2\pi)^{-3} \alpha^* ((P - \sum_{j=1}^n k_j)^2) \int d^4 q \delta_+(q^2 - \mu^2) \\ & \mathcal{L}((P - \sum_{j=1}^n k_j - q)^2) \bar{M}^{(n+1)}(k_1, \dots, k_n, q; P) + i \alpha ((P - \sum_{j=1}^{n-1} k_j)^2) \\ & \mathcal{L}((P - \sum_{j=1}^{n-1} k_j)^2) \tilde{\Theta}((P - \sum_{j=1}^n k_j)^2) \bar{M}^{(n-1)}(k_1, \dots, k_{n-1}; P) \quad (12) \\ & + i \beta ((P - \sum_{j=1}^n k_j)^2) \mathcal{L}((P - \sum_{j=1}^n k_j)^2) \bar{M}^{(n)}(k_1, \dots, k_n; P) \end{aligned}$$

for $N(s) \geq n \geq 2$ $\bar{M}^{(n)} = 0$ for $n > N(s)$ where the integer $N(s)$ is defined by

$$N(s) \leq \frac{\sqrt{s} - 2m}{\mu} < N(s) + 1 \quad (13)$$

$[N(s)]$ is the integer part of $(\sqrt{s} - 2m)/\mu$. In Eq. (12), $\tilde{\Theta}(P^2) = \Theta(P^2 - 4m^2) \times \Theta(P_0)$, $\delta_+(q^2 - \mu^2) = \delta(q^2 - \mu^2) \Theta(q_0)$, Θ being the step function, and the function $\mathcal{L}(P^2)$ is given by the phase space integral

$$\begin{aligned} \mathcal{L}(P^2) = & 2 (2\pi)^{-2} \int d^4 p_1 \int d^4 p_2 \delta_+(p_1^2 - m^2) \delta_+(p_2^2 - m^2) \delta(P - p_1 - p_2) \\ = & \frac{1}{4\pi} \left(1 - \frac{4m^2}{P^2}\right)^{1/2} \tilde{\Theta}(P^2). \quad (14) \end{aligned}$$

Substituting Eq. (12) for $n = N(s)$ (for this value of n the first term in the right-hand side vanishes) into the same equation with $n = N(s) - 1$ and continuing this procedure down to $n = 2$ one obtains after a simple calculation that $\bar{M}^{(n)}$ can be written as

$$\bar{M}^{(n)} = \left\{ 1 + \mathcal{L}(\alpha_n) \left[-i \beta(\alpha_n) + |\alpha(\alpha_n)|^2 Z^{N(\alpha)-n}(\alpha_n) \right] \right\}^{-1} \quad (15)$$

$$i \alpha(\alpha_{n-1}) \mathcal{L}(\alpha_{n-1}) \tilde{\Theta}(\alpha_n) \bar{M}^{(n-1)}(k_1, \dots, k_{n-1}; P),$$

where we have introduced the notation

$$\alpha_j = \left(P - \sum_{l=1}^j k_l \right)^2 \quad (16)$$

and where the function $Z^j(P^2)$ is given by the recursion relation

$$Z^j(P^2) = (2\pi)^{-3} \int d^4 q \delta_+(q^2 - \mu^2) \mathcal{L}((P-q)^2)$$

$$\left\{ 1 + \mathcal{L}((P-q)^2) \left[-i \beta((P-q)^2) + |\alpha((P-q)^2)|^2 Z^{j-1}((P-q)^2) \right] \right\}^{-1} \quad (17)$$

with $Z^{(0)}(P^2) \equiv 0$.

We now take for $|b\rangle$ in Eq. (4) a state with two nucleons and then a state with one meson and two nucleons and obtain in the same manner that

$$\bar{M}^{(1)}(k; P) = \left\{ 1 + \mathcal{L}(\alpha_1) \left[-i \beta(\alpha_1) + |\alpha(\alpha_1)|^2 Z^{N(\alpha)-1}(\alpha_1) \right] \right\}^{-1} \quad (18)$$

$$(2i \alpha(\alpha) + i \alpha(\alpha) \mathcal{L}(\alpha) \tilde{\Theta}(\alpha_1) \bar{M}^{(0)}(\alpha))$$

and

$$\bar{M}^{(0)}(\alpha) = (1 - i \mathcal{L}(\alpha) \beta(\alpha))^{-1} (2i \beta(\alpha)$$

$$+ i (2\pi)^{-3} \alpha^*(\alpha) \int d^4 q \delta_+(q^2 - \mu^2) \mathcal{L}((P-q)^2) \bar{M}^{(1)}(q; P)). \quad (19)$$

The substitution of Eq. (18) into Eq. (19) gives for the elastic amplitude that

$$\bar{M}^{(0)}(\alpha) = \frac{2i \beta(\alpha) - 2|\alpha(\alpha)|^2 Z^{N(\alpha)}(\alpha)}{1 + \mathcal{L}(\alpha) (-i \beta(\alpha) + |\alpha(\alpha)|^2 Z^{N(\alpha)}(\alpha))} \quad (20)$$

Putting this back first into Eq. (18) and then the result back into Eq. (15), one obtains for the production amplitudes that

$$\bar{M}^{(n)}(k_1, \dots, k_n; P) = \frac{2(i)^n}{\mathcal{L}(s)} \frac{1}{1 + \mathcal{L}(s_n)(-i\beta(s_n) + |\alpha(s_n)|^2 Z^{N(s)-n}(s_n))} \quad (21)$$

$$\prod_{j=0}^{n-1} \frac{\alpha(s_j) \mathcal{L}(s_j)}{1 + \mathcal{L}(s_j)(-i\beta(s_j) + |\alpha(s_j)|^2 Z^{N(s)-j}(s_j))}.$$

This expression can be simplified. First of all by doing the trivial integrations over the angles and over $|q|$ and by introducing the integration variable $x = (P-q)^2$ in (17), we can write

$$Z^j(s) = (2\pi)^{-3} \frac{\pi}{2s} \theta(\sqrt{s} - 2m - \mu) \int_{4m^2}^{(\sqrt{s}-\mu)^2} dx \frac{\sqrt{\lambda(x, s, \mu^2)} \mathcal{L}(x)}{1 + \mathcal{L}(x)(-i\beta(x) + |\alpha(x)|^2 Z^{j-1}(x))} \quad (22)$$

where $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$ and $Z^{(0)} = 0$. From this follows the property

$$Z^{N(s)+n}(s) = Z^{N(s)} \quad (23)$$

for integer $n \geq 0$. This is so because, according to the recursion formula, $Z^j(x) = 0$ for $x \leq (2m + \mu)^2$ and the first n iterations for $Z^{N(s)+n}(s)$ involve the $Z^j(x)$, $j \leq n$, for values of $x \leq (2m + \mu)^2$, i.e., the expression for $Z^{N(s)+n}(s)$, $n \geq 0$, implies always just $N(s)$ non-trivial iterations independently of n .

Using this relation and the fact that $\sqrt{s_j} \leq \sqrt{s} - j\mu$ [i.e., $N(s_j) \leq N(s) - j$] we can write Eq. (21) in the form

$$\bar{M}^{(n)}(k_1, \dots, k_n; P) = \frac{2(i)^n}{\mathcal{L}(s)} \frac{1}{1 + \mathcal{L}(s_n)(-i\beta(s_n) + |\alpha(s_n)|^2 Z^{N(s)-n}(s_n))} \quad (24)$$

$$\prod_{j=0}^{n-1} \frac{\alpha(s_j) \mathcal{L}(s_j)}{1 + \mathcal{L}(s_j)(-i\beta(s_j) + |\alpha(s_j)|^2 Z^{N(s)-j}(s_j))}$$

which, together with Eq. (20), is our final result in terms of an arbitrary $\alpha(s)$ and the real function $\beta(s)$. In view of Eqs. (22) and (23) the function $Z^{N(s)}(s)$ is the solution of the non-linear integral equation

$$Z^{N(s)}(s) = (2\pi)^{-3} \frac{\pi}{2s} \theta(\sqrt{s} - 2m - \mu) \int_{4m^2}^{(\sqrt{s}-\mu)^2} dx \frac{\sqrt{\lambda(x, s, \mu^2)} \mathcal{L}(x)}{1 + \mathcal{L}(x)(-i\beta(x) + |\alpha(x)|^2) Z^{N(x)}(x)} \quad (25)$$

This equation has a unique solution ⁸⁾. Because of the θ function, $Z^{N(s)}(s) \equiv 0$ for $s < (2m + \mu)^2$. At the same time the above equation permits one to calculate the values of $Z^{N(s)}(s)$ in the interval $s_0 \leq s \leq (\sqrt{s_0} + \mu)^2$ from the values of $Z^{N(x)}(x)$ in the interval $4m^2 \leq x \leq s_0$. Therefore in order to calculate $Z^{N(s)}(s)$ in the interval $(2m + n\mu)^2 < s < (2m + (n+1)\mu)^2$ we have to iterate Eq. (25), with the initial condition $Z^{N(x)}(x) = 0$ for $x < (2m + \mu)^2$, n times, i.e., we have to do n integrations.

That our solution for the amplitude as given by Eqs. (20), (24) and (25) indeed satisfies the original Heitler equations [Eqs. (12), (18) and (19)] can be verified by direct substitution. That the unitarity relation for the elastic amplitude is satisfied

$$2\text{Re } \bar{M}^{(0)} = - (2\pi)^{-2} \int d^4 p_1'' \int d^4 p_2'' \delta_+(p_1''^2 - m^2) \delta_+(p_2''^2 - m^2) \left\{ \delta(P - p_1'' - p_2'') \right. \quad (26)$$

$$\left. |\bar{M}^{(0)}|^2 + \sum_{n=1}^{N(s)} (2\pi)^{-3n} \left(\prod_{j=1}^n \int d^4 q_j \delta_+(q_j^2 - \mu^2) \right) \delta(P - p_1'' - p_2'' - \sum_{j=1}^n q_j) |\bar{M}^{(n)}|^2 \right\}$$

can also be checked by straightforward calculation ⁹⁾.

The many particles $\rightarrow$ many particles (many meaning here more than two) amplitudes could also be calculated.

Our production amplitude has first order short-range correlation between the mesons in the terminology of Ref. 10). This seems to be related to the fact that the connected part of the K matrix used by us had only 2 particle $\rightarrow$ 2 particle and 2 particle $\rightarrow$ 3 particle vertices. Higher order vertices in the K matrix will introduce higher order correlations.

It is amusing to note that the crossed amplitude obtained by replacing $P = p_1 + p_2$ by the nucleon momentum transfer $p_1 - p_1'$ in Eq. (24) is of the Q factorizable form in the terminology used in Ref. 11) which is the form one has in the extensively used multiperipheral model.

The production amplitude we obtained is not symmetric in the meson momenta. One could of course always symmetrize the amplitudes, but this would spoil unitarity unless $\bar{M}^{(n)}(k_1, \dots, k_n; P)$ happens to be already symmetric or if

$$\left(\prod_{j=1}^n \int d^4 q_j \delta_+(q_j^2 - \mu^2) \right) \mathcal{L}((P - \sum q_i)^2) \bar{M}^{(n)}(Q \bar{M}^{(n)})^* = 0 \quad (27)$$

for any arbitrary non-trivial permutation Q of the meson momenta. This means that our amplitude will be a good approximation to a unitary model for indistinguishable particles if either

$$Q \bar{M}^{(n)} \sim \bar{M}^{(n)}$$

or

$$\left(\prod_{j=1}^n \int d^4 q_j \delta_+(q_j^2 - \mu^2) \right) \mathcal{L}((P - \sum q_i)^2) \bar{M}^{(n)}(Q \bar{M}^{(n)})^* \sim 0.$$

Let us now discuss the analytic properties of the amplitudes. We begin with the elastic amplitude $\bar{M}^{(0)}(s)$. For constant $\alpha(s)$ and $\beta(s)$ (independent of s) the $\bar{M}^{(0)}(s)$ will be a continuous function of s but it will not be the boundary value of a single analytic function. Rather, in the different segments $(2m+n\mu)^2 < s < (2m+(n+1)\mu)^2$, $n=0,1,\dots$, it will be the boundary value of different analytic functions of s . (The situation, in this respect, is similar to that which was found in paper I.) In order to have an analytic $\bar{M}^{(0)}(s)$ (with branch points at the physical s channel thresholds) it is necessary to have a more complicated, non-analytic $\beta(s)$. It is simple to formulate an equation for $\beta(s)$, the function $\alpha(s)$ being given, for this to be the case. Because of Eq. (20), $\bar{M}^{(0)}(s)$ is the boundary value of an analytic function, if and only if the function

$$(-i\beta(s) + |\alpha(s)|^2) Z^{N(s)}(s)$$

is. For this to happen, the real and the imaginary parts of

$$(-i\beta(s) + |\alpha(s)|^2) Z^{N(s)}(s)$$

have to be related through a dispersion integral which gives the condition that

$$\beta(s) = |\alpha(s)|^2 \operatorname{Im} Z^{N(s)} + \frac{g(s)}{\pi} P \int_{(2m+\mu)^2}^{\infty} ds' \frac{|\alpha(s')|^2 \operatorname{Re} Z^{N(s')}}{g(s')(s'-s)}, \quad (28)$$

$$+ g(s)$$

where $g(s)$ and $g(s)$ are arbitrary real analytic functions regular for $\operatorname{Im} s = 0$, $\operatorname{Re} s \geq 4m^2$. Since $Z^{N(s)}(s)$ in the right-hand side of this equation involves $\beta(s)$ in a non-linear manner, this is a very complicated equation for $\beta(s)$. However, since $g(s)$ and $g(s)$ are to a large extent arbitrary functions, the equation does not seem to be very restrictive, apart from containing information about the singularities of $\beta(s)$ at the production thresholds.

With $\beta(s)$ satisfying the equation above the production amplitude $\bar{M}^{(n)}$ becomes also the boundary value of a single analytic function of the sub-energy s_i , $i=0,1,\dots,n$, the other s_j , $j \neq i$, being held fixed, with branch points at elastic and production thresholds. In a more complete theory there would be some other more complicated singularities in the production amplitudes in addition to the singularity structure in the subenergies s_i found in the present model. The existence of these singularities follows from general arguments involving crossing symmetry and cluster decomposition¹²⁾. Their absence in our model is related to the fact that the K matrix used here had no elastic pion-pion and pion-nucleon scattering terms. If terms of this kind were included in the model, a more complicated singularity structure would emerge for the production amplitudes.

In the next section we shall briefly look at the high energy behaviour of the model.

3. HIGH ENERGY LIMIT

We shall investigate the limit of the model as $s \rightarrow \infty$. To simulate the experimental situation, we assume that $\beta(s)$ and $\alpha(s)$ have been chosen in such a way that $\sigma_{\text{el}}/\sigma_{\text{total}}$ tends to a non-trivial limit (different from 1) while $\operatorname{Im} \bar{M}^{(0)}(s) \rightarrow 0$. (It is to be remembered here that our definition of $\bar{M}$ differed from the more customary one by a factor of i .) For this to happen one must have

$$\beta(s) \rightarrow 0$$

and

$$|\alpha(s)|^2 Z^{N(s)}(s) \longrightarrow A = \text{constant} \quad (29)$$

This gives, through Eqs. (14) and (20), for the elastic cross-section as $s \rightarrow \infty$ that

$$\sigma_{el} = \frac{1}{\pi s} \left(\frac{A}{1 + \frac{1}{4\pi} A} \right)^2 \quad (30)$$

and for the total cross-section through the optical theorem that

$$\sigma_{tot} = \frac{4}{s} \frac{A}{1 + \frac{1}{4\pi} A} \quad (31)$$

from which we express A as the ratio

$$A = 4\pi \frac{\sigma_{el}}{\sigma_{in}} \quad (32)$$

where $\sigma_{in} = \sigma_{tot} - \sigma_{el}$.

Substituting now Eq. (29) and $\beta(s) \rightarrow 0$ into Eq. (22), we can calculate the high s behaviour of $Z^{N(s)}(s)$ and $\alpha(s)$ separately. In fact we obtain that

$$Z^{N(s)}(s) \longrightarrow \frac{1}{2} (4\pi)^{-3} \frac{1}{1 + \frac{1}{4\pi} A} s. \quad (33)$$

Putting this and Eq. (29) into Eq. (24), we obtain a simple expression for the production amplitude in the kinematical region $s \sim s'$ large, $Q^2 \ll s'$ where $s' = (p_1' + p_2')^2$ and $Q^2 = (p_1 + p_2 - p_1' - p_2')^2$ is the invariant mass transferred to the produced mesons :

$$|\bar{M}^{(n)}(k_1, \dots, k_n; P)| = \frac{8\pi}{1 + \frac{1}{4\pi} A} (4\pi)^n \left(\frac{\frac{1}{2\pi} A}{1 + \frac{1}{4\pi} A} \right)^{n/2} \left(\frac{1}{s'} \right)^{n/2} \quad (34)$$

valid in the kinematical region stated above.

The model in its present state is far too simple to be able to give a realistic picture of high energy scattering. (There is only S wave scattering in the elastic channel.) Nevertheless, it is amusing and may be instructive to calculate a two-particle distribution function for the two nucleons in a certain kinematical region where one gets a particularly simple result which exhibits a kind of scaling behaviour ¹³⁾ at high energy.

We define the partial two-nucleon distribution function $f^{(n)}$

$$f^{(n)}(s, s', Q^2) = \frac{1}{\sigma_{tot}} \frac{d^2}{d(\ln s') d(\ln Q^2)} \sigma^{(n)}, \quad n \geq 2 \quad (35)$$

where $\sigma^{(n)}$ is the partial cross-section for producing n mesons and the quantities s' and Q^2 have been defined before. With the normalization used here, $f^{(n)}$ is given by

$$f^{(n)} = 2(2\pi)^{-2} (2\pi)^{-3n} \frac{Q^2 s'}{\sigma_{tot}} \left(1 - \frac{4m^2}{s}\right)^{-\frac{1}{2}} \int d^4 p'_1 \int d^4 p'_2 \delta_+(p_1'^2 - m^2) \delta_+(p_2'^2 - m^2) \left(\prod_{j=1}^n \int d^4 k_j \delta_+(k_j^2 - \mu^2) \right) \delta(P - p'_1 - p'_2 - \sum k_j) \delta(s' - (p'_1 + p'_2)^2) \delta(Q^2 - (P - p'_1 - p'_2)^2) |\bar{M}^{(n)}(k_1, \dots, k_n; P)|^2. \quad (36)$$

In the region $s' \gg Q^2$, $s \sim s'$ large, we may use Eq. (34) for $\bar{M}^{(n)}$, Eq. (32) for A , and write

$$f^{(n)} = \frac{s' Q^2}{s} \frac{32}{\sigma_{tot}} \left(\frac{\sigma_{in}}{\sigma_{tot}}\right)^2 \left(\frac{4}{\pi}\right)^n \left(\frac{\sigma_d}{\sigma_{tot}}\right)^n I(s, s', Q^2) I_n(Q^2) \left(\frac{1}{s}\right)^n, \quad (37)$$

where $I_n(Q^2)$ is the phase space integral

$$I_n(Q^2) = \left(\prod_{j=1}^n \int d^4 q_j \delta_+(q_j^2 - \mu^2) \right) \delta(P - p'_1 - p'_2 - \sum q_j) \quad (38)$$

and where

$$I(s, s', Q^2) = \int d^4 p'_1 \int d^4 p'_2 \delta_+(p_1'^2 - m^2) \delta_+(p_2'^2 - m^2) \delta(s' - (p'_1 + p'_2)^2) \delta(Q^2 - (P - p'_1 - p'_2)^2) \quad (39)$$

The first integral can be evaluated in the limit $Q^2 \gg (n\mu)^2$ (we may then put $\mu = 0$) and one gets that

$$I_n(Q^2) = \left(\frac{\pi}{2}\right)^{n-1} \frac{1}{(n-1)!} \frac{1}{(n-2)!} (Q^2)^{n-2}. \quad (40)$$

The second integral can also be evaluated for large s and s' giving that

$$I(s, s', Q^2) = \frac{\pi^2}{4} \left((1 + \sqrt{\frac{s'}{s}})^2 - \frac{Q^2}{s} \right)^{\frac{1}{2}} \left((1 - \sqrt{\frac{s'}{s}})^2 - \frac{Q^2}{s} \right)^{\frac{1}{2}}. \quad (41)$$

Putting these back into Eq. (37) we get that

$$f^{(n)}(s, s', Q^2) = (2)^n \frac{1}{(n-1)!} \frac{1}{(n-2)!} \left(\frac{\sigma_{in}}{\sigma_{tot}} \right)^2 \left(\frac{\sigma_{el}}{\sigma_{tot}} \right)^{n-1} \left(\frac{Q^2}{s'} \right)^{n-1} \left((1 + \sqrt{\frac{s'}{s}})^2 - \frac{Q^2}{s} \right)^{\frac{1}{2}} \left((1 - \sqrt{\frac{s'}{s}})^2 - \frac{Q^2}{s} \right)^{\frac{1}{2}} \quad (42)$$

valid for $s \sim s'$ large $s' \gg Q^2 \gg (n\mu)^2$ which has now become a function of the ratios s'/s and Q^2/s only and is independent of s .

4. CONCLUSIONS

The model we described incorporated some of the most important properties one expects from the physical scattering amplitude to hold. These were : relativity, unitarity, cluster decomposition and analyticity. One is therefore naturally led to the question : would it be possible to apply the model to some physical process at high energies ? Unfortunately, the model, as it stands, gives a total cross-section decreasing like s^{-1} and a single particle distribution which is isotropic in the centre-of-mass transverse and longitudinal momenta, in blatant contradiction to the most prominent features of high

energy scattering¹⁴⁾. These bad properties are consequences of the simple fact that there is only S wave elastic scattering in the model. Consequently, from the point of view of possible applications at high energy, one would have to generalize the model to have more realistic angular behaviour in the elastic channel. How difficult this is going to be, we do not know, but it is one of the things we would like to investigate. It would also be of interest to see whether the inclusion of internal quantum numbers like isospin is possible in the model.

On the theoretical side, the model had two shortcomings : the distinguishability of the produced particles and the lack of crossing symmetry. With respect to the first one, there does not seem to be much else to do than possibly to investigate the conditions under which this can be regarded as a reasonable approximation. The combination of cluster decomposition and indistinguishability seems to lead to such a complicated equation for the multi-particle amplitudes as to exclude any possibility of a solution in closed form. The requirement of crossing symmetry or rather some approximate form of it could perhaps be used to constrain the input functions of the model.

There seem to be some straightforward generalizations of the model. One could, for example, have vertices with higher number of particle lines in the connected part of the K matrix and still be able to obtain an explicit solution. This would give higher order correlations in the production amplitude.

A low or medium energy approximation could be obtained in the model by truncating it in such a way that only a finite number of particles can be produced. Physically, this would correspond to neglecting the effect of high multiplicity production thresholds.

A model, similar in spirit to that of paper I (based on the phase operator rather than on the K matrix), has been put forward recently¹⁵⁾.

ACKNOWLEDGEMENTS

The author would like to thank Dr. D.I. Olive and Dr. M. Toller for valuable discussions and useful criticism. He would also like to thank the Theoretical Study Division of CERN for its hospitality.

R E F E R E N C E S

- 1) L.B. Rédei, CERN preprint TH.1454 (1972) - to be published in Nuovo Cimento.
- 2) For a model with cluster decomposition and unitarity at infinite energy, see the recent preprint by :
R. Aviv, R.L. Sugar and R. Blankenbecler, Unitary models at multi-particle amplitudes, University of California, Santa Barbara preprint (1971). See also :
G. Calucci and R. Jengo, Eikonal formalism for multiparticle production, Trieste preprint (1972).
- 3) For a general discussion of the Heitler equations, see, e.g., in S.S. Schweber, "An Introduction to Relativistic Quantum Field Theory", Row, Peterson and Co., New York (1961).
- 4) For a discussion of cluster decomposition and of other general properties of the S matrix, we refer to :
R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne, "The Analytic S-Matrix", Cambridge University Press, Cambridge (1966).
- 5) See, e.g., in J.C. Slater, "Quantum Theory of Matter", McGraw-Hill, Inc., New York (1968).
- 6) L. Bertocchi, S. Fubini and M. Tonin, Nuovo Cimento 26, 626 (1962).
- 7) D. Amati, S. Fubini and A. Stanghellini, Nuovo Cimento 26, 896 (1962).
- 8) This was pointed out to the author in a discussion with Dr. O. Brander.
- 9) In the calculations, one has to make repeated use of the real part of Eq. (17) combined with Eq. (23).
- 10) G.F. Chew, M.L. Goldberger and F.E. Low, Phys.Rev.Letters 22, 208 (1969).
- 11) A. Bassetto, M. Toller and L. Sertorio, Nuclear Phys. B34, 1 (1971).
- 12) D.I. Olive, Phys.Rev. 135, B745 (1964), and also
S. Coleman and R.E. Norton, Nuovo Cimento 38, 438 (1965).
- 13) R.P. Feynman, Phys.Rev.Letters 23, 1415 (1969), and
J. Benecke, T.T. Chan, C.N. Yang and E. Yen, Phys.Rev. 188, 2159 (1969).
- 14) See, e.g., the review articles :
L. Van Hove, Phys.Reports 10, No.7 (1971), and
F. Zachariasen, Phys.Reports 20, No.1 (1971).

- 15) K. Rzażewski, Solvable relativistic unitary model for multiparticle production, University of Warsaw preprint, IFT/5/72 (1972).

Schematic Representation of W

$W =$

FIG.1