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ROLE OF CROSSING AND THE ρ MESON
IN LOW ENERGY $\pi\pi$ AMPLITUDES

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A B S T R A C T

The crossing conditions on physical region $\pi\pi$ partial wave amplitudes are used to study the implications of the existence of the ρ . We show that the mere existence of the ρ does not constrain appreciably the s wave scattering lengths and that no correlation from crossing between the ρ and the $I = 0$ ϵ resonance can be established without further physical assumptions.

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In this series of papers, we undertake the dispersion relation analysis of low energy $\pi\pi$ scattering. Our eventual ambition is to get a better description of $\pi\pi$ scattering amplitudes by putting together the available experimental information and properties derived from fundamental principles : analyticity, unitarity and crossing.

The best known feature of $\pi\pi$ scattering below 1 GeV is the existence of the ρ resonance in the $I = 1$ p wave. In this paper we investigate the classes of $\pi\pi$ amplitudes which satisfy unitarity and crossing, accommodate the ρ meson and no $I = 2$ s wave resonance below 1 GeV, and are elastic up to the $K\bar{K}$ threshold to a good approximation, as is indicated by experiment ¹⁾.

Another formulation of our question is : how many important parameters determine uniquely the main features of $\pi\pi$ scattering below 1 GeV ? In the forward dispersion analysis of Morgan and Shaw ²⁾, once the ρ is fixed, two extra pieces of information are necessary : for instance the energy at which the $I = 0$ s wave phase δ_0^0 passes through 90° , and the ratio of s wave scattering lengths a_0^0/a_0^2 . However, in that approach, one only imposes s-u crossing at $t = 0$, and the question is whether full s, t, u crossing further reduces the arbitrariness. On the other hand, the unphysical region crossing constraints ³⁾ provide very useful tools, and a number of models have been built with them ⁴⁾⁻⁶⁾. In particular, Le Guillou, Morel and Navelet ⁴⁾ have concluded that the existence of the ρ , put together with some information about Regge ρ parameters lead to more or less uniquely defined $\pi\pi$ s waves whose scattering lengths are roughly compatible with Weinberg's current algebra predictions ⁷⁾.

Here we use a method which expresses full crossing directly on physical amplitudes. As we shall see, we obtain an even larger arbitrariness than Morgan and Shaw. In the following paper we show how the arbitrariness is reduced by incorporating the experimental s wave information.

It has been known for a long time ⁸⁾ (although perhaps not fully appreciated in $\pi\pi$ scattering) that exact partial wave equations can be written by projecting onto partial waves twice-subtracted fixed momentum transfer dispersion relations. Recently, Roy ⁹⁾ has studied the $\pi\pi$ equations, and it has been advocated elsewhere ¹⁰⁾ that since they express the crossing and analyticity properties directly on partial wave amplitudes at physical energies, they are a most convenient tool to perform low energy $\pi\pi$ phenomenology. These partial wave equations are of the form

$$f_e^I(s) = \begin{pmatrix} a_0^0 \\ 0 \\ a_0^2 \end{pmatrix} \delta_{e0} + \left(\frac{2a_0^0 - 5a_0^2}{3} \right) \left(\frac{s-4}{4} \right) \begin{pmatrix} \delta_{e0} \\ \frac{1}{6} \delta_{e4} \\ -\frac{1}{2} \delta_{e0} \end{pmatrix} \\ + \sum_{I'=0}^2 \sum_{\ell'=0}^1 \int_4^\infty K_{\ell,I}^{\ell',I'}(s,s') \operatorname{Im} f_{\ell'}^{I'}(s') ds' + \varphi_e^I(s) \quad (1)$$

where a_0^0 and a_0^2 are the s wave scattering lengths, we have set $m_\pi = 1$, and φ_ℓ^I is a well-defined sum of higher partial wave contributions ($\ell \geq 2$)

$$\varphi_\ell^I(s) = \sum_{I'=0}^2 \sum_{\ell'=2}^\infty \int_4^\infty K_{\ell,I}^{\ell',I'}(s,s') \operatorname{Im} f_{\ell'}^{I'}(s') ds' \quad (2)$$

For all details concerning the equations and the form of the kernels $K_{\ell I}^{\ell' I'}$ we refer the reader to Ref. 10). These equations must be supplemented by another set of relations ^{9), 10)} which, however, constrain only higher partial waves ($\ell \geq 2$). Therefore, for s and p waves ($\ell = 0, 1$) Eq. (1) is a complete expression of the crossing property in the region $-4m_\pi^2 \leq s \leq 60 m_\pi^2$ [see Ref. 10] if the driving term $\varphi_\ell^I(s)$ is known.

In practice, we evaluate these driving terms as follows. We split all s' integrals in Eqs. (1) and (2) into two parts $\int_4^N + \int_N^\infty$ where N is an energy squared above which a Regge representation of the total amplitude is convenient. For $s' > N$ we compute the Regge contribution in the original fixed t dispersion relations, and project the result on partial waves ^{*)}. For $s' < N$ and $\ell' \geq 2$ we directly evaluate the resonance contributions. With reasonable values for asymptotic cross-sections, and with the experimental parameters of the f_0 resonance, the resulting driving terms $\varphi_\ell^I(s)$ for $\ell = 0, 1$ are small below 1 GeV (see next paper ¹¹⁾ for numerical details). Furthermore, at low energies, and in particular for $0 \leq s \leq 4$, the supplementary crossing conditions are satisfied to a good accuracy. Our qualitative results are somewhat independent of the exact values of the driving terms, provided the order of magnitude is correct, and therefore our experimental ignorance of $\pi\pi$ Regge parameters is not a practical drawback here.

*) In full rigour, by doing this we of course include some s and p wave contributions (for $s' > N$) in the definition of the driving terms.

We now define what we call a unitary and crossing symmetric amplitude at low energies.

Criterion Since $f_\ell^I(s)$ as defined by Eq. (1) for $\ell = 0, 1$, in terms of $\text{Im } f_\ell^I(s')$, (a) satisfies all crossing and positivity constraints whatever the value of $\text{Im } f_\ell^I(s') > 0$ ¹⁰⁾, and (b) has by construction ¹⁰⁾ the imaginary part $\text{Im } f_\ell^I(s)$, we define a solution to Eq. (1) by requiring that all three amplitudes f_0^0, f_0^2, f_1^1 also satisfy unitarity $\text{Im } f_\ell^I(s) = (s-4/s)^{\frac{1}{2}} |f_\ell^I(s)|^2$ to a given accuracy (e.g., 1%) in the elastic region. We choose the elastic region to be $s < 4m_K^2$ on phenomenological grounds ¹⁾ (indeed 4π inelasticity seems weaker than 1%, below the $K\bar{K}$ threshold).

Once s and p waves are known, higher partial wave amplitudes $\ell \geq 2$ are also known in the low energy region without any further iteration of the equations ¹⁰⁾, and the resulting total amplitude satisfies fixed t dispersion relations, crossing and unitarity for $s \lesssim 60 m_\pi^2$, to a very good accuracy.

In order to realize this, we first choose $\text{Im } f_\ell^I(s)$ to be the imaginary part of an amplitude $g_\ell^I(s)$ parametrized so that it satisfies elastic unitarity exactly below $s = 4m_K^2$. We then compute $f_\ell^I(s)$ by Eq. (1) and we fix the parameters in $g_\ell^I(s)$ by minimizing the difference $|f_\ell^I(s) - g_\ell^I(s)|$ in the physical region $4 \leq s \leq 60 m_\pi^2$ (the CERN program MINUIT was used).

To prevent the parametrization from playing a crucial role, we have used two very different analytic forms, which both lead to the same results in practice. The first is a simple K matrix approach, elastic up to infinity,

$$g_\ell^I(s) = \frac{K_\ell^I(s)}{1 + CM(s) K_\ell^I(s)} \quad (3)$$

where $CM(s)$ is the Chew-Mandelstam function

$$CM(s) = \frac{2}{\pi} \sqrt{\frac{s-4}{s}} \text{Log} \left(\frac{\sqrt{4-s} + \sqrt{s}}{2} \right) \quad (4)$$

and $K_\ell^I(s)$ is taken as a rational function in s

$$K_\ell^I(s) = \frac{a_0^I \delta_{\ell 0} + b_1^I (s-4) + b_2^I (s-4)^2 + \dots}{1 + c_1^I (s-4) + c_2^I (s-4)^2 + \dots} \quad (5)$$

This method is simple and practical, however, it is not well suited for treating inelasticity, and another method consists in expressing the partial wave S matrix elements as products of rational elements in a conformal variable z

$$\mathcal{S}_\ell^I(s) = (1 + 2i\sqrt{\frac{s-4}{s}} g_\ell^I(s)) = \left(\frac{z - z_R^*}{z - z_R} \cdot \frac{z + z_R}{z + z_R^*} \right) \left(\frac{i - a z^n}{i + a z^n} \right) \dots \quad (6)$$

[where z_R is complex (resonance pole), a is real and n is an odd integer], z being related to s by

$$q^2 = \frac{s-4}{4} = \frac{\lambda^2 z^2}{(z^2 - z_0^2)(z^2 - z_0^{*2})} \quad (7)$$

where $z_0 = e^{i\theta}$ and λ are determined by requiring that $z = i$ corresponds to $s = 0$ ($q = i$), and $z = 1$ to $s = 4m_K^2$ (inelastic threshold). Within this parametrization, it is easy to insert a given inelasticity for $s \geq 4m_K^2$, in particular the S^* effect¹⁾. The two s wave scattering lengths a_0^o and a_0^2 , which determine the subtraction terms in Eq. (1), are of course among the parameters in Eqs. (5) and (6).

Our physical ingredients, besides unitarity and crossing, are :

- a) the parameters of the ρ resonance, m_ρ and $\sqrt{s}$, in f_1^1 (the p wave scattering length itself a_1^1 is computed from the equations, it is not imposed) ;
- b) that to a good approximation (1%) inelasticity starts in $\pi\pi$ at $s = 4m_K^2$ and not at $s = 16m_\pi^2$;
- c) the Regge and higher wave parameters which determine the driving terms ;
- d) we also assume that the $I = 2$ s wave is non-resonant below 1 GeV.

We do not claim to obtain a complete enumeration of $\pi\pi$ amplitudes satisfying these criteria, we want to have an estimate of the multiplicity of solutions, and of the number of important parameters.

We summarize our results.

- 1) We have found no published model which satisfies our criterion. In particular, simple analytic forms which fit the data will in general violate Eq. (1) grossly. However, for several models which incorporate the ρ meson by construction^{4),5),12)}, we can produce amplitudes which are close to the model predictions near threshold ($s < m_\rho^2$) and do satisfy the criterion.

2) Figure 1 shows the region in the (a_0^0, a_0^2) plane which we have explored and where we have found solutions. This band is much larger than the range obtained by GMN ⁴⁾. It is clear that the existence of the ρ meson does not put a stronger constraint on a_0^0 and a_0^2 than to lie in our band. We have reasons to believe that our assumption about the absence of an s wave exotic resonance makes it impossible to reach the point $a_0^0 = a_0^2 = 0$, and therefore that the upper boundary of the band has a physical meaning. However, we have not explored to the right or to the left of the band which is by no means restrictive.

3) At each point in this region we have an infinity of solutions. In fact, we can choose the energy where δ_0^0 passes through 90° , and this gives us a family of solutions. This is exemplified in Fig. 2.

4) However, once a_0^0, a_0^2, m_ρ and τ_ρ are fixed, the very low energy parameters (i.e., the p wave scattering length, and the s wave effective ranges) are essentially determined. Figure 2 gives an impression of the spread of solutions. In particular: (a) the $I = 2$ s wave is very tightly constrained, and (b) the s and p wave amplitudes in the unphysical region $0 \leq s \leq 4$ are essentially determined, up to very slight differences.

5) Within this band, all amplitudes are such that $\delta_0^0(m_\rho) > 0$ and $\delta_0^2(m_\rho) < 0$. Consequently, each s wave amplitude possesses a zero in the region $0 \leq s \leq 4$ or near it [e.g., if $a_0^2 > 0$ (or $a_0^0 < 0$), the zero of f_0^2 (or of f_0^0) lies in the physical region $s > 4$].

6) In the GMN calculation, a fundamental ingredient is the unsubtracted dispersion sum rule of Olsson ¹³⁾

$$L \equiv 2a_0^0 - 5a_0^2 = \frac{3}{4\pi^2} \int_4^\infty \frac{ds}{\sqrt{s(s-4)}} \left[\sigma_{tot}^{\pi^+\pi^-}(s) - \sigma_{tot}^{\pi^+\pi^+}(s) \right] \quad (8)$$

which we may insert in our analysis. It is natural to split Eq. (8) as

$$L = L_{sp} + L_\infty \quad (9)$$

where L_{sp} is the s and p wave contribution to the right-hand side of (8) for $s \leq N$, and L_∞ includes the higher wave, $l \geq 2$, and asymptotic, $s > N$, contributions. In this case, the asymptotic contribution L_∞ is quite important (20 to 30% of the total), and this requires a good knowledge

of the Reggeized ρ residue at $t = 0$. Imposing Eq. (8) with a given L_ω in our calculation has the effect of essentially choosing the energy where $\delta_0^0(E) = 90^\circ$ at fixed a_0^0 and a_0^2 , and of slightly reducing the width of our band. It does not restrict a_0^0 and a_0^2 to a small range like in GMN, and the results are quite sensitive to L_ω . Therefore, we prefer to operate backwards and, for each of our solutions, to define L_ω as the value which saturates Eqs. (8) and (9). We may then compare it with the available (rather ill defined) experimental information.

We draw the following conclusions.

- 1) Our results demonstrate the failure of the uniqueness claim 4),5). Although we can construct solutions close to those of Refs. 4), 5), and in that sense these models indeed satisfy crossing, it is not true that the existence of the ρ and Eq. (8) suffices to fix the s wave amplitudes unambiguously *).
- 2) Our findings are somewhat similar to those of Morgan and Shaw 2). They differ in the following respects : (a) we have a larger arbitrariness, i.e., we must fix the energy where $\delta_0^0 = 90^\circ$ and both scattering lengths, and (b) the $I = 2$ s wave phase is strongly correlated to the $I = 0$ one (this, we believe, stems from our more complete treatment of crossing).
- 3) Our findings are also similar to those of Piguet and Wanders 6) who work with the unphysical region constraints, and have noticed a similar arbitrariness. For phenomenological purposes, however, the present method which applies directly to physical amplitudes, is much more convenient in practice (in particular above 500 MeV).
- 4) Another interesting result is the absence of any crossing correlation between the ρ and ϵ resonances. In this sense, crossing does not constrain the striking features of low energy $\pi\pi$ s and p waves. This may be understood by referring to the Born term of a Lagrangian containing a contact $\lambda (\vec{\pi} \cdot \vec{\pi})^2$ term, an s wave $I = 0$ ϵ pole and a p wave ρ pole. In Born approximation the amplitude depends on five parameters, i.e., λ , m_ρ , m_ϵ and the coupling constants g_ρ^2 and g_ϵ^2 , all uncorrelated by crossing, and which may be eliminated in favour of a_0^0 , a_0^2 , r_ρ , m_ρ and some combination of r_ϵ and m_ϵ . If the coupling constants are small, one expects that unitarity

*) The uniqueness claim of GMN has been discussed by various people 6),14).

effects ^{*}) will not change the basic character of the Born term. This fact may also be related to the CDD ambiguity noticed by Atkinson and Warnock ¹⁵⁾.

5) In principle the asymptotic behaviour of amplitudes might discriminate between our various possibilities. However, this is difficult in practice here since asymptotic parameters are badly known and appear through the driving terms which are small in the regions of interest.

6) The inelasticity, however, is very useful, and we shall see in the next paper how the $K\bar{K}$ threshold effect can be used to reduce the possible classes of $\pi\pi$ amplitudes.

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^{*}) Notice that on an absolute scale, e.g., compared to the axiomatic bounds $a_0^0 \geq -7.5$, $a_0^2 \geq -7.1$ [B. Bonnier and R. Vinh Mau - Phys.Letters 24B, 9 (1967) ; A.K. Common - Nuovo Cimento 63, 451 (1969)] our band corresponds to very small scattering lengths.

REFERENCES

- 1) M. Alston Garnjost et al. - Phys.Letters 36B, 152 (1971).
- 2) D. Morgan and G. Shaw - Phys.Rev. D2, 520 (1970) ; Nuclear Phys. B10, 261 (1969).
- 3) A. Martin - Nuovo Cimento 47A, 265 (1967) ;
A.P. Balachandran and J. Nuyts - Phys.Rev. 172, 1821 (1968) ;
R. Roskies - Nuovo Cimento 65A, 467 (1970).
- 4) J.C. Le Guillou, A. Morel and H. Navelet - Nuovo Cimento 5A, 659 (1971).
- 5) B. Bonnier and P. Gauron - Nuclear Phys. B21, 465 (1970).
- 6) O. Piguet and G. Wanders - to appear in Nuclear Phys.
- 7) S. Weinberg - Phys.Rev.Letters 17, 616 (1966).
- 8) R. Oehme - Phys.Rev. 100, 1503 (1955) ; 102, 1174 (1956) ;
R.H. Capps and G. Takeda - Phys.Rev. 103, 1877 (1956) ;
G.F. Chew, M.L. Goldberger, F.E. Low and Y. Nambu - Phys.Rev. 106, 1337 (1957) ;
F. Steiner - Fortschr.Phys. 19, 115 (1971).
- 9) S.M. Roy - Phys.Letters 36B, 353 (1971).
- 10) J.L. Basdevant, J.C. Le Guillou and H. Navelet - Nuovo Cimento 7A, 363 (1972).
- 11) J.L. Basdevant, C.D. Froggatt and J.L. Petersen - CERN Preprint TH. 1519 (1972).
- 12) J.L. Basdevant and J. Zinn-Justin - Phys.Rev. D3, 1865 (1971).
- 13) M.G. Olsson - Phys.Rev. 162, 1338 (1967).
- 14) D. Morgan - "mini-Rapporteur" talk at Amsterdam Conference (1971) ;
Rutherford Laboratory Preprint RPP/C/30 (1971) ;
E.P. Tryon - Columbia University Preprint NYO-1932(2)-196 (1971) ;
- 15) D. Atkinson and R.L. Warnock - Phys.Rev. 188, 2098 (1969).

FIGURE CAPTIONS

Figure 1 Band in the a_0^0, a_0^2 plane inside of which we have found solutions to the crossing equations with our assumptions. We also indicate the Weinberg predictions and the range obtained by Le Guillou, Morel and Navelet (GMN).

Figure 2 Family of s wave phases corresponding to the same scattering lengths (here $a_0^0 = 0.37$, $a_0^2 = 0.0$) and the same ρ meson, and satisfying our criterion (here the inelasticity and the driving term were taken to be zero, however, the qualitative picture is the same if driving terms are included).

FIG.1