CM-P00058917

Archives

Ref.TH.1454-CERN

SOLVABLE RELATIVISTIC UNITARY MODEL FOR MULTIPARTICLE PRODUCTION

L.B. Rédei *)
CERN -- Geneva

ABSTRACT

By making a Hermitian ansatz for the K matrix and solving the resulting equation for the scattering amplitude we derive a model for multiparticle production for which the amplitude can be given in explicit closed form. The amplitude satisfies many-particle unitarity exactly (there is a non-vanishing amplitude for the production of not more than N particles, where N is arbitrary but fix). The requirement of Lorentz invariance is also satisfied.

Analytic properties are discussed and the elastic amplitude is shown to be the boundary value of a single analytic function of s (s being the centre-of-mass energy squared) having branch points at production thresholds.

*) Permanent address : Department of Physics, Umeå University, Umeå, Sweden.

1. INTRODUCTION

In a previous short report ¹⁾, subsequently referred to as I, we have presented a solvable model for many-particle production for which the amplitude could be given in closed explicit form. The amplitude obtained satisfied many-particle unitarity and also the requirement of Lorentz invariance. However, apart from not being crossing symmetric, the model in its simple form, as discussed in I, had some unorthodox analytic properties. The elastic amplitude was not the boundary value of a single analytic function of s (s being the centre-of-mass energy squared), but rather, in different segments of the positive real axis it was the boundary value of different analytic functions. On the other hand, one knows from the results of field theory ²⁾ that the elastic amplitude is the boundary value of a single analytic function of s if the requirement of microcausality is satisfied. The purpose of this paper is, apart from providing some details omitted in I, to put forward a slight but non-trivial generalization of the previous model which is such that while preserving the good features of I it gives rise to an amplitude with more reasonable analytic properties. In other words, the model to be described here gives a non-trivial amplitude for the production of not more than N particles, (N being an arbitrary but fixed positive integer) it is solvable, by which we mean that the amplitude is given in terms of a finite number of convergent, finite-dimensional, definite integrals. The amplitude satisfies many-particle unitarity exactly and it is Lorentz invariant. In addition, the elastic amplitude is now the boundary value of a single analytic function, regular for $\text{Im } s \neq 0$.

Again we restrict ourselves to scalar particles and to S waves in the elastic scattering. Generalizations to have dependence on angle, spin and isospin are possible but will not be discussed here.

The model to be described here consists again of making a relatively simple Hermitian ansatz (more complicated than that of I, however) for the K matrix and solving the resulting linear integral equations ³⁾ for the on the energy shell amplitudes.

In the next Section, we shall first discuss very briefly a way to construct an interpolating Heisenberg field from a given K matrix and also touch upon the power series expansion of the scattering amplitude. This will be followed by Section 3, where our model is going to be presented, solved and its properties discussed.

For the meagre list of references to previous known examples of solvable unitary models for particle production, paper I can be consulted. [In fact there are no previous examples of a solvable, relativistic, unitary model for the production of more than one particle ⁴⁾.]

The article is self-contained in the sense that familiarity with the results of the previous paper I is not required.

2. FORMULATION OF QUANTUM FIELD THEORY IN TERMS OF A GIVEN K MATRIX

For simplicity, we consider a system of stable scalar particles with mass $m > 0$, the generalization of the content of this section to more complicated kinematical situations being obvious. We denote by $\psi(x)$ the corresponding standard free-field acting on the appropriate Fock space, i.e.,

$$(\square - m^2) \psi(x) = 0 . \quad (1)$$

We write the scattering operator S as

$$S = (I - ig K)^{-1} (I + ig K) , \quad (2)$$

where K is an operator, the matrix elements of which give the usual K matrix ³⁾ and g is a real parameter defining the strength of the interaction. We write

$$K = \int d^4x \mathcal{K}(x) , \quad (3)$$

where $\mathcal{K}(x)$ is some functional of the field $\psi(x)$. In order to have a unitary and Lorentz invariant S matrix, we assume $\mathcal{K}(x)$ to be a Hermitian scalar density. Let us now discuss briefly a simple way to construct an interpolating Heisenberg field ⁵⁾ from a given K matrix.

We may define, first of all,

$$K(y) = \int d^4x e(y-x) \mathcal{K}(x) , \quad (4)$$

where $e(x)$ is the characteristic function for the inside of the backward light cone at the point 0 [i.e., $e(x) = 1$ if $x^2 \geq 0$ and $x_0 \leq 0$, it is zero otherwise ⁶⁾]. The operator $S(y)$ defined as

$$S(y) = (I - ig K(y))^{-1} (I + ig K(y)) \quad (5)$$

is then a unitary Lorentz scalar quantity with asymptotic behaviour

$$\lim_{y_0 = \pm \infty} S(y) = \begin{Bmatrix} S \\ I \end{Bmatrix} . \quad (6)$$

The Heisenberg-field $\psi_H(x)$ defined as

$$\psi_H(x) = S^\dagger(x) \psi(x) S(x) \quad (7)$$

behaves like a scalar under Lorentz transformation and it fulfils the asymptotic condition

$$\lim_{x_0 = \pm \infty} \psi_H(x) = \begin{cases} \psi_{out}(x) \\ \psi_{in}(x) \end{cases} \quad (8)$$

with

$$\psi_{in}(x) = \psi(x)$$

and

$$\psi_{out}(x) = S^\dagger \psi_{in}(x) S . \quad (9)$$

Thus, the field $\psi_H(x)$ is indeed a Lorentz scalar interpolating field. In general [unless $K(x)$ happens to have some very special properties] it will not satisfy local commutativity, i.e.,

$$[\psi_H(x), \psi_H(y)] \neq 0 \quad \text{for} \quad (x-y)^2 < 0 \quad (10)$$

and therefore our model is not going to be microcausal, nor will it be crossing symmetric.

From Eq. (2) follows the power series expansion for the S matrix

$$\langle b | S | a \rangle = \langle b | a \rangle + 2 \sum_{n=1}^{\infty} (i g)^n \langle b | \left(\prod_{j=1}^n \int d^4 x_j K(x_j) \right) | a \rangle . \quad (11)$$

From this, one can if one so wishes, derive a diagrammatic representation of the S matrix by normal ordering of the right-hand side of this equation. Because of the lack of time ordering compared to the usual Feynman-Dyson expression for the S matrix in terms of a local Lagrangian, the occurrence of $\langle 0 | \Psi(x) \Psi(y) | 0 \rangle$ as particle propagator, rather than the usual time ordered $\langle 0 | T \Psi(x) \Psi(y) | 0 \rangle$ will be characteristic for this expansion. In momentum space, this implies the replacement of the Feynman propagator $(p^2 - m^2)^{-1}$ by $\delta_+(p^2 - m^2)$, i.e., in this representation the virtual particles are restricted to be on their positive energy mass shell. Since in the following we shall not need to make use of the power series expansion, we do not discuss this any further.

3. SOLVABLE UNITARY MODEL FOR MANY-PARTICLE PRODUCTION

We discuss first the connection between the K matrix and the scattering amplitude. Introducing the operator T by

$$T = S - I \quad (12)$$

we obtain from Eq. (2) that ³⁾

$$T = 2igK + igKT. \quad (13)$$

Taking the matrix element of this equation and inserting a complete set of states in the right-hand side, we obtain that

$$\langle b | T | a \rangle = 2ig \langle b | W | a \rangle + ig \sum_c \delta(P_a - P_c) \langle b | W | c \rangle \langle c | T | a \rangle \quad (14)$$

for $P_a = P_b$, where P is the total four-momentum and where $\langle b | T | a \rangle$ is the on the energy shell scattering amplitude

$$\langle b | T | a \rangle = \delta(P_b - P_a) \langle b | M | a \rangle \quad (15)$$

and the quantity W is defined by

$$\langle b | K | a \rangle = \delta(P_b - P_a) \langle b | W | a \rangle. \quad (16)$$

For a given W Eq. (14) constitutes a coupled system of linear integral equations for the scattering amplitude on the energy shell. Our model will consist of making an ansatz for W (which, because of the above equation amounts to the same as making an ansatz for K) and solving Eq. (14) for the scattering amplitude.

We consider two kinds of scalar particles with masses m and μ different from zero. Let $m > \mu$ and we refer to the particles correspondingly as heavy, respectively as light. We consider the scattering of two heavy particles with four-momenta p_1 and p_2 . We denote the momenta of the outgoing heavy particles by p'_1 and p'_2 (the number of heavy particles will be conserved in the model) and the momenta of the produced light particles by k_j . The total momentum will be denoted by P , i.e., $P = p_1 + p_2$ and we shall use the variable $s = P^2$.

We define W (and therefore K) by the following ansatz for its matrix elements

$$\begin{aligned} \langle k_1, \dots, k_n; p'_1, p'_2 | W | p_1, p_2 \rangle &= \langle p_1, p_2 | W | k_1, \dots, k_n; p'_1, p'_2 \rangle^* = \\ &= \frac{1}{2} (2\pi)^{-2} (2\pi)^{-3n/2} \sqrt{\frac{n!}{E'_1 E'_2 E_1 E_2}} \left(\prod_{j=1}^n \frac{1}{\sqrt{2\omega_j}} \right) \alpha_n(s), \quad n=1,2,\dots, \end{aligned} \quad (17)$$

$$\langle k'_1, \dots, k'_n; p'_1, p'_2 | W | k_1, \dots, k_n; p_1, p_2 \rangle = \frac{\beta}{g} \frac{1}{2} (2\pi)^{-2} \frac{1}{n!} \frac{1}{\sqrt{E'_1 E'_2 E_1 E_2}} \quad (18)$$

$$\sum_{Q_n} \alpha_{Q_n} \delta(k_1 - k'_1) \dots \delta(k_n - k'_n), \quad n=1,2,\dots,$$

and

$$\langle p'_1, p'_2 | W | p_1, p_2 \rangle = \frac{1}{2} (2\pi)^{-2} \frac{1}{\sqrt{E'_1 E'_2 E_1 E_2}} f(s) \quad (19)$$

where

- 1) $E = (p^2 + m^2)^{\frac{1}{2}}$, $\omega = (k^2 + \mu^2)^{\frac{1}{2}}$;
- 2) Q_k are the permutations of the light particle momenta k_j and $\beta > 0$ is a real constant;
- 3) the $\alpha_n(s)$ are functions such that $\alpha_n^{*}(s^*) \alpha_n(s)$ is regular for $\text{Im } s \neq 0$ and for $\text{Re } s > (2m)^2$ and also $\alpha_n(s) \equiv 0$ for $n > N$ where N is some fixed positive integer. Apart from these requirements the functions $\alpha_n(s)$ and also the number N need not be specified;

- 4) $f(s)$ is some function which is real for $\text{Im } s = 0$, $\text{Re } s \geq 4m^2$ to be determined later by demanding the elastic amplitude to be given by the boundary value of a single analytic function.

The other matrix elements of W we put equal to zero. The K matrix defined this way is Hermitian and Lorentz invariant giving rise to a unitary and Lorentz invariant S matrix. There will be a non-vanishing amplitude for the production of not more than N light particles, but there will be no production of heavy particles in our model. For $\alpha_n(s) = \text{constant}$, $\beta = g$ and $f(s) = 1$, one recovers the model already discussed in I.

It is illuminating to write down the corresponding expression for the density of the K matrix in terms of the free-field operators $\psi(x)$ and $\phi(x)$ for the heavy and light particles. In fact

$$\begin{aligned} K(x) = & \frac{1}{2} \left(\frac{1}{\square} (\psi^{(+)}(x))^2 (\psi^{(-)}(x))^2 0 + \frac{1}{2} \frac{\beta}{g} (\psi^{(+)}(x))^2 (\psi^{(-)}(x))^2 (1-0) \right. \\ & \left. + \sum_{n=1}^{\infty} (\alpha_n(\square) (\phi^{(+)}(x))^n (\psi^{(+)}(x))^2 (\psi^{(-)}(x))^2 0 + \text{Hermitian conjugate} \right), \end{aligned} \quad (20)$$

where $\square$ is the d'Alembertian, 0 is the projection operator, projecting onto the subspace containing no light particles and $\psi^{(+)}$, $\phi^{(+)}$ are the creation operators (positive frequency parts); the $\psi^{(-)}$, $\phi^{(-)}$ the annihilation operators (negative frequency parts) for the heavy, respectively, for the light particles; gives rise through Eqs. (3) and (16) to the matrix elements of W as given by Eqs. (17), (18) and (19). From the above expression one can show for example that the K matrix used by us is a well-defined mathematical object, in fact a bounded self-adjoint operator, at least at finite energies (by finite energies we mean that we restrict the operator K to the invariant subspace formed out of plane waves with centre-of-mass energies less than some arbitrary but finite constant C). This implies through Eq. (2) that the scattering operator S is also a well-defined quantity (no divergences) and that the power series expansion in g has a non-zero radius of convergence as long as C is kept finite. Complications may arise at infinite energy, depending on the detailed properties of $\alpha_n(s)$. That the scattering operator indeed has these good properties will be obvious from the explicit expression for the amplitude we are now going to derive.

We introduce the invariant amplitudes $\bar{M}^{(n)}$ on the energy shell for the production of n light particles, $n=0,1,\dots,N$, [$\bar{M}^{(0)}$ being the amplitude for the elastic scattering of two heavy particles] by

$$\langle k_1, \dots, k_n; p'_1, p'_2 | M | p_1, p_2 \rangle = \frac{1}{2} (2\pi)^{-2} (2\pi)^{-3n/2} \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{E_2' E_1' E_2 E_1}} \quad (21)$$

$$\left(\prod_{j=1}^n \frac{1}{\sqrt{2\omega_j}} \right) \bar{M}^{(n)}(k_1, \dots, k_n; p'_1, p'_2; p_1, p_2),$$

for

$$p_1 + p_2 = P = p'_1 + p'_2 + \sum_j k_j.$$

With the simple ansatz for K we have made, the on the energy shell amplitudes $\bar{M}^{(n)}$ depend on the heavy particle momenta only through the sums $p'_1 + p'_2 = p_1 + p_2 - \sum_j k_j$ and $p_1 + p_2 = P$. We may therefore write

$$\bar{M}^{(n)} = \bar{M}^{(n)}(k_1, \dots, k_n; P). \quad (22)$$

[One can show this, e.g., by looking at the normal ordered form of the expansion given by Eq. (11).]

We are now ready to solve Eq. (14) for the scattering amplitude. We take $|a\rangle$ and $|b\rangle$ to be two states with two heavy particles each and obtain the equation for the elastic amplitude

$$\begin{aligned} \bar{M}^{(0)}(s) &= 2ig f(s) + 2ig (2\pi)^{-2} \int d^3 p_1'' \int d^3 p_2'' \frac{1}{4E_2'' E_1''} \delta(P - p_1'' - p_2'') \\ & f(s) \bar{M}^{(0)}(s) + 2ig (2\pi)^{-2} \sum_{n=1}^N \theta(\sqrt{s} - 2m - n\mu) (2\pi)^{-3n} \int d^3 p_1'' \int d^3 p_2'' \end{aligned} \quad (23)$$

$$\frac{1}{4E_2'' E_1''} \left(\prod_{j=1}^n \int d^3 q_j \frac{1}{2\omega_j} \right) \delta(P - p_1'' - p_2'' - \sum_{\ell} q_{\ell}) \alpha_n^*(s) \bar{M}^{(n)}(q_1, \dots, q_n; P),$$

where θ is the step function and where we have used Eqs. (14), (17) and (19). We can carry out the integration over the heavy particle momenta p_1'' and p_2'' and get

$$\begin{aligned} \bar{M}^{(0)}(s) &= 2ig f(s) + ig f(s) \mathcal{L}(P) \bar{M}^{(0)}(s) + ig \sum_{n=1}^N \theta(\sqrt{s} - 2m - n\mu) \\ & (2\pi)^{-3n} \alpha_n^*(s) \left(\prod_{j=1}^n \int d^4 q_j \delta_+(q_j^2 - \mu^2) \right) \mathcal{L}(P - \sum_{\ell=1}^n q_{\ell}) \bar{M}^{(n)}(q_1, \dots, q_n; P), \end{aligned} \quad (24)$$

where $\delta_+(q^2 - \mu^2) = \delta(q^2 - \mu^2) \theta(q_0)$ and the invariant function $\mathcal{L}(P)$ is given by

$$\mathcal{L}(P) = \frac{2}{(2\pi)^2} \int d^4 p_1 \int d^4 p_2 \delta_+(p_1^2 - m^2) \delta_+(p_2^2 - m^2) \delta(P - p_1 - p_2). \quad (25)$$

The function $\mathcal{L}(P)$ can be evaluated by standard methods ⁷⁾ and one gets that

$$\mathcal{L}(P) = \Gamma(s) \theta(P_0) \theta(\sqrt{s} - 2m)$$

where the function $\Gamma(s)$ is given by

$$\Gamma(s) = (4\pi)^{-1} \left(1 - \frac{4m^2}{s}\right)^{\frac{1}{2}}. \quad (26)$$

For later purposes we note that $\Gamma(s)$ is regular, apart from two branch points at $s=0$ and at $s=4m^2$.

In a similar fashion, by taking for $|a\rangle$ a state with two heavy particles and for $|b\rangle$ a state with two heavy particles + n light particles, we obtain from Eq. (14) for the production amplitudes $\bar{M}^{(n)}$, $n > 0$, that

$$\begin{aligned} \bar{M}^{(n)}(k_1, \dots, k_n; P) = & i g n! \alpha_n(s) (2 + \mathcal{L}(P) \bar{M}^{(0)}(s)) \\ & + i \beta \mathcal{L}(P - \sum_{j=1}^n k_j) \bar{M}^{(n)}(k_1, \dots, k_n; P) \end{aligned} \quad (27)$$

for $\sqrt{s} \geq 2m + n\mu$. This can be solved for $\bar{M}^{(n)}$ in terms of the elastic amplitude $\bar{M}^{(0)}$

$$\bar{M}^{(n)}(k_1, \dots, k_n; P) = i g n! \alpha_n(s) (2 + \mathcal{L}(P) \bar{M}^{(0)}(s)) \left(1 - i \beta \mathcal{L}(P - \sum_{j=1}^n k_j)\right)^{-1}. \quad (28)$$

Substituting this into Eq. (24), we obtain that

$$\bar{M}^{(0)}(s) = i g f(s) (2 + \mathcal{L}(P) \bar{M}^{(0)}(s)) - g^2 (2 + \mathcal{L}(P) \bar{M}^{(0)}(s)) \bar{F}(s), \quad (29)$$

where

$$\bar{F}(s) = \sum_{n=1}^N \theta(\sqrt{s} - 2m - n\mu) (2\pi)^{-3n} n! |\alpha_n(s)|^2 F_n(s) \quad (30)$$

and

$$F_n(s) = \left(\prod_{j=1}^n \int d^4 q_j \delta_+(q_j^2 - \mu^2) \right) \frac{\mathcal{L}(P - \sum_{\ell=1}^n q_\ell)}{1 - i \beta \mathcal{L}(P - \sum_{\ell=1}^n q_\ell)}. \quad (31)$$

Equation (29) can now be solved, giving that

$$\bar{M}^{(0)}(s) = (2igf(s) - 2g^2\bar{F}(s))(1 - ig\mathcal{L}(P)f(s) + g^2\mathcal{L}(P)\bar{F}(s))^{-1} \quad (32)$$

which, together with Eq. (28), is an explicit expression for the scattering amplitude in terms of the definite integral $\bar{F}(s)$ as defined by Eqs. (30) and (31). The function $\bar{F}(s)$ is well defined because the integration in the right-hand side of Eq. (31) is, for finite s , over a compact domain of the momenta q_j due to the Θ functions appearing in the expression for $\mathcal{L}(P)$ [Eq. (25)]. The amplitude obtained is manifestly covariant and it also satisfies many-particle unitarity, for any real $f(s)$ and arbitrary $\alpha_n(s)$, since the input expression used for the K matrix was Hermitian. That unitarity is indeed satisfied by our solution is checked explicitly in Appendix A.

Let us now look at the analytic properties of the elastic amplitude $\bar{M}^{(0)}(s)$. For constant $f(s)$ and $\alpha_n(s)$ (this was the situation discussed in I) $\bar{M}^{(0)}(s)$ is continuous and piecewise regular, the analytic continuations from the different pieces on the real axis giving different analytic functions. However, by choosing a more complicated $f(s)$, it is possible to tie up the different pieces in such a way that $\bar{M}^{(0)}(s)$ does become the boundary value of a single analytic function which is regular for $\text{Im}s \neq 0$ and also for at least $\text{Re}s \geq (2m)^2$ [the properties for $\text{Re}s < (2m)^2$, $\text{Im}s = 0$ depend on the concrete form of the $\alpha_n(s)$] apart from branch points at the elastic and production thresholds. The remaining part of this section will be devoted to constructing such an $f(s)$. In the following discussion, we shall use two theorems about the functions $F_n(s)$ which are stated and proved in Appendix B.

Instead of investigating $\bar{M}^{(0)}(s)$, it is convenient to consider first the analytic properties of the related function $A(s)$ defined as

$$A(s) = 2(2 + \mathcal{L}(P)\bar{M}^{(0)}(s))^{-1} \quad (33)$$

which because of Eq. (32) is simply given by

$$A(s) = 1 - ig\mathcal{L}(P)f(s) + g^2\mathcal{L}(P)\bar{F}(s). \quad (34)$$

We denote the values of $f(s)$ in the different segments $\text{Im}s = 0$, $(2m+n\mu)^2 < \text{Re}s < (2m+(n+1)\mu)^2$ by $f_n(s)$ for $0 \leq n < N$ (we assume now that $N > 0$, the case $N=0$ being trivial) and the values of $f(s)$ for $\text{Im}s = 0$, $\text{Re}s > (2m+N\mu)^2$ we denote by $f_N(s)$. We put two conditions on $f_N(s)$, viz.,

a) $f_N(s)$ can be analytically continued into the upper complex half-plane. The analytic continuation, which we denote by the same symbol, is regular for $\text{Im } s > 0$ and its boundary value exists as $\text{Im } s \rightarrow 0_+$ for $\text{Re } s \geq 4m^2$;

b) $f_N(s)$ is real for $\text{Im } s = 0$, $\text{Re } s > (2m+N\mu)^2$. [This is already contained in condition (4) at the beginning of this section.]

We introduce the function $X(s)$ by

$$X(s) = A(s)$$

for $\text{Im } s = 0$, $\text{Re } s > (2m+N\mu)^2$.

Clearly, from Eq. (34), one has that

$$X(s) = 1 - i g \Gamma(s) f_N(s) + g^2 \Gamma(s) \sum_{n=1}^N (2\pi)^{-3n} n! |\alpha_n(s)|^2 F_n(s). \quad (35)$$

Because of Theorem 2 of Appendix B and condition (3) on the functions $\alpha_n(s)$ and of condition a) on $f_N(s)$, the function $X(s)$ can be analytically continued, giving a function which is regular for $\text{Im } s > 0$ and whose boundary value exists for $\text{Im } s \rightarrow 0_+$, $\text{Re } s \geq 4m^2$. We denote this function by the same symbol $X(s)$. From Eqs. (30) and (34) it then follows that $A(s)$ will be the boundary value of a single analytic function [namely of $X(s)$] if and only if $f_n(s)$ satisfy, for $n=0,1,\dots,N-1$, the equation

$$-i g \Gamma(s) f_n(s) = -i g \Gamma(s) f_N(s) + g^2 \Gamma(s) \sum_{\ell > n}^N (2\pi)^{-3\ell} \ell! |\alpha_\ell(s)|^2 L_\ell(s) \quad (36)$$

for $\text{Im } s = 0_+$, $(2m+n\mu)^2 < \text{Re } s < (2m+(n+1)\mu)^2$, where $L_\ell(s)$ is the analytic continuation of $F_\ell(s)$ discussed in Appendix B.

Consequently, we put

$$f_n(s) = f_N(s) + i g \sum_{\ell > n}^N (2\pi)^{-3\ell} \ell! |\alpha_\ell(s)|^2 L_\ell(s). \quad (37)$$

In addition, we must have $\text{Im } f_n(s) = 0$ (otherwise unitarity would no longer be satisfied). This gives the condition on $f_N(s)$ that

$$\text{Im } f_N(s) = -g \sum_{l>n}^N (2\pi)^{-3l} l! |\alpha_l(s)|^2 \text{Re } L_l(s) \quad (38)$$

for $\text{Im } s = 0_+$, $(2m+n\mu)^2 < \text{Re } s < (2m+(n+1)\mu)^2$. The above equation and also conditions a) and b) will be satisfied by any $f_N(s)$ of the form

$$f_N(s) = -g H(s) + g \lambda(s) \quad (39)$$

where $\lambda(s)$ is real analytic, regular for $\text{Im } s \neq 0$ and for $\text{Re } s \geq (2m)^2$, otherwise arbitrary and where $H(s)$ is given by the dispersion integral

$$H(s) = \frac{1}{\pi} \int_{4m^2}^{(2m+N\mu)^2} ds' \sum_{n=1}^N \theta((2m+n\mu)^2 - s') (2\pi)^{-3n} n! \frac{|\alpha_n(s')|^2 \text{Re } L_n(s')}{s' - s} \quad (40)$$

Having made this choice for $f_N(s)$, we obtain the expression for the function $A(s)$:

$$A(s) = 1 - g^2 \Gamma(s) (i \lambda(s) - i H(s) - F(s)) \quad (41)$$

where now $F(s)$ is simply given by

$$F(s) = \sum_{n=1}^N (2\pi)^{-3n} n! \alpha_n^*(s^*) \alpha_n(s) L_n(s) \quad (42)$$

the $L_n(s)$ being as before the analytic continuation of the right-hand side of Eq. (31). In view of Theorem 2 of Appendix B and because of the conditions on $\alpha_n(s)$ and $\lambda(s)$, the function $A(s)$ is now the boundary value of a single analytic function which is regular for $\text{Im } s \neq 0$ and also for $\text{Re } s \geq 4m^2$ apart from the physical thresholds at $s = (2m+n\mu)^2$, $n=0,1,\dots,N$, provided we restrict the mass of the light particle to satisfy $\mu \leq 2m/N$. From Eq. (33) follows the expression for the elastic amplitude

$$\bar{M}^{(0)}(s) = \frac{2g^2 \Gamma(s) (i \lambda(s) - i H(s) - F(s))}{1 - g^2 \Gamma(s) (i \lambda(s) - i H(s) - F(s))} \quad (43)$$

which, together with Eq. (28) for the production amplitude, is our final result. The amplitude $\bar{M}^{(0)}(s)$ is now the boundary value of a single analytic function and, of course, it still satisfies unitarity and Lorentz invariance. The branch points of $\bar{M}^{(0)}(s)$ will be the same as those of $A(s)$ which we have discussed already. In addition, $\bar{M}^{(0)}(s)$ will have poles at the zeros of $A(s)$. Naturally, one would like to avoid the occurrence of complex poles in $\bar{M}^{(0)}(s)$. It would be very difficult to give general conditions on $\alpha_n(s)$ and $\lambda(s)$ for this not to happen. However, in order to show that $\bar{M}^{(0)}(s)$ does not necessarily have poles (complex or real), we quote the following : if one puts $\alpha_n(s) \equiv 0$ for odd n then one can choose the real analytic functions $\alpha_n^*(s^*)\alpha_n(s)$ different from zero for even n and the real analytic function $\lambda(s)$ in such a way that the function $(i\lambda(s) - iH(s) - F(s))\Gamma(s)$ is bounded in the entire complex plane, including infinity. From Eq. (41), it follows then that $\bar{M}^{(0)}(s)$ will have no poles provided $|g| < C$, where C is some positive constant different from zero. One can show this by using lemmas 1) and 2) of Appendix B.

Finally we give the formula for the S wave elastic phase shift

$$e^{2i\delta(s)} = 1 + \Gamma(s)\bar{M}^{(0)}(s) = \frac{1 + g^2 \Gamma(s)(i\lambda(s) - iH(s) - F(s))}{1 - g^2 \Gamma(s)(i\lambda(s) - iH(s) - F(s))} \quad (44)$$

In the limit of switching off particle production [all $\alpha_n(s) \rightarrow 0$] this becomes simply

$$e^{2i\delta(s)} = \frac{1 + i g^2 \Gamma(s) \lambda(s)}{1 - i g^2 \Gamma(s) \lambda(s)} \quad (45)$$

Various generalizations of the model are possible. In fact the essential feature of K which permits one to obtain the scattering amplitude in closed form is that the operator K couples a many-particle state only to a many-particle state with the same number of particles or to a state containing two heavy particles only.

The amplitude for m light particles + two heavy ones going into n light particles + two heavy particles for $m, n > 0$ can also be given explicitly.

ACKNOWLEDGEMENTS

The author would like to thank Dr. M. Toller for useful criticism and valuable discussions. He would also like to thank the Theoretical Study Division of CERN for its kind hospitality.

A P P E N D I X A

Here we shall check by explicit calculation that the scattering amplitude given by Eqs. (28) and (32) satisfy unitarity. With the normalization used in this article, the unitarity relation reads for the elastic amplitude as

$$\begin{aligned} \text{Re } \bar{M}^{(0)}(\omega) = & -\frac{1}{4} (2\pi)^{-2} \int d^3 p_1'' \int d^3 p_2'' \frac{1}{E_2'' E_1''} \delta(P - p_1'' - p_2'') |\bar{M}^{(0)}(\omega)|^2 \\ & -\frac{1}{4} (2\pi)^{-2} \sum_{n=1}^{\infty} \theta(\sqrt{s} - 2m - n\mu) (2\pi)^{-3n} \frac{1}{n!} \int d^3 p_1'' \int d^3 p_2'' \frac{1}{E_2'' E_1''} \end{aligned} \quad (\text{A.1})$$

$$\left(\prod_{j=1}^n \int d^4 q_j \delta_+(q_j^2 - \mu^2) \right) \delta(P - p_1'' - p_2'' - \sum_{\ell=1}^n q_{\ell}) |\bar{M}^{(n)}(q_1, \dots, q_n; P)|^2.$$

Denoting the right-hand side of this equation by Z and integrating over p_1'' and p_2'' one obtains, after using Eqs. (25) and (28), that

$$\begin{aligned} Z = & -\frac{1}{2} \mathcal{L}(P) |\bar{M}^{(0)}(\omega)|^2 - \frac{1}{2} g^2 \sum_{n=1}^{\infty} \theta(\sqrt{s} - 2m - n\mu) (2\pi)^{-3n} n! |\alpha_n(\omega)|^2 \\ & |2 + \mathcal{L}(P) \bar{M}^{(0)}(\omega)|^2 \left(\prod_{j=1}^n \int d^4 q_j \delta_+(q_j^2 - \mu^2) \right) \frac{\mathcal{L}(P - \sum_{\ell=1}^n q_{\ell})}{|1 - i\beta \mathcal{L}(P - \sum_{\ell=1}^n q_{\ell})|^2} \end{aligned}$$

A comparison with Eqs. (30) and (31) gives that

$$Z = -\frac{1}{2} \mathcal{L}(P) |\bar{M}^{(0)}(\omega)|^2 - \frac{1}{2} g^2 |2 + \mathcal{L}(P) \bar{M}^{(0)}(\omega)|^2 \text{Re } \bar{F}(\omega)$$

which, because of Eq. (32), can be written as

$$Z = -2 \frac{g^2 \text{Re } \bar{F}(\omega) + \mathcal{L}(P) |ig f(\omega) - g^2 \bar{F}(\omega)|^2}{|1 - ig \mathcal{L}(P) f(\omega) + g^2 \mathcal{L}(P) \bar{F}(\omega)|^2}$$

Calculating $\text{Re } \bar{M}^{(0)}(s)$ directly from Eq. (32), we see that indeed

$$\text{Re } \bar{M}^{(0)}(s) = \mathbb{Z}$$

i.e., unitarity is satisfied.

A P P E N D I X B

In this Appendix we shall state and prove two theorems and state two lemmas concerning the functions $F_n(s)$ defined by Eqs. (25) and (31) for $n=1,2,\dots$ and $\text{Im } s = 0$.

Theorem 1

$$F_n(s) = 0 \quad \text{for } \text{Re } s \leq (2m+n\mu)^2.$$

Theorem 2

Putting

$$F_n(s) = \theta(\sqrt{s} - 2m - n\mu) L_n(s)$$

one has that the functions $L_n(s)$ can be analytically continued from the segment $\text{Im } s = 0$, $\text{Re } s > (2m+n\mu)^2$ into the entire complex s plane giving a function which is regular everywhere, except possibly for branch points at $s = (2m+j\mu)^2$ and $s = (j\mu)^2$ where j can be any integer such that $|j| \leq n$. We mention also that the $L_n(s)$ are continuous everywhere apart from the point $s=0$ (a pole type singularity). A simple consequence of this theorem is that $L_n(s)$ for $n=1,2,\dots,N$ is regular for $\text{Re } s \geq (2m)^2$ apart from branch points at $s = (2m+k\mu)^2$, $k=0,1,\dots,N$, provided we restrict μ to satisfy $\mu \leq 2m/N$.

Theorem 1 follows trivially from the definition of $F_n(s)$ and from the occurrence of the θ functions in the right-hand side of Eq. (25).

For the proof of Theorem 2 we use the method of induction. From Eq. (31) and Theorem 1 follows the relation

$$F_n(s) = \int d^4 q \delta_+(q^2 - \mu^2) \theta((P-q)^2 - (2m + (n-1)\mu)^2) L_{n-1}((P-q)^2) \quad (\text{B.1})$$

for $n=1,2,\dots$, provided we define $L_0(x)$ as

$$L_0(x) = \frac{\Gamma(x)}{1 - i\beta\Gamma(x)} \quad (\text{B.2})$$

where $\Gamma(x)$ is given by Eq. (26). We may put $\underline{P}=0$. The integration over the angles of $\underline{q}$ is then trivial. The integration over $|\underline{q}|$ can be carried out by means of the δ function. For the remaining integration, we choose the variable $x = (P-q)^2 = s - 2\sqrt{s} q_0 + \mu^2$. After some simple calculation we obtain that

$$F_n(s) = \frac{\pi}{2s} \int_{(2m+(n-1)\mu)^2}^{(\sqrt{s}-\mu)^2} dx ((\sqrt{s}-\mu)^2 - x)^{\frac{1}{2}} ((\sqrt{s}+\mu)^2 - x)^{\frac{1}{2}} \theta(s - (2m+n\mu)^2) L_{n-1}(x)$$

giving the recursion formula

$$L_n(s) = \frac{\pi}{2s} \int_{(2m+(n-1)\mu)^2}^{(\sqrt{s}-\mu)^2} dx ((\sqrt{s}-\mu)^2 - x)^{\frac{1}{2}} ((\sqrt{s}+\mu)^2 - x)^{\frac{1}{2}} L_{n-1}(x) \quad (B.3)$$

For $n=1$ this becomes

$$L_1^{(\epsilon)}(s) = \frac{\pi}{2s} \int_{4m^2}^{(\sqrt{s}-\mu)^2} dx ((\sqrt{s}-\mu)^2 - x - i\epsilon)^{\frac{1}{2}} ((\sqrt{s}+\mu)^2 - x)^{\frac{1}{2}} L_{n-1}(x), \quad (B.4)$$

where we have introduced a small positive parameter ϵ which we shall eventually let go to zero. This formula is suitable for the analytical continuation of $L_1^{(\epsilon)}(s)$. We simply let s become complex with the understanding that the path of integration is such that it does not encounter any of the singularities of the integrand (some of which are fixed and some of which depend on s), as we let $s \rightarrow 4m^2$ along the path. In this way we obtain a function which is regular for all s apart from the branch points at $s = (2m+\mu)^2$, coming from the singularity of $L_0(x)$ at $x = 4m^2$, at $s = (2m-\mu)^2$ from the singularity of the factor $((\sqrt{s}+\mu)^2 - x)^{\frac{1}{2}}$ and at $s = \mu^2$ from the singularity of $L_0(x)$ at $x = 0$. In addition, there is of course a pole at $s = 0$, on top of a branch point. Also

$$\lim_{\epsilon \rightarrow 0} L_1^{(\epsilon)}(s) = L_1(s)$$

uniformly in any compact domain, excluding the singular points of $L_1^{(\epsilon)}(s)$. Consequently $L_1(s)$ is also analytic and it satisfies Theorem 2. Applying now the same kind of argument to the general recursion formula given by Eq. (B.3), one sees that what happens is that $L_n(s)$ will have singularities at the points $s = (\sqrt{s_0} \pm \mu)^2$ where s_0 is any of the singular points of $L_{n-1}(s)$. Since Theorem 2 has already been proved for $n=1$, it follows from this that the theorem holds for any $n=1,2,\dots$.

From the recursion formula given by Eq. (B.3), follow two simple lemmas :

Lemma 1

$$\lim_{z \rightarrow \infty} \frac{1}{z^{n+1}} L_n(z) = 0.$$

Lemma 2

For even n one has that

$$\operatorname{Re} L_n(s) = 0$$

for $\operatorname{Im} s = 0$, $(2m+(n-2)\mu)^2 \leq \operatorname{Re} s \leq (2m+n\mu)^2$.

There is a complement to this Lemma which says that

$$\operatorname{Im} L_n(s) = 0$$

for $\operatorname{Im} s = 0$, $(2m+(n-2)\mu)^2 \leq \operatorname{Re} s \leq (2m+n\mu)^2$ if n is odd.

R E F E R E N C E S

- 1) L.B. Rédei, Umeå University preprint (1971) - unpublished.
- 2) See, N.N. Bogoliubov and D.V. Shirkov, "Introduction to the Theory of Quantized Fields", (Interscience Publishers, Inc., New York, 1959), and references therein.
- 3) These are the well-known Heitler equations. See, e.g., in S.S. Schweber, "An Introduction to Relativistic Quantum Field Theory", (Row, Peterson and Company, New York, 1961).
- 4) See, however, the recent preprint by R. Aviv, R.L. Sugar and R. Blankenbecler, Unitary models of multiparticle amplitudes, University of California, Santa Barbara, preprint (1972), where a model is presented which satisfies many-particle unitarity at infinite energy.
- 5) Another way to construct an interpolating field would be to use the logarithm of the scattering operator.
H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento 6, 319 (1957).
- 6) We use the metric $xy = x_0 y_0 - \underline{x} \cdot \underline{y}$.
- 7) See, e.g., G. Källén, "Elementary Particle Physics", (Addison-Wesley Publishing Company, Inc., New York, 1964).