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INFINITESIMAL INVARIANCE TRANSFORMATIONS

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A B S T R A C T

The condition for infinitesimal invariance of the action integral is formulated as a set of differential identities. Two examples demonstrate the type of calculations involved in applications of these conditions.

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1. INTRODUCTION

Noether's theorem ¹⁾ allows one to construct a first integral of the Euler-Lagrange equations of a variational problem if one knows an infinitesimal invariance transformation of that variational problem. Its importance in physics stems from the possibility of deriving the equations of motion of a physical system as the Euler-Lagrange equations corresponding to an action integral. The first integrals that Noether's theorem provides have then the physical significance of constants of the motion. Unfortunately, the application of the framework of a canonical Lagrangian field theory to particle physics is beset by many conceptual, and practical, difficulties. Therefore, there has been an increasing trend to use it only as a guide for guessing of properties of some alternative theoretical set-ups. Examples of this trend are the S matrix approach, axiomatic field theory, and the algebraic approach. We adopt a similar attitude here and consider Lagrangian field theory without any further comment.

If we require that the equations of motion for systems we wish to describe be Poincaré invariant and that, therefore, the action integral which gives rise to these equations be Poincaré invariant, then Noether's theorem supplies us with the corresponding 10 linearly independent conserved currents. On the other hand, if we observe certain degeneracies in our spectrum and want to attribute them to the conservation of certain currents then a very natural question to ask is whether the converse of Noether's theorem is true. In other words, does every conserved current (whose charge does not vanish identically) follow from an infinitesimal invariance property of the action integral? This question was answered affirmatively several years ago by Horn ²⁾, Dass ³⁾ and Boyer ⁴⁾. We thus have, through Noether's theorem and its inverse, a one to one correspondence between infinitesimal invariance transformations of the action integral and conserved currents.

This brings us now to a third question that we want to answer in the present work. Namely, given a particular action integral what are all possible infinitesimal transformations leaving invariant that particular action integral. Once these transformations are found we can apply Noether's theorem and construct the corresponding conserved currents.

Since the problem to which we address ourselves can be posed in a classical context we shall indeed treat it classically. In transcribing the results to the quantum domain we face immediately two problems: the ordering to non-commuting operators and the regularization of singular products of operators. We do not know a general procedure to overcome these difficulties. On the other hand, we expect it to be true that if the classical quantities form a Lie algebra under the Poisson bracket operation then their quantum analogues form an isomorphic Lie algebra under commutation. Therefore, as far as suggesting interesting Lie algebras, the classical theory should suffice.

The answer to the problem we raise is implicit in Boyer's paper ⁴⁾. But the first explicit answer that the present author knows appears in a work of Hermann ⁵⁾. In fact Hermann shows how to convert Boyer's remark into a calculational tool. The treatment in Herman's work is restricted to generalized momentum conservation. In other words the currents J_α he is able to calculate are linear in the canonical momentum four vector $\pi_{\alpha a} = \partial \mathcal{L} / (\partial (\partial_\alpha \varphi_a))$ and the canonical energy momentum tensor $\mathcal{G}_{\alpha\beta} = \pi_{\alpha a} \partial_\beta \varphi_a - g_{\alpha\beta} \mathcal{L}$. More explicitly his J_α are restricted to the form

$$J_\alpha = \pi_{\alpha a} A_a(\varphi, x) + \mathcal{G}_{\alpha\beta} A^\beta(x) + j_\alpha(\varphi, x) \quad (1)$$

(repeated indices are summed over).

As we shall see one may drop certain restrictions that appear in the work of Herman and get a generalized result. The fact that a generalization is needed becomes clear if one considers the Kepler problem for instance. The Runge-Lenz vector is not a generalized momentum and, therefore, the corresponding transformation ⁶⁾ violates Herman's restrictions.

We therefore discuss in Section 2 the concept of infinitesimal invariance of the action integral without imposing Herman's restrictions. This discussion leads to a formulation of the condition for infinitesimal invariance in Eq. (17).

In Section 3 we demonstrate the content of the infinitesimal invariance condition by considering the special case of Weinberg's Lagrangian.

In Section 4 we make the connection with Noether's theorem. As a by-product we get another formulation of the infinitesimal invariance conditions in Eq. (61). This formulation is slightly more general in that it can be set up directly in terms of the equations of motion without even mentioning a Lagrangian. In Section 5 we treat as an example a conformal invariant theory. In the concluding Section 6 we sum up the work.

2. INFINITESIMAL INVARIANCE

In this Section we derive the conditions for infinitesimal invariance of the action integral. Although we consider the case of a classical field theory in four-dimensional space-time it is easy to apply the results to other situations like classical mechanics. We write the action integral in the form

$$A[\varphi] = \int_R \mathcal{L}(x_\alpha; \varphi_a(x), \partial_\beta \varphi_a(x)) d^4x \quad (2)$$

Here $\mathcal{L}$ is a Lagrangian density, x_α $\alpha=0,1,2,3$ are the co-ordinates of a space time point x , φ_a $a=1,\dots,n$ are fields describing the system and $\partial_\mu \varphi_b$ $\mu=0,\dots,3$, $b=1,\dots,n$ are their first derivatives. We restrict $\mathcal{L}$ to be independent of higher derivatives of the field because the equations of motion one usually considers involve at most the second derivatives of the fields. (As far as mathematics is concerned Herman's method is generalizable to problems involving higher order derivatives.)

To make our notation manageable we use the following abbreviated symbols

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} \quad \partial_a = \frac{\partial}{\partial \varphi_a} \quad \partial_{\mu b} = \frac{\partial}{\partial (\partial^\mu \varphi_b)} \quad (3)$$

we also denote

$$\partial_\alpha \varphi_a = \varphi_{\alpha a} \quad \partial_\mu \partial_\nu \varphi_a = \varphi_{\mu\nu a} \quad \partial_\mu \partial_\nu \partial_\delta \varphi_a = \varphi_{\mu\nu\delta a} \quad \text{etc.} \quad (4)$$

We write the Euler-Lagrange equations of motion for the fields φ_a in the form

$$d^\alpha \partial_{\alpha a} \mathcal{L} - \partial_a \mathcal{L} = 0 \quad (5)$$

The derivative d^α is a total derivative which can be expressed in terms of partial derivatives as follows

$$d^\alpha = \partial^\alpha + \varphi^\alpha_a \partial_a + \varphi^\alpha_{\mu b} \partial_{\mu b} + \varphi^\alpha_{\mu\nu\delta c} \partial_{\mu\nu\delta c} + \dots \quad (6)$$

When performing the indicated differentiations the Lagrangian density $\mathcal{L}(x_\alpha; \varphi_a, \varphi_{\mu b})$ is considered as a function of $5n+4$ independent variables appearing as its arguments.

We now perform a general infinitesimal transformation on $A[\varphi]$ as follows

$$x_\alpha \rightarrow x'_\alpha = x_\alpha + \epsilon B_\alpha(x_\mu; \varphi_b, \varphi_{\mu c}) \quad (7a)$$

$$\varphi_a(x) \rightarrow \varphi'_a(x') = \varphi_a(x) + \epsilon C_a(x_\mu; \varphi_b, \varphi_{\mu c}) \quad (7b)$$

where ε is an infinitesimal parameter. These equations determine the transformation of $\partial_\alpha \varphi_a$ to be ⁷⁾

$$\partial_\alpha \varphi_a(x) \rightarrow \partial'_\alpha \varphi'_a(x') = \partial_\alpha \varphi_a(x) + \varepsilon [d_\alpha C_a - \varphi_{pa} d_\alpha B^p] \quad (8)$$

The transformed action is then

$$A[\varphi'] = \int_{R'} \mathcal{L}(x'; \varphi'_a(x'), \partial'_b \varphi'_b(x')) d^4 x' \quad (9)$$

Define a function $\mathcal{L}'$ of the untransformed arguments by

$$\mathcal{L}'(x; \varphi, \partial \varphi, \partial^2 \varphi) = \mathcal{L}(x'; \varphi', \partial' \varphi') \left| \frac{dx'}{dx} \right| \quad (10)$$

where $|dx'/dx|$ is the Jacobian determinant of the transformation, and define similarly an action A' of the untransformed fields by

$$A'[\varphi] = A[\varphi'] \quad (11)$$

The action $A'[\varphi]$ can now be written as

$$A'[\varphi] = \int_R \mathcal{L}'(x; \varphi, \partial \varphi, \partial^2 \varphi) d^4 x \quad (12)$$

To first order in ε we have

$$\left| \frac{dx'}{dx} \right| = 1 + \varepsilon d^\alpha B_\alpha \quad (13)$$

$$\begin{aligned} \mathcal{L}'(x; \varphi, \partial \varphi, \partial^2 \varphi) &= \mathcal{L}(x; \varphi, \partial \varphi) (1 + \varepsilon d^\alpha B_\alpha) + \\ &+ \varepsilon \{ B^\alpha \partial_\alpha \mathcal{L} + C_a \partial_a \mathcal{L} + [d^\alpha C_a - \varphi_{pb} d^\alpha B^p] \partial_{pb} \mathcal{L} \} \end{aligned} \quad (14)$$

And, therefore

$$A'[\varphi] = \int_R \mathcal{L} d^4x + \varepsilon \left\{ B^\alpha \partial_\alpha \mathcal{L} + C_a \partial_a \mathcal{L} + \right. \\ \left. + [d^\beta C_\beta - \varphi_{\gamma\beta} d^\beta B^\gamma] \partial_{\beta b} \mathcal{L} + \mathcal{L} d^\alpha B_\alpha \right\} d^4x \quad (15)$$

The action $A'[\varphi]$ is defined on the same fields φ as the action $A[\varphi]$. In general it describes a system with different dynamics corresponding to a Lagrangian density given by $\mathcal{L}'(x; \varphi, \partial\varphi, \partial^2\varphi)$. Note in particular that this expression may depend on second derivatives of the fields as indicated.

The Lagrangian density $\mathcal{L}'$ gives rise to equations of motion for the fields which are in general different from those corresponding to the Lagrangian density $\mathcal{L}$. We can now pose the question: what infinitesimal transformations leave the equations of motion invariant? To answer this question we note that the integrand of the action $A'[\varphi]$ has a term which is of zero order in ε . This term being $\mathcal{L}$ will, of course, reproduce the original equations of motion. In order that this is not altered by the term which is linear in ε it is clearly sufficient that the coefficient of ε in the integrand of $A'[\varphi]$ (the expression in curly brackets) fulfill the Euler-Lagrange equations as identities. That this condition is also necessary is a theorem in the calculus of variation⁸⁾. We denote the expression in curly brackets by $\mathcal{M}$

$$\mathcal{M} = \mathcal{L} d^\alpha B_\alpha + B^\alpha \partial_\alpha \mathcal{L} + C_a \partial_a \mathcal{L} + [d^\beta C_\beta - \varphi_{\gamma\beta} d^\beta B^\gamma] \partial_{\beta b} \mathcal{L} \quad (16)$$

and the condition for infinitesimal invariance can be written as

$$d^\alpha d^\beta \partial_{\alpha\beta a} \mathcal{M} - d^\alpha \partial_{\alpha a} \mathcal{M} + \partial_a \mathcal{M} \equiv 0 \quad (17)$$

These identities are not equations of motion for the fields φ_a but partial differential equations for the quantities B_α, C_a to determine their dependence on $x, \varphi, \partial\varphi$. The reason the first term appears is that, as we have already remarked, $\mathcal{M}$ may depend on $\varphi_{\alpha\beta a}$.

Since it is rather important to demonstrate the content of the infinitesimal invariance identities we interrupt here the mathematical development and take up a simplified example. The reader who wishes to skip the example can proceed directly to Section 4.

3. GENERALIZED POINT TRANSFORMATIONS IN WEINBERG'S CHIRAL LAGRANGIAN

We consider the following Lagrangian density (Weinberg's Lagrangian ⁹⁾,
 $n=3$),

$$\mathcal{L} = \frac{1}{2} \varphi_{\alpha a} \varphi^{\alpha}_a (1 + \lambda^2 \varphi_b \varphi_b)^{-2} \quad (18)$$

To simplify the example we restrict B_{α} to depend only on x and C_a to depend only on x and φ . This is the dependence allowed in Herman's work, Ref. 5), and corresponds to a generalized point transformation. We have for this Lagrangian

$$\partial_{\alpha} \mathcal{L} = 0 \quad (19a)$$

$$\partial_a \mathcal{L} = -2\lambda^2 \varphi_a \varphi_{\alpha c} \varphi^{\alpha}_c (1 + \lambda^2 \varphi_b \varphi_b)^{-3} \quad (19b)$$

$$\partial_{\alpha a} \mathcal{L} = \varphi_{\alpha a} (1 + \lambda^2 \varphi_b \varphi_b)^{-2} \quad (19c)$$

and because of our simplifying restrictions on B_{α} and C_a

$$d_{\beta} B_{\gamma} = \partial_{\beta} B_{\gamma} \quad (20a)$$

$$d_{\beta} C_b = \partial_{\beta} C_b + \varphi_{\beta c} \partial C_b \quad (20b)$$

Therefore $\mathcal{M}$ can be written as

$$\begin{aligned} \mathcal{M} = & \frac{1}{2} \varphi_{\alpha a} \varphi^{\alpha}_a (1 + \lambda^2 \varphi_b \varphi_b)^{-2} \partial^{\beta} B_{\beta} - 2\lambda^2 C_a \varphi_a \varphi_{\alpha c} \varphi^{\alpha}_c (1 + \lambda^2 \varphi_b \varphi_b)^{-3} + \\ & + [\partial^{\beta} C_b + \varphi^{\beta}_c \partial C_b - \varphi_{\gamma b} \partial^{\beta} B^{\gamma}] \varphi_{\beta b} (1 + \lambda^2 \varphi_d \varphi_d)^{-2} \end{aligned} \quad (21)$$

This can be rearranged more conveniently to read

$$\begin{aligned} \mathcal{M} = & \varphi_{\alpha a} \varphi_{\beta b} \left\{ \delta_{ab} g^{\alpha\beta} \frac{1}{2} \partial^\gamma B_\gamma (1 + \lambda^2 \varphi_c \varphi_c)^{-2} - 2\lambda^2 \delta_{ab} g^{\alpha\beta} \varphi_d C_d (1 + \lambda^2 \varphi_c \varphi_c)^{-3} \right. \\ & + g^{\alpha\beta} \frac{1}{2} (\partial_a C_b + \partial_b C_a) (1 + \lambda^2 \varphi_c \varphi_c)^{-2} - \delta_{ab} \frac{1}{2} (\partial^\alpha B^\beta + \partial^\beta B^\alpha) (1 + \lambda^2 \varphi_c \varphi_c)^{-2} \left. \right\} + \\ & + \varphi_{\alpha a} \partial^\alpha C_a (1 + \lambda^2 \varphi_c \varphi_c)^{-2} \end{aligned} \quad (22)$$

To make the manipulations more tractable let us denote

$$\begin{aligned} U_{\alpha a \beta b} = & \delta_{ab} g_{\alpha\beta} \left[\frac{1}{2} \partial^\gamma B_\gamma (1 + \lambda^2 \varphi_c \varphi_c)^{-2} - 2\lambda^2 \varphi_d C_d (1 + \lambda^2 \varphi_c \varphi_c)^{-3} \right] + \\ & + g_{\alpha\beta} \frac{1}{2} (\partial_a C_b + \partial_b C_a) (1 + \lambda^2 \varphi_c \varphi_c)^{-2} - \delta_{ab} \frac{1}{2} (\partial_\alpha B_\beta + \partial_\beta B_\alpha) (1 + \lambda^2 \varphi_c \varphi_c)^{-2} \end{aligned} \quad (23a)$$

$$V_{\alpha a} = \partial_\alpha C_a (1 + \lambda^2 \varphi_c \varphi_c)^{-2} \quad (23b)$$

so that $\mathcal{M}$ can be written as

$$\mathcal{M} = U_{\alpha a \beta b} \varphi_a^\alpha \varphi_b^\beta + V_{\alpha a} \varphi_a^\alpha \quad (24)$$

and, because of the restrictions on B_α and C_a we have

$$\partial_{\gamma c} U_{\alpha a \beta b} = 0 \quad \partial_{\gamma c} V_{\alpha a} = 0 \quad (25)$$

also, by definition

$$U_{\alpha a \beta b} = U_{\beta b \alpha a} \quad (26)$$

The identities to be satisfied by $\mathcal{M}$ read

$$d^{\chi} \partial_{\chi c} \mathcal{M} - \partial_c \mathcal{M} \equiv 0 \quad (27)$$

or

$$\begin{aligned} d^{\chi} (U_{\alpha a \chi c} \varphi^{\alpha}_a + U_{\chi c \beta b} \varphi^{\beta}_b + V_{\chi c}) - \varphi^{\alpha}_a \varphi^{\beta}_b \partial_c U_{\alpha \beta b} - \\ - \varphi^{\alpha}_a \partial_c V_{\alpha a} \equiv 0 \end{aligned} \quad (28)$$

Carrying out the differentiation taking the symmetry of U into account gives

$$\begin{aligned} 2 U_{\chi c \beta b} \varphi^{\beta}_b + 2 \varphi^{\beta}_b d^{\chi} U_{\chi c \beta b} + \partial^{\chi} V_{\chi c} + \varphi^{\alpha}_a \partial_a V_{\chi c} - \\ - \varphi^{\alpha}_a \varphi^{\beta}_b \partial_c U_{\alpha \beta b} - \varphi^{\alpha}_a \partial_c V_{\alpha a} \equiv 0 \end{aligned} \quad (29)$$

Since $U_{\chi c \beta b}$ is independent of φ^{α}_a its total derivative $d^{\chi} U_{\chi c \beta b}$ is independent of φ^{α}_b and, therefore, the only dependence on φ^{α}_b is the one exhibited explicitly. Therefore, in order that the left-hand side vanish identically we must have

$$U_{\chi c \beta b} = 0 \quad (30)$$

and

$$\partial^{\chi} V_{\chi c} + \varphi^{\alpha}_a (\partial_a V_{\alpha c} - \partial_c V_{\alpha a}) \equiv 0 \quad (31)$$

Again, the only dependence on φ^{β}_b is the one exhibited explicitly and we must have

$$\partial^{\chi} V_{\chi c} = 0 \quad (32)$$

$$\partial_a V_{\alpha c} - \partial_c V_{\alpha a} = 0 \quad (33)$$

We now have the following sets of partial differential equation for $B_\alpha(x)$ and $C_a(x, \varphi)$

$$\begin{aligned} \delta_{ab} [g_{\alpha\beta} \frac{1}{2} \partial^\gamma B_\gamma - \frac{1}{2} (\partial_\alpha B_\beta + \partial_\beta B_\alpha)] = \\ = g_{\alpha\beta} [\delta_{ab} 2\lambda^2 \varphi_d C_d (1 + \lambda^2 \varphi_c \varphi_c)^{-1} - \frac{1}{2} (\partial_\alpha C_\beta + \partial_\beta C_\alpha)] \end{aligned} \quad (34a)$$

$$\partial^\alpha \partial_\alpha C_a = 0 \quad (34b)$$

$$\partial_a [\partial_\alpha C_\beta (1 + \lambda^2 \varphi_c \varphi_c)^{-2}] = \partial_\beta [\partial_\alpha C_a (1 + \lambda^2 \varphi_c \varphi_c)^{-2}] \quad (34c)$$

Contracting a and b in equations (34a) we get

$$\begin{aligned} \frac{3}{2} [g_{\alpha\beta} \partial^\gamma B_\gamma - (\partial_\alpha B_\beta + \partial_\beta B_\alpha)] = \\ = g_{\alpha\beta} [6\lambda^2 \varphi_d C_d (1 + \lambda^2 \varphi_c \varphi_c)^{-1} - \partial_\alpha C_\beta] \end{aligned} \quad (35)$$

Contracting α and β we get

$$3\partial^\gamma B_\gamma = 4 [6\lambda^2 \varphi_d C_d (1 + \lambda^2 \varphi_c \varphi_c)^{-1} - \partial_\alpha C_\alpha] \quad (36)$$

We therefore get the following equations for B_α

$$\partial_\alpha B_\beta + \partial_\beta B_\alpha = g_{\alpha\beta} \frac{1}{2} \partial^\gamma B_\gamma \quad (37)$$

The most general solution of these equations is ⁵⁾

$$B_\alpha(x) = a_\alpha (x_\delta x^\delta) - 2x_\alpha (a_\delta x^\delta) + b_{\alpha\delta} x^\delta + c x_\alpha + f_\alpha \quad (38)$$

where a_α , $b_{\alpha\delta}$, c , f_α are constants and

$$b_{\alpha\delta} = -b_{\delta\alpha} \quad (39)$$

Contracting α and β in Equations (34a), we have

$$\partial^\gamma B_\gamma \delta_{ab} = \delta_{ab} 8\lambda^2 \varphi_d C_d (1 + \lambda^2 \varphi_c \varphi_c)^{-1} - 2(\partial_a C_b + \partial_b C_a) \quad (40)$$

But, from Eq. (36), we have

$$8\lambda^2 \varphi_d C_d (1 + \lambda^2 \varphi_c \varphi_c)^{-1} = \partial^\gamma B_\gamma + \frac{4}{3} \partial_c C_c \quad (41)$$

and, therefore,

$$\partial_a C_b + \partial_b C_a = \delta_{ab} \frac{2}{3} \partial_c C_c \quad (42)$$

This set of equations determines the φ dependence of the functions C_a . Again the most general solution of these equations is ⁵⁾

$$C_a(x, \varphi) = k_a(x) (\varphi_c \varphi_c) - 2\varphi_a (k_c(x) \varphi_c) + l_{ac}(x) \varphi_c + n(x) \varphi_a + r_a(x) \quad (43)$$

As indicated $k_a(x)$, $l_{ac}(x)$, $n(x)$, $r_a(x)$ are independent of φ and are functions of x only, also

$$l_{ac}(x) = -l_{ca}(x) \quad (44)$$

From Eq. (36) we get the following condition on the coefficient functions in $C_a(x, \varphi)$

$$3(-2a_\delta x^\delta + c) = 6\lambda^2 (1 + \lambda^2 \varphi_c \varphi_c)^{-1} [-k_a(x) \varphi_a \varphi_b \varphi_b + n(x) \varphi_b \varphi_b + l_a(x) \varphi_a] + 6k_a(x) \varphi_a - 3n(x) \quad (45)$$

Therefore,

$$h_a(x) + \lambda^2 h_a(x) = 0 \quad (46a)$$

$$u(x) = 0 \quad (46b)$$

$$a = 0 \quad (46c)$$

$$c = 0 \quad (46d)$$

We now have

$$B_\alpha(x) = b_{\alpha\delta} x^\delta + f_\alpha \quad (47a)$$

$$C_a(x, \varphi) = h_a(x) \varphi_c \varphi_c - 2 \varphi_a h_c(x) \varphi_c + l_{ac}(x) \varphi_c - \frac{1}{\lambda^2} h_a(x) \quad (47b)$$

Let us rewrite the set of equations (34c) in the form

$$\partial_\alpha \left\{ \partial_a [C_b (1 + \lambda^2 \varphi_c \varphi_c)^{-2}] - \partial_b [C_a (1 + \lambda^2 \varphi_c \varphi_c)^{-2}] \right\} = 0 \quad (48)$$

Substituting the expression we have for $C_a(x, \varphi)$ we get the equations

$$\begin{aligned} 4 [\varphi_b \partial_\alpha h_a(x) - \varphi_a \partial_\alpha h_b(x)] + (1 + \lambda^2 \varphi_c \varphi_c) \partial_\alpha l_{ab}(x) - \\ - 2 \lambda^2 [\varphi_b \varphi_d \partial_\alpha l_{ad}(x) - \varphi_a \varphi_d \partial_\alpha l_{bd}(x)] = 0 \end{aligned} \quad (49)$$

And, therefore

$$\partial_\alpha l_{ab}(x) = 0 \quad (50a)$$

$$\partial_\alpha h_a(x) = 0 \quad (50b)$$

We therefore write

$$C_a(\varphi) = k_a \varphi_c \varphi_c - 2\varphi_a k_c \varphi_c - \frac{1}{\lambda^2} k_a + l_{ac} \varphi_c \quad (51)$$

where now k_a and l_{ac} are independent of x . We see that now C_a , since it is independent of x , fulfils the conditions (34b) automatically.

Incidentally, we note that actually

$$\partial_\alpha C_a = 0 \quad (52)$$

and, therefore, the quantity $\mathcal{M}$ from which we started vanishes for the infinitesimal invariance transformations we calculated.

We note that $B_\alpha(x)$ is the most general infinitesimal Poincaré transformation. As for $C_a(\varphi)$ the terms proportional to l_{ac} induce infinitesimal rotations in the space of the φ_a and are related to isospin current conservation. The terms proportional to k_a induce a specific linear combination of an infinitesimal translation and an infinitesimal conformal transformation in the space of the φ_a and are related to the conservation of the axial current. After this demonstration of the content of the infinitesimal invariance conditions we proceed with the mathematical development.

4. NOETHER'S THEOREM

In this Section we give a brief derivation of Noether's theorem. As a by-product we get a modified version of the infinitesimal invariance condition.

We start by observing that a necessary and sufficient condition for a function $\mathcal{M}(x_\alpha; \varphi_a, \varphi_{pb}, \varphi_{xc})$ to satisfy the Euler-Lagrange equations identically is to be a total divergence of some four vector $d^\gamma \mathcal{V}_\gamma(x_\alpha; \varphi_a, \varphi_{pb}, \varphi_{xc})$. Therefore, if $\mathcal{M}$ calculated from a given Lagrangian satisfies the infinitesimal invariance condition it must be of this form. Namely, if Eq. (17) is satisfied then

$$\begin{aligned} d d^\alpha B_\alpha + B_\alpha \partial^\alpha \mathcal{L} + C_a \partial_a \mathcal{L} + [d^\alpha C_b - \varphi_{xb} d^\alpha B^\alpha] \partial_b \mathcal{L} = \\ = d^\gamma \mathcal{V}_\gamma \end{aligned} \quad (53)$$

It should be noted that here there arises a difference between point mechanics and field theory. More technically there is a difference between the case in which x is one dimensional, namely, time, and the case in which x is in a space with more than one dimension, namely, time and space. In the case of point mechanics, equations of motion which are non-singular allow the expression of all accelerations (second time derivatives of the canonical variables) as functions of velocities co-ordinates and time. Therefore, constants of the motion will be functions of velocities, co-ordinates and time.

However, in the case of field theory non-singular equations of motion do not express all second derivatives of the fields in terms of the fields, their first derivatives and the space-time point x . Only certain linear combinations of second derivatives are expressible in this form. Therefore, conserved currents may involve second derivatives of the fields (and in principle even higher derivatives).

We now want to exhibit the left-hand side of Eq. (53) as a sum of two terms: a term which is a total divergence and a term which vanishes if the φ_a satisfy their equations of motion.

To do so we note that

$$d^\alpha(d B_\alpha) = d d^\alpha B_\alpha + B_\alpha \partial^\alpha d + B_\alpha \varphi_a^\alpha \partial_a d + B_\alpha \varphi_a^{\alpha\beta} \partial_\beta d \quad (54a)$$

$$d^\beta(C_b \partial_\beta d) = d^\beta C_b \partial_\beta d + C_b d^\beta \partial_\beta d \quad (54b)$$

$$d^\beta(B_\alpha \varphi_b^\alpha \partial_\beta d) = d^\beta B_\alpha \varphi_b^\alpha \partial_\beta d + B_\alpha \varphi_b^{\alpha\beta} \partial_\beta d + B_\alpha \varphi_b^\alpha d^\beta \partial_\beta d \quad (54c)$$

Therefore, $\mathcal{M}$ can be written as

$$\mathcal{M} = d^\alpha(d B_\alpha + C_a \partial_a d - B^\beta \varphi_{\beta a} \partial_a d) - (C_a - B^\beta \varphi_{\beta a})(d^\alpha \partial_a d - \partial_a d) \quad (55)$$

Indeed, the first term is a four-divergence. The second term is proportional to the expressions $d^\alpha \partial_a d - \partial_a d$. Since d involves the first derivative of the fields these expressions will generally depend on second derivatives of the fields. Therefore, $(d^\alpha \partial_a d - \partial_a d)^{-1}$ cannot be a factor of the corresponding $(C_a - B^\beta \varphi_{\beta a})$ which is independent of second derivatives and higher derivatives of the fields.

And so this second term will vanish when the equations of motion for the φ_a , Eqs. (5), are satisfied.

We now can exploit the fact that if $\mathcal{M}$ satisfies the infinitesimal invariance condition then it is a total four-divergence. We have then

$$\begin{aligned} d^\alpha [d B_\alpha + (C_a - B^\beta \varphi_{\beta a}) \partial_{\alpha a} d - \mathcal{V}_\alpha] &= \\ &= (C_a - B^\beta \varphi_{\beta a}) (d^\alpha \partial_{\alpha a} d - \partial_a d) \end{aligned} \quad (56)$$

We see that we shall have conserved currents whenever the right-hand side vanishes. We should now distinguish two ways in which the right-hand side can vanish. First, it can vanish by virtue of the fact that the φ_a satisfy their equations of motion, namely Eq. (5). We therefore get a conservation equation

$$d^\alpha [d B_\alpha + (C_a - B^\beta \varphi_{\beta a}) \partial_{\alpha a} d - \mathcal{V}_\alpha] = 0 \quad (57)$$

which holds only for those φ_a which satisfy their equations of motion.

The other way the right-hand side of the divergence equation can vanish is possible if the equations of motion are not linearly independent. In this case there exists a non-zero vector $C_a - B^\beta \varphi_{\beta a}$ such that the scalar product $(C_a - B^\beta \varphi_{\beta a}) (d^\alpha \partial_{\alpha a} d - \partial_a d)$ vanishes. In this case we get a conservation equation for arbitrary φ_a which may have nothing to do with the dynamics of the problem at hand. Rather it is related to the mathematical nature of the fields we chose to describe the system with.

Let us now return to the divergence equation which we rewrite as

$$d^\alpha U_\alpha = E_a (d^\alpha \partial_{\alpha a} d - \partial_a d) \quad (58)$$

where

$$U_\alpha = d B_\alpha + (C_a - B^\beta \varphi_{\beta a}) \partial_{\alpha a} d - \mathcal{V}_\alpha \quad (59a)$$

$$E_a = C_a - B^\beta \varphi_{\beta a} \quad (59b)$$

Since C_a and B^β were allowed to depend on $x, \varphi, \partial\varphi$ we see that E_a is also a function of $x, \varphi, \partial\varphi$. Also, since the left-hand side is a total divergence the right-hand side must satisfy the Euler-Lagrange equation as identities. This will be

the form of the infinitesimal invariance criterion we shall use. Stated more explicitly given a definite Lagrangian density $\mathcal{L}$ as an explicit function of $x, \varphi, \partial\varphi$ we are first to form the expression $\mathcal{H}$

$$\mathcal{H} = E_a (d^\alpha \partial_{\alpha a} \mathcal{L} - \partial_a \mathcal{L}) \quad (60)$$

where E_a are unknown functions of $x, \varphi, \partial\varphi$. (Actually, one can write down the expression $\mathcal{H}$ once equations of motion are given without mentioning $\mathcal{L}$.) The functions E_a are to be determined by imposing on $\mathcal{H}$ the conditions

$$d^\alpha d^\beta \partial_{\alpha\beta a} \mathcal{H} - d^\alpha \partial_{\alpha a} \mathcal{H} + \partial_a \mathcal{H} \equiv 0 \quad (61)$$

as identities in $x, \varphi, \partial\varphi, \partial^2\varphi, \partial^3\varphi$. This will give us a set of conditions on the E_a . For any E_a that satisfy these conditions there corresponds a four-vector such that $\mathcal{H}$ is its total divergence. We look for the most general solution of the conditions on E_a . For this most general solution we also have a four-vector J_α such that

$$d^\alpha J_\alpha = E_a (d^\alpha \partial_{\alpha a} \mathcal{L} - \partial_a \mathcal{L}) \quad (62)$$

We now see that for fields φ_a satisfying the equations of motion J_α is a conserved current. We note that J_α need not coincide with U_α but only that

$$d^\alpha (U_\alpha - J_\alpha) = 0 \quad (63)$$

Therefore, U_α may differ from J_α by a four-vector which is conserved without assuming that the φ_a satisfy the equations of motion, namely

$$J_\alpha = U_\alpha + d^\beta F_{\beta\alpha} \quad (64)$$

where $F^{\alpha\beta}$ is an arbitrary function of $x, \varphi, \partial\varphi, \partial^2\varphi$ which is antisymmetric in α and β . The conditions in Eq. (61) which determine E_a do not determine C_a and B_β . This shows that we have some freedom in the definitions of C_a and B_β . Indeed, if we substitute

$$B_\beta \rightarrow B'_\beta = B_\beta - Y_\beta \quad (65a)$$

$$C_a \rightarrow C'_a = C_a - \varphi_{\beta a} Y_\beta \quad (65b)$$

E_a does not change. We are therefore free to choose $Y^A = B^A$ for instance and interpret E_a as transforming only the field variables φ_a while the co-ordinates x are not transformed.

5. CONFORMAL INVARIANCE IN A MASSLESS φ^4 THEORY

As a demonstration of the procedure outlined above we consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{\lambda}{4!} \varphi^4 \quad (66)$$

The expression $\mathcal{H}$ Eq. (60) is given by

$$\mathcal{H} = E(\varphi^\alpha{}_\alpha + \frac{\lambda}{3!} \varphi^3) \quad (67)$$

The unknown function E is allowed to depend on x , φ and $\partial_\mu \varphi$.

The infinitesimal invariance condition is

$$\begin{aligned} d^\alpha d_\alpha E - d^\alpha \left[\frac{\partial E}{\partial \varphi_\alpha} (\varphi^\alpha{}_\alpha + \frac{\lambda}{3!} \varphi^3) \right] + \frac{\partial E}{\partial \varphi} (\varphi^\alpha{}_\alpha + \frac{\lambda}{3!} \varphi^3) + \\ + E \frac{\lambda}{2!} \varphi^2 \equiv 0 \end{aligned} \quad (68)$$

When the differentiations are worked out one finds that the terms involving third derivatives of the field cancel. Terms which are quadratic in second derivatives of the field give the condition

$$\begin{aligned} g^{\mu\delta} \frac{\partial^2 E}{\partial \varphi_\mu \partial \varphi_\delta} + g^{\alpha\delta} \frac{\partial^2 E}{\partial \varphi_\alpha \partial \varphi_\delta} + g^{\mu\delta} \frac{\partial^2 E}{\partial \varphi_\mu \partial \varphi_\delta} + g^{\alpha\delta} \frac{\partial^2 E}{\partial \varphi_\alpha \partial \varphi_\delta} - \\ - 2g^{\alpha\mu} \frac{\partial^2 E}{\partial \varphi_\mu \partial \varphi_\alpha} - 2g^{\mu\delta} \frac{\partial^2 E}{\partial \varphi_\mu \partial \varphi_\delta} = 0 \end{aligned} \quad (69)$$

This can be reduced by a few contractions to the condition

$$\frac{\partial^2 E}{\partial \varphi_\delta \partial \varphi_\delta} = 0 \quad (70)$$

and, therefore, E must be linear in φ_α

$$E = S(x, \varphi) + V_\alpha(x, \varphi) \varphi^\alpha \quad (71)$$

The terms in Eq. (68) which are linear in second derivatives of the field give now the conditions

$$g_{\alpha\beta} \frac{\partial V_\gamma}{\partial \varphi} + g_{\alpha\gamma} \frac{\partial V_\beta}{\partial \varphi} + g_{\beta\gamma} \frac{\partial V_\alpha}{\partial \varphi} = 0 \quad (72a)$$

$$\partial_\beta V_\alpha + \partial_\alpha V_\beta + g_{\alpha\beta} \left(2 \frac{\partial S}{\partial \varphi} - \partial^\gamma V_\gamma \right) = 0 \quad (72b)$$

From (72a) it follows that

$$\frac{\partial V_\alpha}{\partial \varphi} = 0 \quad (73)$$

so that V_α is a function of x only and does not depend on φ . From (72b) it follows that

$$\frac{\partial S}{\partial \varphi} = \frac{1}{4} \partial^\gamma V_\gamma \quad (74a)$$

and

$$\partial_\beta V_\alpha + \partial_\alpha V_\beta - g_{\alpha\beta} \frac{1}{2} \partial^\gamma V_\gamma = 0 \quad (74b)$$

These equations then give

$$V_\alpha = a_\alpha X_\beta X^\beta - 2a_\beta X^\beta X_\alpha + b_{\alpha\beta} X^\beta + cX_\alpha + e_\alpha \quad (75a)$$

with an antisymmetric $b_{\alpha\beta}$ and

$$S = (c - 2a_\beta X^\beta) \varphi + h(x) \quad (75b)$$

where a_α , $b_{\alpha\beta}$, c , e_α are constants and $h(x)$ is a function of x to be determined. The remaining condition to be satisfied turns out to be

$$\partial^\alpha \partial_\alpha h + \frac{\eta}{2!} \varphi^2 h \equiv 0 \quad (76)$$

whose only solution is

$$h(x) = 0 \quad (77)$$

Thus E is determined to be of the following form

$$E = (c - 2a_\beta x^\beta) \varphi + a_\alpha x_\beta x^\beta \varphi^\alpha - 2a_\alpha x^\alpha x_\beta \varphi^\beta + b_{\alpha\beta} x^\beta \varphi^\alpha + c x_\alpha \varphi^\alpha + e_\alpha \varphi^\alpha \quad (78)$$

With this expression for E one can check that $\mathcal{H}$ in Eq. (67) has the form

$$\mathcal{H} = d^\alpha \left[-a^\beta (2x_\beta x^\gamma \vartheta_{\alpha\gamma} - x_\gamma x^\gamma \vartheta_{\alpha\beta} + 2x_\beta \varphi \varphi_\alpha - 2g_{\alpha\beta} \varphi^2) + \frac{1}{2} b^{\gamma\beta} (x_\beta \vartheta_{\alpha\gamma} - x_\gamma \vartheta_{\alpha\beta}) + c (x^\beta \vartheta_{\alpha\beta} + \varphi \varphi_\alpha) + e^\beta \vartheta_{\alpha\beta} \right] \quad (79)$$

where $\vartheta_{\alpha\beta}$ is the canonical energy momentum tensor

$$\vartheta_{\alpha\beta} = \varphi_\alpha \varphi_\beta - g_{\alpha\beta} \left(\frac{1}{2} \varphi_\gamma \varphi^\gamma - \frac{\lambda}{4!} \varphi^4 \right) \quad (80)$$

We have in Eq. (79), 15 linearly independent conserved currents. As is well known the terms in (79) linear in a^α are the conformal currents, the terms linear in $b^{\alpha\beta}$ are the Lorentz currents, the term linear in c is the dilation current and the terms linear in e^β are the translation currents.

6. DISCUSSION

We have shown that there exists a procedure which allows the calculation of infinitesimal invariance transformations. The examples we treated were not very rich in invariance transformation and gave rise to a finite number of linearly independent conserved currents. As is well known the case of a free massive scalar field gives rise to an infinite number of conserved currents. They include the Poincaré currents similar to the ones appearing in Eq. (79) and the creation and annihilation currents A_α^f ,

$$A_\alpha^f = \varphi \vartheta_\alpha^f - \varphi_\alpha f \quad (81)$$

where $f(x)$ is any function of x which solves the field equations

$$\partial^2_2 f + m^2 f = 0 \quad (82)$$

These annihilation and creation currents generate an infinite dimensional Heisenberg algebra. It would be interesting to find other theories possessing an infinite number of conserved currents which generate more complicated Lie algebras.

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