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DUAL MODELS WITH $SL(2,C)$ SYMMETRY

L. Brink and A. Kihlberg ^{*)}
CERN - Geneva

A B S T R A C T

Making use of homogeneous space techniques, we construct a class of dual models, which is a generalization of the Virasoro-Shapiro type of model. The integrand in the integral representation for the N point function depends not only on the modulus of the distances between two-dimensional "Koba-Nielsen variables", but also on the corresponding phases. This is in fact the most general $SL(2,C)$ invariant amplitude that can be constructed using complex integration variables. The extra phase factors in the integrand provide a possible means of avoiding tachyons both as external particles and as intermediate states in the amplitude. When factorized in a simple-minded fashion the intercepts are fixed to be integers. Although the external particles can be chosen not to be tachyons, such states appear as intermediate states. Within this factorization one can show that there are gauge conditions for the amplitude that can provide a ghost-killing mechanism.

^{*)} On leave from Institute of Theoretical Physics,
Göteborg.

1. INTRODUCTION

Since the appearance of the first dual models ^{1),2)}, a whole set of different generalizations has appeared. The Veneziano type of amplitude was rapidly generalized to an arbitrary number of external (spin zero) particles ³⁾. Much later the Virasoro type was generalized by Shapiro ⁴⁾ to an arbitrary number of external (spin zero) particles.

The Veneziano type is characterized by the $SU(1,1)$ symmetry. As a result, the fundamental integration domain for the integral representation of the amplitude is the circle in the Koba-Nielsen formulation ³⁾. The circle, on the other hand, is a homogeneous space of $SU(1,1)$. Similarly, as shown by Shapiro, the fundamental integration domain of the Virasoro type is the sphere, which is a homogeneous space of $SL(2,C)$. This group is also the invariance group of the model.

The exact role of the group structure is, however, not known. One may, for instance, ask oneself, if there is anything sacred in choosing one group or the other. Brower and Goddard have shown that dual models can be constructed using the groups $SO(1,n)$ ⁵⁾. On the other hand, if one tries to interpret the integration variables as some sort of internal co-ordinates for particles, then it is natural to look for homogeneous spaces of the Lorentz group ⁶⁾. Among these there is one, which is isomorphic to the sphere. Therefore, it is interesting to examine the most general dual model that can be constructed using such a homogeneous space of the Lorentz group. In such a scheme the $SL(2,C)$ symmetry would be an expression for Lorentz invariance and the group structure would become less mysterious. Furthermore, the way might be opened to more fundamental derivations of dual models via say field theories on homogeneous spaces ⁷⁾. One notices in this connection the interesting results by Scherk ⁸⁾ relating the limit of a dual model with zero slope trajectories to a field theory. One would naïvely expect to get only generalizations of the Virasoro type. However, it turns out that one can also get amplitudes breaking the symmetry between the different channels, thus making them more attractive from the phenomenological point of view. In addition, there is a natural way of including spinning particles in the models.

Attempts to connect the integration variables of the dual models to some internal variables have also been done by Domokos et al. ⁹⁾, and by Bars and Gürsey ¹⁰⁾. They arrive, however, only at the Veneziano or Virasoro-Shapiro type of model.

2. SL(2,C) AND ITS HOMOGENEOUS SPACES

A homogeneous space H of a topological group G has the following properties:

- i) it is a topological space on which the group G acts continuously, i.e., let y be a point in H then $y' = gy$ ($g \in G$) is defined and belongs to H and $y \rightarrow y'$ is a continuous transformation;
- ii) the action is transitive, i.e., given any two points y_1 and y_2 in H , then it is always possible to find a group element $g \in G$ such that

$$y_2 = g y_1$$

Now denote by S_0 the maximal subgroup of G , which leaves the point y_0 invariant. S_0 is called the stabilizer of y_0 . The stabilizer of another point y is conjugate to S_0 in G and one can establish a mapping between the points of H and the points in the coset space G/S_0 ¹¹⁾. Therefore, the enumeration of the different homogeneous spaces of G amounts to an enumeration of all closed subgroups of G up to conjugation.

In the case of $SL(2,C)$ the classification of the homogeneous spaces was done by Finkelstein ¹²⁾. One gets a parametrization of the space by writing the general group element as a product of two group elements of which the last one belongs to the stabilizer

$$g = g' g_0 \quad \begin{array}{l} g_0 \in S_0 \\ g' \in G \end{array}$$

If $x_1, \dots, x_n$ are the parameters of the (Lie-) group G and $x_{K+1}, \dots, x_n$ are the parameters of S_0 then g' can be chosen to depend only on $x_1, \dots, x_K$. $x_1, \dots, x_K$, therefore, parametrize H . The Table gives the different stabilizers S_0 (which can be generated from the Lie algebra) of $SL(2,C)$ together with the parametrizations of the corresponding spaces.

Since hadrons have a rich internal structure it is tempting to try to relate it to some internal space. The Table offers a set of such spaces. Since the homogeneous Lorentz group acts (in the natural way) on these spaces one can construct wave functions, which depend on the variables of the homogeneous space. This was done in Ref. 13) for the spaces [4] and [2] in the Table. Furthermore, one can construct $SL(2,C)$ invariant amplitudes using these wave functions ¹⁴⁾. We will be particularly interested in space [2], because it can be identified with the Shapiro sphere ⁴⁾.

It is also the smallest of the $SL(2,0)$ homogeneous spaces and, therefore, imposes more restrictions on the possible amplitudes.

We now proceed to a more detailed description of the space [2]. Topologically it is a sphere. Using polar co-ordinates θ, φ that part of the wave function that depends on the internal variables is

$$\Psi_{\vec{p}, \lambda}^S(\varphi, \theta) = \mathcal{D}_{\lambda n}^S(\varphi', \theta', 0) \quad (2.1)$$

where the primed variables are given by

$$\cos \frac{\theta'}{2} e^{-i\varphi'/2} = e^{-\varepsilon/2} \left[\cos \frac{\vartheta}{2} e^{i\phi/2} \cos \frac{\theta}{2} e^{-i\varphi/2} + \sin \frac{\vartheta}{2} e^{-i\phi/2} \sin \frac{\theta}{2} e^{i\varphi/2} \right] \quad (2.2)$$

$$\sin \frac{\theta'}{2} e^{i\varphi'/2} = e^{\varepsilon/2} \left[\cos \frac{\vartheta}{2} e^{i\phi/2} \sin \frac{\theta}{2} e^{-i\varphi/2} - \sin \frac{\vartheta}{2} e^{-i\phi/2} \cos \frac{\theta}{2} e^{i\varphi/2} \right] \quad (2.3)$$

Then momentum $p = (p_0, \vec{p})$ is given by

$$p^\mu = m [\cosh \varepsilon, \sinh \varepsilon \sin \vartheta \cos \phi, \sinh \varepsilon \sin \vartheta \sin \phi, \sinh \varepsilon \cos \vartheta] \quad (2.4)$$

$\mathcal{D}_{\lambda n}^S$ is the Wigner $\mathcal{D}$ function, where S is the spin, λ the helicity and n a quantum number, which ranges between $-S$ and S . [Notice that we have taken a specific value of the parameter $1 - i\nu$ in Ref. 13), namely $-S$. This is necessary in order to obtain the usual analyticity properties of the scattering amplitude.]

Following Ref. 13) we now construct a relativistically invariant scattering amplitude

$$A_n = \int \dots \int dQ_1 \dots dQ_n \Psi_{\vec{p}_1, \lambda_1}^{(1) S_1}(\varphi_1, \theta_1) \Psi_{\vec{p}_2, \lambda_2}^{(2) S_2}(\varphi_2, \theta_2) \dots \Psi_{\vec{p}_n, \lambda_n}^{(n) S_n}(\varphi_n, \theta_n) \\ K(\vec{p}_1, \theta_1, \varphi_1; \vec{p}_2, \theta_2, \varphi_2; \dots; \vec{p}_n, \theta_n, \varphi_n) \quad (2.5)$$

$\Psi_{\vec{p}_i, \lambda_i}^{(i)}$ is the wave function for particle (i) (considered as incoming) and K is a kernel function to be determined such that A_n is relativistically invariant. The most general form of K is then

$$K = M \prod_{i < j} (\kappa_i \cdot \kappa_j)^{a_{ij}} \pi(d_{ij})^{b_{ij}} \pi(p_i \cdot \kappa_j)^{c_{ij}} \quad (2.6)$$

where κ_i is the four-component object

$$\kappa_i = (1, \sin \theta_i \cos \varphi_i, \sin \theta_i \sin \varphi_i, \cos \theta_i) \quad (2.7)$$

$\kappa_i \cdot \kappa_j$ denotes the scalar product $g_{\mu\nu} \kappa_i^\mu \kappa_j^\nu$ and

$$d_{ij} = \frac{\cos \frac{\theta_i}{2} \sin \frac{\theta_j}{2} e^{i/2(\varphi_i - \varphi_j)} - \sin \frac{\theta_i}{2} \cos \frac{\theta_j}{2} e^{-i/2(\varphi_i - \varphi_j)}}{\cos \frac{\theta_i}{2} \sin \frac{\theta_j}{2} e^{-i/2(\varphi_i - \varphi_j)} - \sin \frac{\theta_i}{2} \cos \frac{\theta_j}{2} e^{i/2(\varphi_i - \varphi_j)}} \quad (2.8)$$

M is an arbitrary function of the invariant scalar products $(p_i \cdot p_j)$. The exponents a_{ij} , b_{ij} and c_{ij} satisfy

$$\sum_{l=1}^n c_{lj} + \sum_{l \neq j}^n (a_{lj} + a_{jl}) = -2S_j - 4 \quad j = 1, \dots, n \quad (2.9)$$

$$\sum_{l \neq j}^n (b_{lj} + b_{jl}) = -n_j \quad j = 1, \dots, n \quad (2.10)$$

Furthermore, the exponents b_{ij} have to be integral or half-integral in order to get a one-valued integrand.

3. THE n -POINT FUNCTION FOR SPINLESS PARTICLES; A COMPUTATION OF THE FOUR-POINT FUNCTION

In this paragraph we shall consider special cases of formula (2.5), namely, we choose

- i) $M = \text{constant}$;
- ii) spinless particles;
- iii) all $c_{ij} = 0$.

Since we are interested in seeing what structure of the amplitude is implied by the homogeneous space we make the simplest choice for M , namely, constant. Only non-negative integers c_{ij} are allowed, if one wants to maintain normal crossing properties of the amplitude ¹⁴⁾. The simplest choice is therefore $c_{ij} = 0$. This choice is supported by the fact

that the factor $(p_i \cdot u_j)^{c_{ij}}$ is the one which is coupling external degrees of freedom to internal ones. However, as in this paragraph we consider only spinless particles, we are not forced to use them and shall, therefore, omit them.

In order to make the connection to the work of Shapiro, we now express our equations using a complex variable z instead of the polar angles θ and φ . The connection is then

$$z_j = \cot \frac{\theta_j}{2} e^{i\varphi_j} \quad (3.1)$$

yielding the following expression up to a constant for A_n

$$A_n = \int \dots \int d^2 z_1 \dots d^2 z_n \prod_{i < j} |z_i - z_j|^{-\alpha_{ij}} \prod_{i < j} \left[\frac{z_i - z_j}{\bar{z}_i - \bar{z}_j} \right]^{\frac{\beta_{ij}}{2}} \quad (3.2)$$

where the exponents now have to satisfy

$$\sum_{i \neq j} \alpha_{ij} = 4 \quad (3.3)$$

$$\sum_{i \neq j} \beta_{ij} = 0, \quad \beta_{ij} \text{ integer} \quad (3.4)$$

Now our amplitude can depend on the kinematical variables only through the exponents. Lorentz invariance then implies that α_{ij} and β_{ij} only depend on invariant quantities such as $(p_i \cdot p_j)$. The natural choice is then to let α_{ij} depend only on $p_i \cdot p_j$ (β_{ij} being discrete cannot depend on $p_i \cdot p_j$). The linear condition (3.3) together with energy momentum conservation then restricts the dependence to a linear one i.e.,

$$\alpha_{ij} = \alpha_{ij}^0 + \alpha'_{ij} (p_i \cdot p_j) \quad (3.5)$$

The Shapiro model is now obtained apart from a scale factor by choosing $\alpha_{ij}^0 = 0$, $\alpha'_{ij} = 2$, $\beta_{ij} = 0$. We shall in this paragraph make the same choice for α'_{ij} but leave α_{ij}^0 arbitrary.

The choice for the α_{ij} implies that

$$4 = \sum_{j \neq i} \alpha_{ij} = \sum_{j \neq i} (2 p_i \cdot p_j + \alpha_{ij}^0) = -2 p_i^2 + \sum_{j \neq i} \alpha_{ij}^0$$

i.e., the α_{ij}^0 's determine the possible masses of the external particles.

Due to the $SL(2, \mathbb{C})$ invariance of the integrand one can always transform three of the z variables, say z_a , z_b and z_c , to fixed positions so that the integrand only depends on the remaining ones. This then implies that the integral over these three z 's yields an infinite constant. The well-known recipe to renormalize the integral is to omit the integrations on z_a , z_b and z_c and multiply by $|z_a - z_b|^2 |z_b - z_c|^2 |z_c - z_a|^2$, which conserves Lorentz invariance. Putting $z_a = z_1 = 0$, $z_b = z_3 = 1$, $z_c = z_4 = \infty$ and $z_2 = z$ we get, for example,

$$A_4 = \int d^2 z |z|^{-\alpha_{12}} |1-z|^{-\alpha_{23}} \left(\frac{z}{\bar{z}}\right)^{\frac{\beta_{12}}{2}} \left(\frac{1-z}{1-\bar{z}}\right)^{\frac{\beta_{23}}{2}} \quad (3.6)$$

Let us investigate the pole structure of this amplitude. The possible poles come from the integration around $z=0$, $z=1$ and $z=\infty$. In order to study the poles arising from the integration around $z=0$ we expand the factors $|1-z|^{-\alpha_{23}}$ and $((1-z)/(1-\bar{z}))^{\beta_{23}/2}$, around $z=0$. We then get by integrating up to $|z| = \epsilon \ll 1$

$$\begin{aligned} A_4^\epsilon &= \int_0^\epsilon r dr \int_0^{2\pi} d\theta \sum_k \sum_l \binom{-\frac{\alpha_{23}}{2} + \frac{\beta_{23}}{2}}{k} \binom{-\frac{\alpha_{23}}{2} - \frac{\beta_{23}}{2}}{l} (-1)^{k+l} \\ &\quad r^{-\alpha_{12}+k+l} e^{i\theta(\beta_{12}+k-l)} = \\ &= \sum_k \sum_l \binom{-\frac{\alpha_{23}}{2} + \frac{\beta_{23}}{2}}{k} \binom{-\frac{\alpha_{23}}{2} - \frac{\beta_{23}}{2}}{l} (-1)^{k+l} \\ &\quad \frac{e^{2i\pi(\beta_{12}+k-l)} - 1}{i(\beta_{12}+k-l)} \frac{\epsilon^{-\alpha_{12}+k+l+2}}{(-\alpha_{12}+k+l+2)} \end{aligned} \quad (3.7)$$

Continuing this to $\text{Re } \alpha_{12} > 2$, we see that we only get poles when $\beta_{12} + k = l$. This is the only case for which the angular integration is non-zero. Then we can conclude that we obtain poles in α_{12} for

$$\alpha_{12} = 2 + |\beta_{12}| + 2N \quad N = 0, 1, 2, \dots \quad (3.8)$$

Similar results can be obtained for the poles in α_{23} and α_{24} in an analogous fashion.

The integral (3.6) can be solved explicitly yielding

$$A_4 = \frac{\Gamma(\alpha_{12}/2 + \alpha_{23}/2 + \beta_{12}/2 + \beta_{23}/2 - 1) \Gamma(1 - \alpha_{12}/2 + \beta_{12}/2) \Gamma(1 - \alpha_{23}/2 + \beta_{23}/2)}{\Gamma(\alpha_{12}/2 + \beta_{12}/2) \Gamma(\alpha_{23}/2 + \beta_{23}/2) \Gamma(2 - \alpha_{12}/2 - \alpha_{23}/2 + \beta_{12}/2 + \beta_{23}/2)} \quad (3.9)$$

For a derivation see the Appendix. We can rewrite the amplitude by using

$$\alpha_{12}/2 = s/2 - m_1^2/2 - m_2^2/2 + \alpha_{12}^0/2 \quad (3.10)$$

$$\alpha_{23}/2 = t/2 - m_2^2/2 - m_3^2/2 + \alpha_{23}^0/2 \quad (3.11)$$

We then get

$$A_4 = \frac{\Gamma(-u/2 + m_1^2/2 + m_4^2/2 + 1 - \alpha_{24}^0/2 - \beta_{24}/2) \Gamma(-s/2 + m_1^2/2 + m_2^2/2 + 1 - \alpha_{12}^0/2 + \beta_{12}/2)}{\Gamma(u/2 - m_2^2/2 - m_4^2/2 + \alpha_{24}^0/2 - \beta_{24}/2) \Gamma(s/2 - m_1^2/2 - m_2^2/2 + \alpha_{12}^0/2 + \beta_{12}/2)} \\ \times \frac{\Gamma(-t/2 + m_2^2/2 + m_3^2/2 + 1 - \alpha_{23}^0/2 + \beta_{23}/2)}{\Gamma(t/2 - m_2^2/2 - m_3^2/2 + \alpha_{23}^0/2 + \beta_{23}/2)} \quad (3.12)$$

Due to the β_{ij} 's in (3.12), which are just constrained by (3.4), we can have an amplitude which is asymmetric in the different channels even if the α_{ij}^0 's are the same as well as the masses.

We have six parameters that determine where the poles in the s, t and u channels start. If we want to fix the values where the poles start we get three conditions besides the two conditions among the α 's and β 's (3.3) and (3.4). This implies that we can let the poles start anywhere we like. We can for example choose one channel to be exotic by letting the poles in this channel start arbitrarily high up. We can in this way recover a beta function for the four-point function. If we do this for the amplitude (3.12) we get poles with a spacing of two units in the squared mass. We can, however, by choosing α ' in (3.5) to be four, get poles with a spacing of one unit in the squared mass and thus recover the Veneziano four-point function. By using the solution (3.9) to the integral (3.6) we obtain that

$$\lim_{\delta_{24} \rightarrow \infty} \gamma_{24}^{1/2} 2^{-\delta_{24}} \cdot 2^{-s-t-1+\sum m_i^2} \frac{\sqrt{2\pi}}{e} \cdot$$

$$\int d^2z |z|^{-2s-2+2\delta_{12}-\delta_{24}/2} |1-z|^{-2t+\sum m_i^2-1-\delta_{12}+\delta_{23}-\delta_{24}/2}$$

$$\left(\frac{z}{2}\right)^{\delta_{24}/4} \left(\frac{1-z}{1-\bar{z}}\right)^{\frac{1}{2}(-\sum m_i^2-1+\delta_{12}+\delta_{23}+\delta_{24}/2)} = \frac{\Gamma(-s+\delta_{12}) \Gamma(-t+\delta_{23})}{\Gamma(-s-t+\delta_{12}+\delta_{23})}$$

(3.13)

We have had to multiply the integral with a factor 2^{-s-t} . This is possible, as we saw in Eq. (2.6).

This result for the four-point function is not generalizable directly to an amplitude with an arbitrary number of external particles. In general one has $2^{n-1}-n-1$ channels with poles in an n point function, while the corresponding number is $\frac{1}{2}n(n-3)$ for an amplitude with planar duality. We have $n(n-1)$ parameters α and β with $2n$ conditions (3.3) and (3.4). In order to fix the values, where the poles start in every channel we must have

$$n(n-1) - 2n \geq 2^{n-1} - n - 1$$

$$n \leq 5$$

(3.14)

If we just want to fix these values in the exotic channels the corresponding condition is $n \leq 6$. These conditions just tell us that for these n values it is always possible to get planar duality. We have, however, not found any way of generalizing this procedure to any n point function.

We also remark that it is possible to write a four-point function for external masses zero, where the poles in the s and t channels start at $m^2 = \frac{1}{2}$ with spin 1 (a $\pi\pi$ amplitude), namely

$$A_4 = \frac{\Gamma(-4/2+5/2) \Gamma(-s/2+1/4) \Gamma(-t/2+1/4)}{\Gamma(s/2+7/4) \Gamma(t/2+7/4) \Gamma(4/2+1/2)}$$

(3.15)

If we think of this as an amplitude for $\pi^+ \pi^- \rightarrow \pi^+ \pi^-$ we see that we have poles in the exotic channel, but the lowest resonance there has $m^2 = 5$ and spin 2 and might be very difficult to find experimentally.

4. A FACTORIZATION OF THE n PARTICLE AMPLITUDE FOR SPINLESS PARTICLES

In order to factorize the model we generalize to five dimensions the methods used by Del Giudice and Di Vecchia¹⁵⁾ and by Yoshimura¹⁶⁾ to factorize the Shapiro model¹⁷⁾. The extra dimension corresponds to some sort of conserved quantum numbers and this extra freedom provides a means of carrying the phase factors in the amplitude. The method used for factorization is perhaps the most simple-minded one. It is certainly not unique. It will be shown that it allows for asymmetry between different channels and external particles with real mass. We will, however, find tachyons as intermediate states. The factorization thus puts great restrictions on the possibilities and it would be very interesting to find a more general factorization scheme.

As the invariance group for the integrand is $SL(2, \mathbb{C})$, we introduce six generators to span the Lie algebra. With conventional notation we call them $L_0, L_+, L_-, \bar{L}_0, \bar{L}_+, \bar{L}_-$ with the commutation relations

$$[L_0, L_{\pm}] = L_{\pm} \quad [L_+, L_-] = -2L_0 \quad (4.1)$$

$$[\bar{L}_0, \bar{L}_{\pm}] = \bar{L}_{\pm} \quad [\bar{L}_+, \bar{L}_-] = -2\bar{L}_0 \quad (4.2)$$

$$[L, \bar{L}] = 0 \quad (4.3)$$

The set of operators L and $\bar{L}$ satisfies the algebra of $SU(1,1) \otimes SU(1,1)$, which is isomorphic to $SL(2, \mathbb{C})$.

We now introduce five-dimensional creation and annihilation operators $a_n^{\mu\dagger}, a_n^{\mu}, b_n^{\mu\dagger}$ and b_n^{μ} , where $\mu = 0, 1, 2, 3, 4$ and n ranges over all positive integers, together with the operators $q_0^{\mu}, \bar{q}_0^{\mu}, p_0^{\mu}$ and $\bar{p}_0^{\mu}$ with the commutation rules

$$[a_n^{\mu}, a_m^{\nu\dagger}] = -g^{\mu\nu} \delta_{nm} \quad [q_0^{\mu}, p_0^{\nu}] = -g^{\mu\nu} \cdot i \quad (4.4)$$

$$[b_n^\mu, b_m^{\nu\dagger}] = -g^{\mu\nu} \delta_{nm} \quad [\bar{q}_0^\mu, \bar{p}_0^\nu] = -g^{\mu\nu} \cdot i \quad (4.5)$$

where $g^{00} = -g^{11} = 1$ and $g^{\mu\nu} = 0$, when $\mu \neq \nu$. All other combinations commute. With these operators we can write

$$\begin{aligned} L_0 &= -\frac{p_0^2}{2} - \sum_{n=1}^{\infty} n a_n^\dagger a_n = -\frac{p_0^2}{2} + R \\ L_+ &= -p_0 \cdot a_1^\dagger - \sum_{n=1}^{\infty} \sqrt{n(n+1)} a_{n+1}^\dagger a_n \\ L_- &= -p_0 \cdot a_1 - \sum_{n=1}^{\infty} \sqrt{n(n+1)} a_n^\dagger a_{n+1} \end{aligned} \quad (4.6)$$

and similar expressions for $\bar{L}$ with $\bar{p}_0$ and b_n 's instead of p_0 and a_n 's. Furthermore, we introduce

$$\begin{aligned} Q_a^0(z) &= q_0 + i p_0 \ln z \\ Q_a^+(z) &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} a_n z^n \\ Q_a^-(z) &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} a_n^\dagger z^{-n} \end{aligned} \quad (4.7)$$

and similar operators Q_b with $\bar{q}_0$, p_0 and b_n 's instead of q_0 , p_0 and a_n 's. Finally we define five-dimensional momentum vectors $k_a = (k, c)$ and $k_b = (k, d)$, where k is the ordinary four-vector and c and d correspond to the fifth dimension. We so define a vertex operator

$$\begin{aligned} V(z, \bar{z}, k, c, d) &= e^{-i k_a \cdot Q_a^-(z)} e^{-i k_a \cdot Q_a^0(z)} e^{-i k_a \cdot Q_a^+(z)} \\ &\quad e^{-i k_b \cdot Q_b^-(\bar{z})} e^{-i k_b \cdot Q_b^0(\bar{z})} e^{-i k_b \cdot Q_b^+(\bar{z})} \end{aligned} \quad (4.8)$$

By using the above commutation rules we can now show that

$$\begin{aligned}
 \langle 0 | \prod_{i=1}^N V(z_i, \bar{z}_i, k_i, c_i, d_i) | 0 \rangle &= \\
 &= \prod_{i < j} |z_i - z_j|^{-2k_i \cdot k_j + c_i c_j + d_i d_j} \left(\frac{z_i - z_j}{\bar{z}_i - \bar{z}_j} \right)^{\frac{c_i c_j - d_i d_j}{2}} \\
 &\quad \prod_{i=1}^N |z_i|^{-k_i^2} z_i^{c_i/2} \bar{z}_i^{d_i/2}
 \end{aligned} \tag{4.9}$$

when $|z_k| \leq |z_\ell|$ where $k < \ell$. To get an expression valid for all z 's we follow Del Giudice and Di Vecchia and define

$$\begin{aligned}
 T(V_1 V_2 \dots V_n) &= V_1 V_2 \dots V_n \theta(|z_n| - |z_{n-1}|) \\
 &\quad \theta(|z_{n-1}| - |z_{n-2}|) \dots \theta(|z_2| - |z_1|) + \text{permutations}
 \end{aligned}$$

(4.10)

where

$$V_i \equiv V(z_i, \bar{z}_i, k_i, c_i, d_i)$$

With this operator on the left-hand side of Eq. (4.9) instead of the one used there we get the right-hand side for all z 's.

As we want to write our amplitude (3.2) with the help of (4.9) the exponents must fulfil the $SL(2, \mathbb{C})$ conditions (3.3) and (3.4)

$$\sum_{i \neq j} 2k_i \cdot k_j - c_i c_j - d_i d_j = 4 \tag{4.11}$$

$$\sum_{i \neq j} c_i c_j - d_i d_j = 0 \tag{4.12}$$

The c 's and d 's are defined so as to be conserved. So by using

$$\sum_{i=1}^N c_i = \sum_{i=1}^N d_i = \sum_{i=1}^N k_i = 0$$

one gets

$$c_i^2 = d_i^2 = 2 + m_i^2 \quad (4.13)$$

where m_i is the mass of the i^{th} external particle. We also notice that $c_i c_j - d_i d_j$ has to be integer in order to have a one-valued amplitude. This implies among other things that m_i^2 must be half-integer or integer in case of equal external masses. Equation (4.13) can be used to simplify Eq. (4.9) somewhat. Consider then the N point function

$$A_N = \lim_{\substack{z_i \rightarrow 0 \\ z_{N-1} \rightarrow 1 \\ z_N \rightarrow \infty}} \int \prod_{i=2}^{N-2} d^2 z_i |z_N - z_{N-1}|^2 |z_1 - z_N|^2 |z_{N-1} - z_1|^2 \prod_{i < j} \frac{1}{|z_i - z_j|^{2k_i \cdot k_j + c_i c_j + d_i d_j}} \left(\frac{z_i - z_j}{\bar{z}_i - \bar{z}_j} \right)^{\frac{c_i c_j - d_i d_j}{2}} \quad (4.14)$$

By using Eq. (4.9) one can show that

$$A_N = \int \prod_{i=2}^{N-2} \frac{d^2 z_i}{|z_i|^2} \langle 0 | e^{-i(k_{1a} \cdot q_0 + k_{1b} \cdot \bar{q}_0)} T(V_2 \dots V_{N-1}) e^{-i(k_{Na} \cdot q_0 + k_{Nb} \cdot \bar{q}_0)} | 0 \rangle \quad (4.15)$$

To solve the z integrations we use

$$V_i(z_i, \bar{z}_i) = z_i^{-L_0} \bar{z}_i^{-\bar{L}_0} V_i(1) z_i^{L_0} \bar{z}_i^{\bar{L}_0} \quad (4.16)$$

Inserting this result into (4.15) one can solve the integrations and get

$$A_N = \langle 0 | e^{-i(k_{1a} \cdot q_0 + k_{1b} \cdot \bar{q}_0)} P(V_2 D V_3 D \dots D V_{N-1}) e^{-i(k_{Na} \cdot q_0 + k_{Nb} \cdot \bar{q}_0)} | 0 \rangle \quad (4.17)$$

where

$$D = \frac{2}{L_0 + \bar{L}_0 - 2} \frac{\sin \pi (L_0 - \bar{L}_0)}{L_0 - \bar{L}_0} \quad (4.18)$$

and the P product is defined by

$$P(V_2 D V_3 \dots D V_{N-1}) = V_2 D V_3 \dots D V_{N-1} + \text{permutations} \quad (4.19)$$

We see thus that the amplitude A_N from (4.14) can be written in a factorized form (4.17). This expression is quite similar to the one in the Shapiro case considered by Del Giudice and Di Vecchia. The main difference is that we have had to use five-dimensional vectors instead of the ordinary four-dimensional ones. As in the Shapiro case we can look at the pole content in a certain channel with $s = (\sum_{i=1}^j k_i)^2$ by writing

$$A_N^s = \langle P | D | q \rangle \quad (4.20)$$

where

$$\langle P | = \langle 0 | e^{-i(k_{1a} \cdot q_0 + k_{1b} \cdot \bar{q}_0)} P(V_2 D \dots D V_j) \quad (4.21)$$

$$| q \rangle = P(V_{j+1} D \dots D V_{N-1}) e^{-i(k_{Na} \cdot q_0 + k_{Nb} \cdot \bar{q}_0)} | 0 \rangle \quad (4.22)$$

We insert a complete set of states

$$| \lambda \rangle = \prod_n \frac{(a_n^\dagger)^{\lambda_n}}{\sqrt{\lambda_n!}} \prod_l \frac{(b_l^\dagger)^{\mu_l}}{\sqrt{\mu_l!}} | 0 \rangle \quad (4.23)$$

We can then compute A_N^s and find

$$A_N^s = \sum_\lambda \langle P | \lambda \rangle \frac{2\pi}{-s - 2 + \frac{1}{2} \left(\sum_{i=1}^j \langle i \rangle^2 + \frac{1}{2} \left(\sum_{i=1}^j d_i \right)^2 + R + \bar{R} \right)} \langle \lambda | q \rangle \quad (4.24)$$

The sum is taken only over the states satisfying

$$L_0 |\lambda\rangle = \bar{L}_0 |\lambda\rangle$$

(4.25)

or in other terms

$$R |\lambda\rangle = (\bar{R} + \frac{1}{2} (\sum_{i=1}^j d_i)^2 - \frac{1}{2} (\sum_{i=1}^j c_i)^2) |\lambda\rangle$$

(4.26)

But

$$\frac{1}{2} \left[\left(\sum_{i=1}^j d_i \right)^2 - \left(\sum_{i=1}^j c_i \right)^2 \right] = \sum_{i < j} (d_i d_j - c_i c_j)$$

(4.27)

which is an integer according to the discussion following Eq. (4.13).

Using the form (4.24) and the condition (4.26) we can now discuss the resonance spectrum in any channel. Since $R + \bar{R}$ is non-negative and integer we see that the poles start at

$$S_{\min} = -2 + \frac{1}{2} \left(\sum_{i=1}^j c_i \right)^2 + \frac{1}{2} \left(\sum_{i=1}^j d_i \right)^2 + (R + \bar{R})_{\min}$$

(4.28)

In general $(R + \bar{R})_{\min}$ is larger than zero because of Eq. (4.26).

In fact

$$(R + \bar{R})_{\min} = \left| \frac{1}{2} \left(\sum_{i=1}^j c_i \right)^2 - \frac{1}{2} \left(\sum_{i=1}^j d_i \right)^2 \right|$$

(4.29)

The spin of the corresponding resonance is equal to $(R + \bar{R})_{\min}$. The parent trajectory is in fact

$$\alpha(s) = s + 2 - \frac{1}{2} \left(\sum_{i=1}^j c_i \right)^2 - \frac{1}{2} \left(\sum_{i=1}^j d_i \right)^2$$

(4.30)

We see thus that in a given channel one can change the intercept of the parent trajectory and at the same time kill the tachyon poles by choosing the c_i 's and d_i 's suitably. Of course, the choice of c_i 's and d_i 's affects all channels. Furthermore, we can see that the poles appear with spacing 2 because R and $\bar{R}$ both must increase by one step according to Eq. (4.26). Also, just as for the ordinary Shapiro case, the parent trajectory is non-degenerate and the degeneracy for $\alpha(s) = N$ grows exponentially with N .

We will consider the possible interpretations of the parameters c_i 's and d_i 's in the next paragraph when looking at the possible four-point functions. The factorization (4.17) gives in a straightforward way the spectrum of states, but is not suitable, when discussing physical states and possible gauge conditions. The difficulty is the propagators that behave in an unnatural way under a Möbius transformation. To find gauge operators we rewrite the amplitude (4.17) as

$$A_N = \langle 0 | e^{-i(k_{1a} \cdot q_0 + k_{1b} \cdot \bar{q}_0)} \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta(L_0 - \bar{L}_0)} d\theta$$

$$P(V_2 DV_3 \dots DV_{N-1}) e^{-i(k_{Na} \cdot q_0 + k_{Nb} \cdot \bar{q}_0)} |0\rangle \quad (4.31)$$

We easily see that the integral that we have put in is one, when the operators act to the left. We now write the propagators as

$$D = \frac{2}{L_0 + \bar{L}_0 - 2} \frac{\sin \pi(L_0 - \bar{L}_0)}{L_0 - \bar{L}_0} = \frac{1}{L_0 + \bar{L}_0 - 2} \int_0^{2\pi} e^{i\theta(L_0 - \bar{L}_0)} d\theta \quad (4.32)$$

We commute all integrands $e^{i\theta_i(L_0 - \bar{L}_0)}$ using (4.16) to the right. When finally acting on the physical ket vector

$$e^{-i(k_{Na} \cdot q_0 + k_{Nb} \cdot \bar{q}_0)} |0\rangle$$

they all just give one. By changing the integration variables properly we can rewrite the factorized amplitude as

$$A_N = \frac{1}{2\pi} \langle 0 | e^{-i(k_{1a} \cdot q_0 + k_{1b} \cdot \bar{q}_0)} P(\tilde{V}_2 \tilde{D} \tilde{V}_3 \dots \tilde{D} \tilde{V}_{N-1})$$

$$e^{-i(k_{Na} \cdot q_0 + k_{Nb} \cdot \bar{q}_0)} |0\rangle \quad (4.33)$$

where

$$\tilde{V}_i = \int_0^{2\pi} V_i(e^{i\theta}, e^{-i\theta}) d\theta \quad (4.34)$$

and

$$\tilde{D} = \frac{1}{L_0 + \bar{L}_0 - 2} \quad (4.35)$$

This factorization is a five-dimensional generalization of the factorization of the Shapiro amplitude by Yoshimura¹⁶⁾. We can now define the proper physical states as

$$\begin{aligned} |p\rangle &= P(\tilde{V}_{j+1}, \tilde{D} \dots \tilde{D} \tilde{V}_{N-1}) e^{-i(k_{Na} \cdot q_0 + k_{Nb} \cdot \bar{q}_0)} |0\rangle \\ &= \int \pi d^2 z_i \int_{N-J} (k_i, z_i) \exp \left[-i \sum_{n=1}^{\infty} \frac{1}{\sqrt{k}} (P_a^{*(n)} a_n^\dagger + \right. \\ &\quad \left. + P_b^{(n)} b_n^\dagger) e^{-i \left(\sum_{i=j+1}^N k_{ia} \cdot q_0 + k_{ib} \cdot \bar{q}_0 \right)} |0\rangle \right] \end{aligned} \quad (4.36)$$

where

$$\begin{aligned} P_a^{(n)} &= \sum_{i=j+1}^N k_{ia} z_i^n \\ P_b^{(n)} &= \sum_{i=j+1}^N k_{ib} z_i^n \end{aligned} \quad (4.37)$$

$f_{N-J}(k_i, z_i)$ is the integrand in the $(N-J)$ point amplitude. We can now follow Yoshimura and derive the gauge operators by making an infinitesimal change of the integration variables. We then get the following gauge operators

$$\begin{aligned} W_a^{(n)} &= L_0 - L_n + n - 1 \\ W_b^{(n)} &= \bar{L}_0 - \bar{L}_n + n - 1 \end{aligned} \quad (4.38)$$

where

$$\begin{aligned} L_n &= -i \sqrt{n} p_0 a_n - \frac{1}{2} \sum_{m=1}^{n-1} \sqrt{m(n-m)} a_m \cdot a_{n-m} - \\ &\quad - \sum_{m=1}^{\infty} \sqrt{m(n+m)} a_m^\dagger a_{n+m} \end{aligned} \quad (4.39)$$

We have thus the same ghost-killing gauge conditions as in the ordinary Shapiro case.

5. APPLICATION TO THE FOUR-POINT FUNCTION FOR SPINLESS PARTICLES

We can now use the restrictions (4.13) together with the conservation of the c_i 's and d_i 's to determine the possible four-point functions for spinless particles and to get an insight to possible interpretations of the c_i 's and d_i 's. We consider the case with the external particles being the same. There are just two different amplitudes one can get directly from (4.14).

where $c = \pm$ means $c = \pm \sqrt{2 + m^2}$ and similarly for d . We have, of course, also all permutations in s , t and u . We then get the following amplitudes [1) and (2) correspond to cases 1 and 2].

$$A_4^{(1)} = \frac{\Gamma(-u/2 - 1) \Gamma(-s/2 + 3 + 2m^2) \Gamma(-t/2 - 1)}{\Gamma(u/2 + 2) \Gamma(s/2 - 2 - 2m^2) \Gamma(t/2 + 2)} \quad (5.1)$$

$$A_4^{(2)} = \frac{\Gamma(-u/2 - 1) \Gamma(-s/2 + 3 + 2m^2) \Gamma(-t/2 + 3 + 2m^2)}{\Gamma(u/2 + 2) \Gamma(s/2 + 2) \Gamma(t/2 + 2)} \quad (5.2)$$

For $A_4^{(1)}$ the poles start in the u and t channels at -2 with a spin -0 resonance, while in the s channel they start at $6 + 4m^2$ with a resonance of spin 0 . For $A_4^{(2)}$ the poles start in the s and t channels at $6 + 4m^2$ with a resonance of spin $4 + 2m^2$ and in the u channel they start at -2 with a spin -0 resonance.

None of these amplitudes is, of course, realistic. Both have tachyon poles and also poles in all three channels. However, for $m^2 \geq 0$ the poles start at first at $s \geq 6$ in at least one channel. We would like to

have exotic channels in a realistic dual model. This is certainly in accord with present experimental data. A feature as above is, however, not excluded and is very difficult to test at today's energies.

By combining different possible amplitudes one can get a somewhat improved amplitude. Consider the following combination (we put $m^2 = 0$ for simplicity).

$$\begin{aligned}
 A_4 &= \frac{\Gamma(-u/2-1) \Gamma(-s/2+3) \Gamma(-t/2-1)}{\Gamma(u/2+2) \Gamma(s/2-2) \Gamma(t/2+2)} - \frac{\Gamma(-u/2-1) \Gamma(-s/2+3) \Gamma(-t/2+3)}{\Gamma(u/2+2) \Gamma(s/2+2) \Gamma(t/2+2)} \\
 &\quad - \frac{\Gamma(-u/2+3) \Gamma(-s/2+3) \Gamma(-t/2-1)}{\Gamma(u/2+2) \Gamma(s/2+2) \Gamma(t/2+2)} \\
 &= \frac{\Gamma(-u/2) \Gamma(-s/2+3) \Gamma(-t/2)}{\Gamma(u/2+2) \Gamma(s/2+2) \Gamma(t/2+2)} \left(u^2 - 2u + \frac{3}{2} ut - 2t + t^2 \right)
 \end{aligned}
 \tag{5.3}$$

The poles in the s channel for this amplitude start at $s=6$ and the first resonance has spin 4. In the u and the t channels the poles start at 0 and have spin 1.

We see that the tachyon poles are gone and that one channel is still "almost" exotic. However, we see that there is no possibility to get half-integer intercept, whatever way we combine possible amplitudes.

We can also see that all these amplitudes are Regge-behaved, which can be found by using the asymptotic properties of the Γ functions.

The most simple-minded attempt to interpret the c 's and d 's, would be to consider them as quantum numbers. (We remember that they are conserved.) Looking, however, at case 2 we see that this interpretation is not at all immediate. Another perhaps more interesting possibility is to associate the c_i 's and d_i 's with a sort of planar duality. The Shapiro model lacks planar duality. In our model we have found that some channels look "almost" exotic, since the poles start only at high energies and spins. This is intimately connected to the choice of the c_i 's and d_i 's. There is thus also a possibility to refer the c_i 's and d_i 's to the external legs of a specific multiperipheral configuration, the important thing being the position of the leg in the diagram and not what kind of particle it is.

6. GENERALIZATIONS AND CONCLUSIONS

The difficulty of getting realistic mass-spectra and Regge trajectories in dual models is a well-known problem. In fact, for positive intercept one usually encounters both tachyons and ghost states¹⁸⁾. The latter are proved to be removed by the gauge conditions in the ordinary dual resonance model with unit intercept and in the Neveu-Schwarz model¹⁹⁾. We have in this paper concentrated on the possibility of avoiding the tachyons. It is indeed possible to construct a dual Regge behaved amplitude, which satisfies the requirement of having no tachyon poles irrespective of the intercept. However, if one also insists on factorization of the amplitude and gauge conditions that can kill the ghost states, the model becomes strongly restricted (at least with the simple-minded factorization procedure we have used). One could speculate, if this amplitude is more appropriate for describing quark scattering. The only indication and admittedly a weak one is the fact that the factorization works easier for an even number of external particles. Now as indicated in Section 3, we have the possibility of constructing a much wider class of models containing for instance external particles with arbitrary spin

$$A_n = \int \dots \int d^2 z_1 \dots d^2 z_n \prod_{i,j} |z_i - z_j|^{-\alpha_{ij}} \left(\frac{z_i - z_j}{\bar{z}_i - \bar{z}_j} \right)^{\beta_{ij}} \prod_i (p_i \cdot \mathcal{U}(z_i))^{\gamma_{ij}} \prod_i \psi_{\vec{p}_i, \lambda_i}^{s_i}(z_i) \quad (6.1)$$

with the restrictions (2.9) and (2.10), where $\mathcal{U}$ is given by (2.7) and (3.1) and the wave functions by (2.1), (2.2) and (2.3). Of course, one has to consider anew the factorization of such amplitudes. This requires certainly a whole new scheme of factorization where the spins of the particles are essential ingredients.

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A P P E N D I X

We want to evaluate the integral

$$A_4 = \int d^2 z |z|^{-\alpha_{12}} |1-z|^{-\alpha_{23}} \left(\frac{z}{2}\right)^{\frac{\beta_{12}}{2}} \left(\frac{1-z}{1-\bar{z}}\right)^{\frac{\beta_{23}}{2}} \quad (\text{A.1})$$

By using $z = x + iy$ we write the amplitude

$$\begin{aligned} A_4 &= \int dx dy (x^2 + y^2)^{-\frac{\alpha_{12}}{2} - \frac{\beta_{12}}{2}} (x^2 - 2x + 1 + y^2)^{-\frac{\alpha_{23}}{2} - \frac{\beta_{23}}{2}} \\ &\quad (x + iy)^{\beta_{12}} (x - 1 + iy)^{\beta_{23}} \\ &= \sum_{k=0}^{\beta_{12}} \sum_{l=0}^{\beta_{23}} \sum_{m=0}^l \binom{\beta_{12}}{k} \binom{\beta_{23}}{l} \binom{l}{m} (-1)^{l-m} \int dx dy (x^2 + y^2)^{-\frac{\alpha_{12}}{2} - \frac{\beta_{12}}{2}} \\ &\quad (x^2 - 2x + 1 + y^2)^{-\frac{\alpha_{23}}{2} - \frac{\beta_{23}}{2}} x^{k+m} (iy)^{\beta_{12} + \beta_{23} - k - l} \end{aligned} \quad (\text{A.2})$$

where we have used the fact that β_{ij} is integer. We then use the formulae

$$(x^2 + y^2)^{-\gamma_1} = \frac{1}{\Gamma(\gamma_1)} \int_0^\infty t_1^{\gamma_1-1} e^{-(x^2 + y^2)t_1} dt_1 \quad (\text{A.3})$$

$$(x^2 - 2x + 1 + y^2)^{-\gamma_2} = \frac{1}{\Gamma(\gamma_2)} \int_0^\infty t_2^{\gamma_2-1} e^{-(x^2 - 2x + 1 + y^2)t_2} dt_2 \quad (\text{A.4})$$

Now the x and y integrations can be solved and finally the t_1 and t_2 integrations can be shown to decouple and lead to one Γ function and one $\mathcal{B}$ function. After a lengthy calculation one thus get

$$A_4 = \pi \sum_{k=0}^{\beta_{12}} \sum_{l=0}^{\beta_{23}} \sum_{m=0}^l \sum_{n=0}^{\{ \frac{k+m}{2} \}} (-1)^{l-m} i^{\beta_{12}+\beta_{23}-k-l} \frac{(2n-1)!! (\beta_{12}+\beta_{23}-k-l-1)!!}{2^{n+\beta_{12}/2+\beta_{23}/2-k/2-l/2}}$$

$$\left(\begin{matrix} \beta_{12} \\ k \end{matrix} \right) \left(\begin{matrix} \beta_{23} \\ l \end{matrix} \right) \left(\begin{matrix} l \\ m \end{matrix} \right) \left(\begin{matrix} k+m \\ 2n \end{matrix} \right) \frac{\Gamma(\alpha_{12}/2 + \alpha_{23}/2 - 1 - n + k/2 + l/2)}{\Gamma(\alpha_{12}/2 + \beta_{12}/2)}$$

$$\frac{\Gamma(-\alpha_{12}/2 + \beta_{23}/2 + 1 - n + m + k/2 - l/2) \Gamma(-\alpha_{23}/2 + \beta_{12}/2 + 1 + n - k/2 - l/2)}{\Gamma(\alpha_{23}/2 + \beta_{23}/2) \Gamma(2 - \alpha_{12}/2 - \alpha_{23}/2 + \beta_{12}/2 + \beta_{23}/2 + m - l)} \quad (A.5)$$

$\{ \}$ means the closest smaller or equal integer. $\beta_{12} + \beta_{23} - k - l$ has to be even. This is, of course, a too complicated expression to be useful directly. However, if we define

$$\mathcal{J}_{cd}^{ab}(\nu) = \int d^2 z \, z^a \bar{z}^b (z-\nu)^c (\bar{z}-\nu)^d \quad (A.6)$$

one easily shows that

$$\mathcal{J}_{cd}^{ab}(\nu) = \nu^{2+a+b+c+d} \mathcal{J}_{cd}^{ab}(1)$$

$$\frac{d}{d\nu} \mathcal{J}_{cd}^{ab}(\nu) = c \mathcal{J}_{c-1,d}^{ab}(\nu) + d \mathcal{J}_{c,d-1}^{ab}(\nu) =$$

$$= (2+a+b+c+d) \nu^{1+a+b+c+d} \mathcal{J}_{cd}^{ab}(1)$$

(A.7)

Put $\nu = 1$. Then

$$c \mathcal{J}_{c-1,d}^{ab}(1) + d \mathcal{J}_{c,d-1}^{ab}(1) = (2+a+b+c+d) \mathcal{J}_{cd}^{ab}(1)$$

Using this recursion formula and (A.5) one can show that

$$A_4 = \frac{\Gamma(\alpha_{12}/2 + \alpha_{23}/2 + \beta_{12}/2 + (\beta_{23}/2 - 1)) \Gamma(1 - \alpha_{12}/2 + \beta_{12}/2)}{\Gamma(\alpha_{12}/2 + \beta_{12}/2) \Gamma(\alpha_{23}/2 + \beta_{23}/2)} \quad (A.8)$$

$$\frac{\Gamma(1 - \alpha_{23}/2 + \beta_{23}/2)}{\Gamma(2 - \alpha_{12}/2 - \alpha_{23}/2 + \beta_{12}/2 + \beta_{23}/2)}$$

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TABLE

Homogeneous spaces of P and $\bar{P}$ belonging to continuous stabilizers of L and $\bar{L}$. The ranges of the parameters are $-\infty < x^\mu, s, t, \tilde{u}, \tilde{t}, u < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \varphi + \psi \leq 4\pi$, $-2\pi \leq \varphi - \psi \leq 2\pi$ for $\bar{P}$ and $0 \leq \varphi, \psi \leq 2\pi$ for P .

Stabilizer generated by	Dimension	Parameters	Invariant measure	Notation of Finkelstein
0	10	$x^\mu, \varphi, \theta, \psi, s, t, u$	$d^4x e^{2s} ds dt du d\varphi d\cos\theta d\psi$	[6]
$L_{02} - L_{23}$	9	$x^\mu, \varphi, \theta, \psi, s, t$	$d^4x e^{2s} ds dt d\varphi d\cos\theta d\psi$	[5]
L_{12}	9	$x^\mu, \varphi, \theta, s, \tilde{t}, \tilde{u}$	$d^4x ds d\tilde{t} d\tilde{u} d\varphi d\cos\theta$	[5 ₀]
$\cos f/2L_{12} + \sin f/2L_{03}$ $0 < f \leq \pi$	9	$x^\mu, \varphi, \theta, \tilde{\psi}, \tilde{t}, \tilde{u}$	$d^4x d\tilde{t} d\tilde{u} d\varphi d\cos\theta d\tilde{\psi}$	[5 _f]
$L_{02} - L_{23}$	8	$x^\mu, \varphi, \theta, \psi, s$	$d^4x e^{2s} ds d\varphi d\cos\theta d\psi$	[4]
$L_{01} + L_{31}$	8	$x^\mu, \varphi, \theta, \psi, \tilde{t}$	No	[4']
$L_{02} - L_{23}, L_{03}$	8	$x^\mu, \varphi, \theta, \tilde{t}, \tilde{u}$	$d^4x d\tilde{t} d\tilde{u} d\varphi d\cos\theta$	[4'']
L_{12}, L_{23}, L_{31}	7	$x^\mu, q^\mu, q^\mu q_\mu = 1$	$d^4x \frac{d^3q}{q_0}$	[3]
L_{12}, L_{01}, L_{02}	7	$x^\mu, q^\mu, q^\mu q_\mu = -1$	$d^4x \frac{dq^0 dq^1 dq^2}{q^3}$	[3]
$L_{12}, L_{02} - L_{23}$	7	$x^\mu, \varphi, \theta, s$	$d^4x e^{2s} ds d\varphi d\cos\theta$	[3 ₀]
$L_{01} + L_{31}$	7	$x^\mu, \varphi, \theta, \tilde{\psi}$	No	[3 _f]
$\cos f/2L_{12} + \sin f/2L_{03}$ $L_{02} - L_{23}$ $L_{01} + L_{31}$ $0 < f \leq \pi$	7	$x^\mu, \varphi, \theta, \tilde{\psi}$	No	[3 _f]
L_{12}, L_{03}	6	x^μ, φ, θ	No	[2]
$L_{02} - L_{23}$ $L_{01} + L_{31}$	6	x^μ, φ, θ	No	[2]
$L_{\mu\nu}$	4	x^μ	d^4x	[0]