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GLOBAL CHARGES AS INTEGRALS OVER DENSITIES

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ABSTRACT

For a conserved local current $\partial^\mu j_\mu(x) = 0$ it is shown that the matrix elements $(\Psi, j_0(f_r)\phi)$ between localized states Ψ and ϕ of the smoothed integral $\int j_0(x) f_r(x) d^4x$ of the "charge" density $j_0(x)$ converge to the corresponding ones $(\Psi, Q\phi)$ of the associated global "charge" Q whenever the latter exists, i.e., if the corresponding symmetry is not spontaneously broken. No assumptions about a gap in the energy momentum spectrum are made.

Local currents play an essential role in quantum field theory both for the formulation of dynamics and for the implementation of symmetry groups. The electrical current j_μ , for instance, is the source of electromagnetic forces. On the other hand, the charge operator, formally given by $\int j_0(x) d^3x$, is the infinitesimal generator of a unitary representation of the group of gauge transformations of the first kind. Both aspects of currents are exploited in the concept of current algebra. A number of investigations ¹⁾⁻⁵⁾ were devoted to clarify the connection between the local field $j_\mu(x)$ and the "charge" operator Q in the framework of axiomatic quantum field theory. Let us just mention one interesting result: despite the fact that locally the "charge" corresponding to some current $j_\mu(x)$ may be conserved - expressed by $\partial^\mu j_\mu(x) = 0$, which means that any change of the amount of "charge" in some finite volume has to be compensated by a flow through the surface - there is no global conservation law and therefore no global "charge" operator Q exists ³⁾⁻⁶⁾. This is due to some remaining flow through the surface even if one shifts it to infinity. In a relativistic theory this kind of flow can only be mediated by particles of mass zero which is the content of the well-known Goldstone theorem ⁶⁾.

Let us, however, concentrate on the case where this rather singular phenomenon does not occur. Thus we assume the existence of a self-adjoint global "charge" operator Q which generates a one-parameter unitary transformation group $V(\tau) = e^{i\tau Q}$ in the Hilbert space of states $\mathcal{H}$ commuting with the Hamiltonian. The dynamics may be described by a family of local fields $\varphi_i(x)$ which are transformed according to $\varphi_i(f) \rightarrow \varphi_{i(\tau)}(f) = V(\tau) \varphi_i(f) V(\tau)^{-1}$ where f denotes a suitable test function. This transformation clearly can be extended to an automorphism of the algebra $\mathcal{P}$ of polynomials in the smeared fields $\varphi_i(f)$. One of those local fields we assume to be a conserved vector current $j_\mu(x)$ corresponding to the symmetry transformation implemented by $V(\tau)$. What does this "corresponding" mean? Since the canonical equation $Q = \int j_0(x) d^3x$ has only a formal meaning in a field theory where fields are represented by operator valued distributions, we have to look for a substitute giving some well-defined meaning to it. One possibility is to require that Q and $\int j_0(x) d^3x$ have the same commutation relations with the fields $\varphi_i(f)$ for test functions with compact support, and therefore both generate the same transformation group $A \rightarrow A_\tau$ on local polynomials of φ_i , i.e., for $A \in \mathcal{P}_0$ we have ^{*)}

*) We denote by $\mathcal{P}_0$ the algebra of polynomials in $\varphi_i(f)$ with $f \in \mathcal{D}(\mathbb{R}^4)$.

$$-i \frac{dA_\tau}{d\tau} \Big|_{\tau=0} = [Q, A] = \lim_{r \rightarrow \infty} [j_0(f_r), A] \quad (1)$$

where f_r is a function from $\mathcal{D}(\mathbb{R}^4)$ of the form

$$f_r(x) = \alpha(x_0) g(|x|/r)$$

with $\int \alpha(x_0) dx_0 = 1$ and

$$g(\rho) = \begin{cases} 1 & \text{for } |\rho| \leq 1 \\ 0 & \text{for } |\rho| \geq 2 \end{cases} \quad (2)$$

Equation (1) is meaningful because the right-hand side becomes independent of r for large r due to spacelike commutativity of $j_\mu(x)$ with $\varphi_\nu(x)$. However, Eq. (1) is unsatisfactory from physical reasons. In most examples, the current j_μ as well as the charge Q are observables and $j_0(f_r)$ should measure the amount of "charge" in the volume described by f_r . Therefore for any reasonable measurement, we expect that the value of $j_0(f_r)$ approaches that of Q if we make r sufficiently large. But, as is already known from investigations in the early days of quantum field theory⁷⁾, there are some problems: if one measures the "charge" in some finite volume, there is always some unavoidable error from vacuum fluctuations in the surface of the volume. This is expressed mathematically by the fact that $\langle j_0(f_r)^2 \rangle_{\text{vac}} \sim r$ for large r ^{*)},⁴⁾. That we get r and not r^2 depends on the smoothing of the surface region by our choice of f_r . Had we chosen some f_r which goes to zero between r and $r + \delta$ (δ fixed), we would have got r^2 ²⁾,^{**)} . One can, however, show that it is impossible to render $\langle j_0(f_r)^2 \rangle_{\text{vac}}$ finite for $r \rightarrow \infty$ by a suitable choice of f_r ⁴⁾. Since $\langle j_0(f_r)^2 \rangle_{\text{vac}} = \|j_0(f_r) - \Omega\|^2$ (Ω denotes the vacuum state), we cannot hope that $j_0(f_r)$ converges to Q in the strong or weak sense, only certain matrix elements may converge. Is this enough for the physical interpretation? If we deal, for example, with

*) This is already so for free fields. There are, however, exceptions for higher rank tensor currents⁸⁾. Incidentally, we always assume that $j_0(x)$ is normalized to $\langle j_0(x) \rangle_{\text{vac}} = 0$.

**) The same consideration shows that one really has to use smooth test functions without sharp edges.

a free field theory, we can prepare states describing a finite number of freely moving particles in a finite volume by applying some $A \in \mathcal{P}_0$ to Ω . If $j_0(f_r)$ is to be interpreted as the amount of charge in the volume described by f_r we have, at least, to require that for all $A \in \mathcal{P}_0$

$$(A\Omega, QA\Omega) = \lim_{r \rightarrow \infty} (A\Omega, j_0(f_r)A\Omega) \quad (3)$$

We do not expect equality for finite r even if A is strictly localized because of vacuum fluctuations which might not cancel in the expectation value (as they do in the vacuum) in the presence of particles. For interacting fields we could make the same consideration for the asymptotic free particles. However, the corresponding free fields are no more relatively local to the current j_μ . In fact, the asymptotic field has no simple physical interpretation in x space (i.e., we cannot prepare a state of strictly free electrons in some finite volume). Therefore we consider only states of the form $A\Omega$ where A is some local polynomial in the interacting fields $\varphi_i(x)$. Such states may have definite "charge" although they have no simply interpretation in terms of particles. They differ only locally from the vacuum and can perhaps be prepared by local measurements, although this point is not at all clarified⁹⁾.

Equation (3) has in fact been proved in general for theories which have a gap in the energy spectrum above the vacuum (which means that there is a particle with smallest mass different from zero). The point is that

$$\begin{aligned} \lim_{r \rightarrow \infty} (A\Omega, j_0(f_r)B\Omega) &= \lim_{r \rightarrow \infty} (\Omega, j_0(f_r)A^*B\Omega) - \\ &- \lim_{r \rightarrow \infty} (\Omega, [j_0(f_r), A^*]B\Omega) = \\ &= \lim_{r \rightarrow \infty} (\Omega, j_0(f_r)A^*B\Omega) - (\Omega, [Q, A^*]B\Omega) = \\ &= \lim_{r \rightarrow \infty} (\Omega, j_0(f_r)A^*B\Omega) + (A\Omega, QB\Omega) \end{aligned}$$

It is therefore sufficient to prove $\lim_{r \rightarrow \infty} (\Omega, j_0(f_r)C\Omega) = 0$ for all $C \in \mathcal{P}_0$ ⁶⁾ which has, in fact, been done for theories with gap in the energy spectrum. What remains to be done is to show the validity of Eq. (3) whenever the global "charge" operator Q exists, irrespective of any additional conditions on the energy spectrum. This will be done in the following. More precisely, we shall

prove that the only intermediary states that contribute in the matrix element $(\Omega, j_0(f_r) A \Omega)$ for $r \rightarrow \infty$ are discrete states with mass zero, no mass continuum whatsoever reaching zero.

Theorem : Let $\varphi_i(x)$ be a family of Wightman fields, $\mathcal{P}_0$ the algebra of $\varphi_i(f)$ smeared with test functions with compact support. Let one of these fields, $j_\mu(x)$, be a conserved current ^{*)}
 $\partial^\mu j_\mu(x) = 0$, normalized to $(\Omega, j_0(x) \Omega) = 0$ then for any $A \in \mathcal{P}_0$

$$\begin{aligned} \lim_{r \rightarrow \infty} (\Omega, j_0(f_r) A \Omega) &= \lim_{r \rightarrow \infty} (\Omega, j_0(f_r) E_0 A \Omega) = \\ &= \frac{1}{2} \lim_{r \rightarrow \infty} (\Omega, [j_0(f_r), A] \Omega) \end{aligned} \quad (4)$$

where E_0 is the projection on the states of mass zero and f_r as defined in Eq. (2).

Remark 1

The above theorem can be generalized to certain currents which are not translation covariant as, for instance, the dilatation current $x^\nu \theta_{\mu\nu}(x)$ ^{**)} .

Remark 2

It is essential to take a spherically symmetric function $g_r(\underline{x})$ ^{***)} as the following example shows : take $j_\mu(x) = \varepsilon_{\mu 12 \nu} \partial^\nu \varphi(x)$ where $\square \varphi(x) = 0$ and $A = \varphi(h)$, $h \in \mathcal{D}(\mathbb{R}^4)$, $\int h(x) d^3x \neq 0$. One finds $\partial^\mu j_\mu = 0$, $\lim_{r \rightarrow \infty} (\Omega, [j_0(f_r), A] \Omega) = 0$, however

$$\lim_{r \rightarrow \infty} (\Omega, j_0(f_r) A \Omega) = \int \frac{\tilde{g}(p) p_3}{2|p|} d^3p \cdot \tilde{h}(0) \neq 0$$

for suitably chosen $g(\underline{x})$.

^{*)} It is not necessary that j_μ be a Lorentz covariant vector field, all we need is translation covariance. Cf., however, Remark 1.

^{**)} H. Reeh, private communication.

^{***)} This was also recognized by R. Ferrari and L.E. Picasso (private communication).

Corollary : If $\lim_{r \rightarrow 0} (\Omega, [j_0(f_r), A] \Omega) = 0$ for all $A \in \mathcal{P}_0$, then

$$\lim_{r \rightarrow \infty} (\Psi, j_0(f_r) \Phi) = (\Psi, Q \Phi)$$

for all $\Psi, \Phi \in \mathcal{P}_0 \Omega$, where Q is the global "charge" operator corresponding to $j_\mu(x)$ which exists in this case ^{3),6)}.

Proof of the Theorem

Put $k_\mu^-(x) = (\Omega, j_\mu(x) A \Omega)$, respectively $k_\mu^+(x) = (\Omega, A j_\mu(x) \Omega)$. $k_\mu^\pm$ are the positive, respectively, negative frequency part of the commutator $k_\mu(x) = (\Omega, [j_\mu(x), A] \Omega) = k_\mu^-(x) - k_\mu^+(x)$. $k_\mu(x)$ possesses a JLD representation ¹⁰⁾ :

$$k_\mu(x) = \int_0^\infty ds \int_D d\underline{x} \left[\rho_{1,\mu}(\underline{x}, s) \Delta(x - \underline{x}; s) + \rho_{2,\mu}(\underline{x}, s) \dot{\Delta}(x - \underline{x}; s) \right]$$

where D is some compact region in $\mathbb{R}^3$ and $\rho_{i,\mu}(\underline{x}; s)$ are measures in s . From that we find

$$\int k_\mu^\pm(x) \omega(x_0) e^{i p \cdot x} d^4x = \int_0^\infty ds \frac{\tilde{k}_\mu^\pm(\underline{p}, \pm \sqrt{p^2 + s}) \tilde{\omega}(\pm \sqrt{p^2 + s})}{2 \sqrt{p^2 + s}}$$

with

$$\tilde{k}_\mu^\pm(\underline{p}, \pm \sqrt{p^2 + s}) = \tilde{\rho}_{1,\mu}(\underline{p}, s) \pm \sqrt{p^2 + s} \tilde{\rho}_{2,\mu}(\underline{p}, s)$$

$\tilde{k}_\mu^\pm(\underline{p}, \pm \sqrt{p^2 + s})$ are in fact the restrictions of the measures $\tilde{k}_\mu^\pm(p) = \int k_\mu^\pm(x) e^{i p x} d^4x$ to the mass shells $\{p^2 = s, p_0 \geq 0, \text{ respectively, } p_0 \leq 0\}$.
From the properties of $\rho_{i,\mu}(\underline{x}, s)$ we deduce that

$$K_\mu(\underline{p}, s) = \frac{\tilde{k}_\mu^-(\underline{p}, -\sqrt{p^2 + s}) \tilde{\omega}(-\sqrt{p^2 + s}) - \tilde{k}_\mu^+(\underline{p}, \sqrt{p^2 + s}) \tilde{\omega}(\sqrt{p^2 + s})}{2 \sqrt{p^2 + s}}$$

respectively

$$L_\mu(\underline{p}, s) = \tilde{k}_\mu^-(\underline{p}, -\sqrt{p^2 + s}) \tilde{\omega}(-\sqrt{p^2 + s}) + \tilde{k}_\mu^+(\underline{p}, \sqrt{p^2 + s}) \tilde{\omega}(\sqrt{p^2 + s}) \quad *)$$

*) L_μ corresponds to $(\Omega, [\frac{\partial}{\partial x_0} j_0(x), A] \Omega)$.

are measures in s and entire analytic functions of $\underline{p}$, i.e., $\int K_{\mu}(\underline{p}, s) g(s) ds$ is an entire analytic function of $\underline{p}$ for any bounded continuous function $g(s)$ and

$$|\int K_{\mu}(\underline{p}, s) g(s) ds| \leq c_{\mu}(\underline{p}) \sup |g(s)|$$

(analogously for L_{μ}).

From $\partial^j j_{\mu}(x) = 0$ it follows that

$$\sqrt{p^2+s} \left(K_0(\underline{p}, s) + \frac{L_0(\underline{p}, s)}{\sqrt{p^2+s}} \right) = p^i \left(K_i(\underline{p}, s) + \frac{L_i(\underline{p}, s)}{\sqrt{p^2+s}} \right) \quad (5)$$

Since K_{μ} and L_{μ} are analytic in $\underline{p}$ this is only possible if

$$L_0(\underline{p}, s) = p^i K_i(\underline{p}, s) \quad (6)$$

and

$$(p^2+s) K_0(\underline{p}, s) = p^i L_i(\underline{p}, s) \quad (7)$$

This can be seen most easily by running once around the branch point of $\sqrt{p^2+s}$ in Eq. (5): the square root $\sqrt{p^2+s}$ changes sign, whereas K_{μ} and L_{μ} remain fixed. Integrating Eq. (6) over the solid angle $d\Omega(\underline{p})$ we get

$$\int d\Omega(\underline{p}) L_0(\underline{p}, s) = p^2 M(p^2, s)$$

where $M(p^2, s)$ is entire analytic in p^2 and a finite measure in s . Next, we split K_0 into $K_0(\underline{p}, s) = K'_0(\underline{p}) \delta(s) + K''_0(\underline{p}, s)$ with

$$\int_{s=0} K''_0(\underline{p}, s) ds = 0, \text{ i.e., } K'_0(\underline{p}) = \int_{s=0} K_0(\underline{p}, s) ds$$

Integrating again over $d\Omega(\underline{p})$ gives

$$\int K'_0(\underline{p}) d\Omega(\underline{p}) = 4\pi K'_0(0) + p^2 N'(p^2)$$

with $N'(p^2)$ entire analytic and

$$\int K''_0(\underline{p}, s) d\Omega(\underline{p}) = p^2 N(p^2, s)$$

and again $N(p^2, s)$ entire analytic in p^2 and a finite measure in s with

$$\int_{s=0} N(p^2, s) ds = 0$$

Putting all this information together, we arrive at

$$\begin{aligned}
 \lim_{r \rightarrow \infty} (\Omega, j_0(f_r) A \Omega) &= \lim_{r \rightarrow \infty} \int_0^\infty ds \int d^3 p \frac{\tilde{b}_0(\underline{p}, -\sqrt{p^2+s}) \tilde{\alpha}(-\sqrt{p^2+s}) \tilde{g}_r(\underline{p})}{2\sqrt{p^2+s}} \\
 &= \lim_{r \rightarrow \infty} \int_0^\infty ds \int d^3 p \frac{\tilde{b}_0(\frac{p}{r}, -\sqrt{\frac{p^2}{r^2}+s}) \tilde{\alpha}(-\sqrt{\frac{p^2}{r^2}+s})}{2\sqrt{\frac{p^2}{r^2}+s}} \tilde{g}(|p|) = \quad *) \\
 &= \lim_{r \rightarrow \infty} \int_0^\infty ds \int_0^\infty d|p| \int d\Omega(p) \left(\frac{1}{2} K_0(\frac{p}{r}, s) + \frac{r L_0(\frac{p}{r}, s)}{2\sqrt{p^2+r^2 s}} \right) \tilde{g}(|p|) |p|^2 = \\
 &= \lim_{r \rightarrow \infty} \left\{ \int_0^\infty d|p| |p|^2 \tilde{g}(|p|) \left(2\pi K_0'(0) + \frac{|p|^2}{2r^2} N'(\frac{|p|^2}{r^2}) \right) + \right. \\
 &\quad \left. + \int_{s>0} ds \int_0^\infty d|p| |p|^2 \tilde{g}(|p|) \left(\frac{|p|^2}{2r^2} N(\frac{|p|^2}{r^2}, s) + \frac{|p|}{r} \frac{M(\frac{|p|^2}{r^2}, s)}{\sqrt{1+\frac{r^2 s}{|p|^2}}} \right) \right\} = \\
 &= \frac{1}{2} K_0'(0) = \lim_{r \rightarrow \infty} (\Omega, j_0(f_r) E_0 A \Omega)
 \end{aligned}$$

since

$$\left| \int_{s>0} \frac{M(\frac{|p|^2}{r^2}, s)}{\sqrt{1+\frac{r^2 s}{|p|^2}}} ds \right| \leq \int_{s>0} |M(\frac{|p|^2}{r^2}, s)| ds \xrightarrow{r \rightarrow \infty} \int_{s>0} |M(0, s)| ds < \infty$$

On the other hand

$$\begin{aligned}
 \lim_{r \rightarrow \infty} (\Omega, [j_0(f_r), A] \Omega) &= \\
 &= \lim_{r \rightarrow \infty} \int_0^\infty ds \int_0^\infty d|p| |p|^2 \tilde{g}(|p|) \int d\Omega(p) K_0(\frac{p}{r}, s) = \\
 &= K_0'(0)
 \end{aligned}$$

This proves all of Eq. (4).

*)

$$\tilde{g}(|p|) = \int g(|x|) e^{i \underline{p} \cdot \underline{x}} d^3 x !$$

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