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THE PADE APPROXIMATION AND ITS PHYSICAL APPLICATIONS *)

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A B S T R A C T

The Padé approximation, its mathematical features, and its practical and physical applications are reviewed. We describe the present status of the convergence problem. Among the physical applications, we give particular emphasis to elementary particle theory. We discuss various proofs of convergence, and show why the approximation is particularly well suited for studying strong interaction dynamics. We finally review the various achievements of the approximation in the determination of $\pi\pi$, πN and NN scattering amplitudes.

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I. - INTRODUCTION

The Padé approximation ¹⁾ is a very simple and powerful alternative to polynomial approximations for analytic functions. Consider a function $f(z)$ analytic around $z = 0$, we can expand it in a Taylor series

$$f(z) = f_0 + z f_1 + z^2 f_2 + \dots \quad (\text{I.1})$$

which converges in a circle $|z| \leq R$ going through the nearest singularity of $f(z)$. Suppose we want to know the value $f(z_1)$ at a point z_1 outside the convergence circle but where f is analytic, the Taylor series (I.1) is useless as such since it diverges, although we know from the theory of analytic functions that $f(z_1)$ is completely determined by the coefficients f_0, f_1 , etc. In other words, given a power series expansion, we want an approximation scheme which is capable of analytically continuing the series outside its circle of convergence, provided this can be done ²⁾. The Padé approximation provides such a method, we define the $[N, M]$ Padé approximant to $f(z)$ as the ratio of two polynomials in z , $P_N(z)$ and $Q_M(z)$ of degree N and M respectively, which has the same $N + M$ first derivatives as $f(z)$ at $z = 0$

$$f^{[N, M]}(z) \equiv \frac{P_N(z)}{Q_M(z)} = f(z) + O(z^{N+M+1}) \quad (\text{I.2})$$

from this point of view, the set of Padé approximants are a generalization of the Taylor series expansion - the $[N, 0]$ approximants being identical to the partial sums of (I.1).

The Padé approximants are constructed with the coefficients of the Taylor series. In being able to continue it outside its circle of convergence, they constitute a summing procedure for divergent series ³⁾. There are many methods available to define the sum of a divergent series ⁴⁾, for instance Borel's method which uses the formal identity

$$\sum_{m=0}^{\infty} a_m z^m = \int_0^{\infty} e^{-xz} \sum_{m=0}^{\infty} a_m \frac{(xz)^m}{m!} dx \quad (\text{I.3})$$

however, it is clear that although $\sum a_n y^n/n!$ may converge while the original series does not, one cannot truncate in a useful way (in n) the integrand on the right-hand side. The virtue of the Padé method is that it actually gives an approximation for the final result, it is not formal. One of the simplest summation procedures in Euler's method ⁴⁾. We explain it on a very simple example : consider a function $f(z)$ which has a cut on the real axis from $-\infty$ to -1 [e.g., $\log(1+z)$], its Taylor series converges for $|z| \leq 1$, and suppose we want to know $f(2)$. If we make the transformation $z = y/(1-y)$ we map the cut on $[1, \infty]$ and $z = 2$ corresponds to $y = 2/3$ which is inside the convergence circle of $\varphi(y) = f(y/(1-y))$. Therefore, we may use in a convergent way the Taylor series to evaluate $f(2)$ provided we use this homographic change of variable [for $\log(1+z)$ we would end up with $-\log(1-y)$ and the result is obvious, but this holds for any analytic function cut on $[-\infty, -1]$]. Using Euler transformations on the variable : $z = \lambda y/(1+\mu y)$ we can map a part of the holomorphy domain into the convergence circle. It is a great virtue that Padé approximants perform the Euler transformations systematically, and choose λ and μ in terms of the coefficients f_i (and not arbitrarily) in the most efficient way as we shall see in Section II.

Another, intuitive, way to understand the motivations for using Padé approximants ⁵⁾ is the Cauchy formula itself

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(z') dz'}{z' - z} \quad (I.4)$$

which expresses that an analytic function can be expressed as the limit of a sum of pole terms. In this respect, we remark that the poles of the Padé approximants represent the singularities of the function. It is interesting to examine some simple examples, from the theory of continued fractions ⁶⁾ which are particular cases of Padé approximants - i.e., $[N, N]$ and $[N \pm 1, N]$ approximants. Consider for instance the continued fraction expansion for $\operatorname{tg} z$

$$\operatorname{tg} z = \frac{z}{1 - \frac{z^2}{3 - \frac{z^2}{5 - \frac{z^2}{\ddots}}}} \quad (I.5)$$

$\operatorname{tg} z$ has poles at $z = (2n+1)\pi/2$ and zeros at $z = n\pi$. If we truncate Eq. (I.5), we see that the $[0,1]$ approximant has poles at $z = \pm\sqrt{3}$, the $[1,1]$ has poles at $z = \pm\sqrt{5/2} = \pm 1.581 \dots$ (within 1% of $\pi/2$) and zeros at $z = \pm\sqrt{15} = \pm 3.86$ which move to $z = \pm\sqrt{21/2} = 3.24$ in the $[1,2]$, etc. As the order increases, one obtains the lower poles and zeros more accurately, and higher poles and zeros appear. In the case of a cut, for instance $\log(1+z)/z$ we have

$$\frac{1}{z} \log(1+z) = \frac{1}{1 + \frac{z}{2 + \frac{z}{3 + \frac{4z}{4 + \frac{4z}{5 + \frac{9z}{\dots}}}}}} \quad (\text{I.6})$$

The $[0,1]$ has a pole at $z = -2$, the $[1,1]$ a pole at $z = -3/2$ and a zero at $z = -2$, etc. The poles and zeros alternate on $[-\infty, -1]$ in order to represent the cut. It is instructive, from these examples, that the position and strength of the singularities can actually be estimated from the knowledge of the coefficients of the Taylor series.

Inside the convergence circle of the Taylor series, Padé approximants usually provide a much better estimate for the function than the series itself. In that case, the Padé approximation acts as an accelerator of convergence. For example, if we want to calculate $\sqrt{2}$ from the continued fraction expansion of $\sqrt{1+z}$ around $z = 0$

$$\sqrt{1+z} = 1 + \frac{z}{2 + \frac{z}{2 + \frac{z}{2 + \frac{z}{\dots}}}} \quad (\text{I.7})$$

we obtain the sequence of fractions : $3/2, 5/7, 17/12, 41/29, 99/70$, etc., the tenth term $8119/5741 = 1.414213551\dots$ giving the exact value $\sqrt{2} = 1.414213562\dots$ up to nine significant figures !

Finally, the Padé approximants may, in certain cases, be used even if the radius of convergence of the series is strictly zero, and the series is purely formal.

All these interesting features, and the particular simplicity of the Padé approximation make it a very convenient tool for practical and physical applications. In theoretical physics, one frequently encounters power series expansions which do not converge or converge very slowly, and this is the natural framework to treat such series. The range of applications to physical problems is very broad, from turbulent diffusion to solid state physics and elementary particle physics ⁷⁾. Several reviews on the subject exist already ^{5),7)-12)}, and we have tried, here, to avoid too much overlap, concentrating mainly on strong interaction physics where great progress has been made in recent years, as we shall see, both in establishing the properties of the approximation and in computing strong interaction scattering amplitudes.

II. - MATHEMATICAL FEATURES

In this section, we want to give the leading ideas about the mathematical properties of Padé approximants, and in particular the convergence problem. We make no attempt at rigour or completeness and we omit most proofs if not all. The reason is that there exist several books or review articles which deal with this subject. The general theory can be found in the book of Wall ⁶⁾, a more detailed analysis, in particular of Stieltjes series, is given in the review articles of Baker ^{7),8)}, finally, a recent and up to date account has been given by Zinn-Justin ¹¹⁾. Further references can be found in these articles.

1. - General properties

From the definition (I.2), or the equivalent formula

$$Q_M(z) f(z) - P_N(z) = O(z^{N+M+1}) \quad (\text{II.1})$$

it follows that the coefficients of the polynomials $P_N(z)$ and $Q_M(z)$ are determined from the first $N + M + 1$ coefficients of the Taylor series by a set of linear equations, up to a constant common factor. Therefore the $[N, M]$ Padé approximant is unique. The Padé denominators, for instance are given by

$$Q_M(z) = \begin{vmatrix} f_{N-M+1} & f_{N-M+2} & \dots & f_N & z^M \\ f_{N-M+2} & \dots & \dots & f_{N+1} & z^{M-1} \\ \vdots & & & & \vdots \\ f_{N+1} & \dots & \dots & f_{N+M} & 1 \end{vmatrix} \quad (\text{II.2})$$

where it is understood that $f_P \equiv 0$ if $P < 0$. Notice, however, that it may happen that for certain values of N and M , the approximant does not exist ⁸⁾.

When $f(z)$ is a matrix, one can build matrix Padé approximants ^{11),13),14)}. In this case, the question of uniqueness is a little bit more involved. For instance one can define right-hand and left-hand approximants

$$P_N(z) [Q_M(z)]^{-1} \quad \text{and} \quad [\tilde{Q}_M(z)]^{-1} P_N(z) \quad (\text{II.3})$$

and, besides these, mixed ones of the form $P_{N_1} Q_{M_1}^{-1} P_{N_2} Q_{M_2}^{-1} \dots P_{N_R} Q_{M_R}^{-1}$. If a function has both a right-hand and a left-hand approximant, they are the same ¹¹⁾. All properties which can be proved in the scalar case also hold in the matrix case which we shall not mention further.

Diagonal Padé approximants are those for which the degree of the numerator is equal to the degree of the denominator, i.e., $N = M$.

2. - Homographic properties

The question asked is : what are the transformation properties which are common to the function $f(z)$ and its Padé approximants, since the Padé algorithm is non linear ? We will consider transformations both on the function and on the variable, and we restrict ourselves to diagonal approximants.

Consider a homographic transformation of the function

$$f(z) \rightarrow \frac{a f(z) + b}{c f(z) + d} \quad (\text{II.4})$$

where a, b, c and d are arbitrary constants. Consider now the same transformation on the diagonal approximant $f_{[N,N]}(z)$ to $f(z)$:

$$R(z) = \frac{a f^{[N,N]} + b}{c f^{[N,N]} + d} = \frac{a P_N(z) + b Q_N(z)}{c P_N(z) + d Q_N(z)} \quad (\text{II.5})$$

It is obvious that : a) $R(z)$ is the ratio of two polynomials of degree N , and that b) it has the same first $2N$ derivatives as the transformed function (II.4). Since the Padé approximant is unique, the diagonal approximants of the transformed function are the transformed of the diagonal approximants of the function.

Similarly, if we perform a homographic transformation on the variable which leaves the origin invariant (since we expand around this point)

$$z = \frac{\alpha z'}{1 + \beta z'} \quad (\text{II.6})$$

If we express the $[\overline{N}, \overline{N}]$ approximant in terms of the transformed variable, and multiply the numerator and denominator by $(1 + \beta z')^N$, we obtain

$$f^{[\overline{N}, \overline{N}]} \left(\frac{\alpha z'}{1 + \beta z'} \right) = \frac{P_N \left(\frac{\alpha z'}{1 + \beta z'} \right) (1 + \beta z')^N}{Q_N \left(\frac{\alpha z'}{1 + \beta z'} \right) (1 + \beta z')^N} = \frac{R_N(z')}{S_N(z')} \quad (\text{II.7})$$

i.e., a ratio of two polynomials in z' of degree N . Exactly as above, we conclude that the transformed approximant is the approximant of the transformed function $f(\alpha z' / (1 + \beta z'))$.

That Padé approximants preserve homographic transformation properties of the function and of the variable is of fundamental importance, it explains many of their powerful properties.

a) A complete study of the homographic properties, i.e., for $N \neq M$, is given in Refs. 8) and 11) we do not repeat it here. These properties can be used to relate contiguous approximants to one another - $[\overline{N} \pm 1, \overline{N}]$ to $[\overline{N}, \overline{N}]$ - and in particular to obtain the coefficients in a recursive way from the Taylor series.

b) These properties yield that poles of $f(z)$ are not to be considered as special points for Padé approximants - as opposed to the Taylor series - since one can regularize them by suitable homographic transformations. In particular, Padé approximant sum geometric series exactly.

c) As a consequence, the Euler transformations mentioned in the Introduction to enlarge convergence are done systematically by the Padé approximation. This will be very important when we come to convergence problems. In fact, the natural domain of convergence for Padé approximants is not a circle as for the Taylor series, it is much larger. Suppose, for example, that we have proven the convergence of Padé approximants inside the convergence circle of the Taylor series. By using the homographic transformation properties on the variable, we immediately extend this region to the union of all circles in the analyticity domain which contain the origin - e.g., if the function has a cut on the real axis, it is the entire cut plane.

d) We finally mention that these covariance properties under homographic transformations are also responsible for the fact that in scattering amplitudes, Padé approximants are unitary. In fact, the partial wave S matrix satisfies $S_\ell S_\ell^* = 1$ or, equivalently

$$S_\ell = [S_\ell^*]^{-1} \quad (\text{II.8})$$

and it is easy to see that this property will be preserved by diagonal Padé approximants, i.e.,

$$S_\ell^{[N,N]} = [S_\ell^{*[N,N]}]^{-1} \quad (\text{II.9})$$

3. - Orthogonality properties of Padé denominators

Let us assume that $f(z)$ is analytic in the vicinity of $z = 0$. From the definition of diagonal approximants

$$Q_N(z) f(z) = P_N(z) + O(z^{2N+1}) \quad (\text{II.10})$$

we can write a Cauchy formula around $z = 0$

$$\oint Q_N(z) f(z) \frac{dz}{z^{N+P+2}} = 0 \quad 0 \leq P < N \quad (\text{II.11})$$

where the contour C is in the analyticity domain and surrounds the origin. Changing variables to $u = 1/z$, we have, since $Q_N(z)/z^N$ is a polynomial of degree N in $1/z$,

$$\oint \tilde{Q}_N(u) f\left(\frac{1}{u}\right) u^p du = 0 \quad 0 \leq p < N \quad (\text{II.12})$$

Equation (II.12) expresses that $\tilde{Q}_N(u)$ is orthogonal to all powers of u less than N on the complex contour, with the weight $f(1/u)$.

Therefore the set of Padé denominators $Q_N(z)$ are a set of orthogonal polynomials on a complex contour with $f(z)$ as measure. If the singularities of $f(z)$ are on the real axis we can distort the contour, and the Q_N are orthogonal polynomials on the real axis with the measure $\rho(z)$ where $2i\rho$ is the discontinuity of $f(z)$.

This property is used to prove convergence of Padé approximants for Stieltjes functions as we shall see. It is also interesting to notice that the classical Gaussian integration method follows from this ¹¹⁾.

4. - Convergence

a) Statement of the problem

One wishes to prove uniform convergence of $[N, M]$ approximants as N and M both tend to infinity. Approximants for which M (or N) is fixed and N (or M) goes to infinity are not so powerful in practice, and convergence is much easier to prove for them. Owing to homographic transformation properties, the $[N+k, N]$ approximants, k being a fixed integer, are deduced from the $[N, N]$ ones, and therefore will have the same convergence properties. We therefore concentrate only on diagonal approximants. As mentioned above, if we can prove convergence in some finite domain, we immediately enlarge the domain using the homographic properties.

We remark that even inside the circle of convergence of the Taylor series there is no general proof up to now that its Padé approximants converge. Intuitively, this is because, to prove convergence of the Taylor series, one only needs absolute constraints on the coefficients as $n \rightarrow \infty$, whereas presumably convergence of Padé approximants more or less implies that the series can be analytically continued outside this circle, and therefore requires relative properties of the coefficients with respect to each other ²⁾.

The basic difficulty lies in localizing the poles of the approximants, i.e., the zeros of Padé denominators ; uniform convergence can only be obtained outside small circles around these poles. We notice that in general, convergence may hold only for a subsequence of the diagonal approximants - some approximants may not exist, poles of one approximant may appear in the convergence region of the rest of the sequence ⁸⁾.

b) Stieltjes functions

For Stieltjes functions, that is to say functions which have a cut on the real axis with a positive discontinuity

$$\operatorname{Im} f(z) \cdot \operatorname{Im} z \geq 0 \quad (\text{II.13})$$

the Padé denominators are orthogonal polynomials with a positive measure (see Section II.3). Consequently, one can apply the theory of orthogonal polynomials ¹⁵⁾ ; one knows that the zeros are located on the cut. Therefore the poles of Padé approximants are localized, and one can prove uniform convergence in the cut plane. For complete details, see Refs. 7) and 8).

c) General case

In general cases, it is likely that the poles of Padé approximants may become dense in the complex plane. One expects weaker forms of convergence.

For meromorphic functions, Nuttall ¹⁶⁾ has proved the following form of convergence. Let $f(z)$ be a function meromorphic in the entire z plane, and let ε , δ and R be three positive arbitrary numbers. Then one can find an integer N such that for all $n \geq N$ we have

$$\left| f(z) - f^{[n,n]}(z) \right| < \varepsilon \quad \text{for } |z| \leq R \quad (\text{II.14})$$

except on an open set D_n of measure smaller than δ . This is called convergence in measure (of course we cannot localize D_n). Zinn-Justin ¹⁷⁾, furthermore, has given a bound for the error ; he has generalized the proof and shown that for some classes of functions one has convergence except on a set of measure zero.

The case of functions meromorphic in a circle of radius R has been investigated by Chisholm¹⁸⁾ and Zinn-Justin¹⁷⁾. One obtains the following results. Let q_n be the number of zeros of the $[n, n]$ Padé denominator inside the circle $|z| \leq R$, if the following condition is satisfied

$$\frac{q_n \log n}{n} \xrightarrow{n \rightarrow \infty} 0 \quad (\text{II.15})$$

(i.e., q_n does not increase too fast), then the Padé approximants converge uniformly in a circle of radius $R/\sqrt{3}$ except for a domain of measure zero. If the condition (II.15) is not satisfied, one can prove (II.14) except for a domain of finite measure in the circle $|z| \leq R$, which depends on n and cannot be localized, although one can give an upper bound for the measure of these domains¹⁹⁾.

d) Johnson's approach

In a recent paper, Johnson²⁰⁾ has proposed an entirely new approach to Padé approximants. His reformulation of the problem may lead to very interesting results. Consider a power series expansion

$$f(z) = f_0 + z f_1 + z^2 f_2 + \dots \quad (\text{II.16})$$

and the sequence $\{S_r\}$ of its partial sums

$$S_r = \sum_{m=0}^r f_m z^m \quad (\text{II.17})$$

We assume that the coefficient f_n of (II.16) is determined from f_{n-1} , $f_{n-2}, \dots, f_0$ by some recurrence relation. Therefore the partial sum S_{r+1} is obtained from S_r by some functional relation g , i.e.,

$$S_{r+1} = g(S_r) \quad (\text{II.18})$$

The existence and uniqueness of the functional g is of fundamental importance here.

The sum S of the series (II.16) is therefore defined as the solution to the fixed point equation

$$S = g(S) \quad (\text{II.19})$$

If the series converges, this is obvious, if it diverges Eq. (II.19) does define its generalized sum.

The important point is that Johnson shows that the Padé method estimates, by contraction, a solution to Eq. (II.19). In order to show this one needs the following : let us denote by α a solution to Eq. (II.19), $\alpha = g(\alpha)$, and by t_n all elements generated by the transformation g , i.e., $t_{n+1} = g(t_n)$. Provided we can define a distance $\|t - t'\|$ between two elements t and t' , and provided we can also define an expansion of g around the value α

$$g(\alpha + \varepsilon) = \alpha + \varepsilon g' + o(\varepsilon^2) \quad (\text{II.20})$$

where g' is independent of $\|\varepsilon\|$, then Padé approximants tend to map the partial sums of the power series (II.16) closer in value to the function which is a fixed point of the generating functional g .

In other words, for convergent series Padé approximants accelerate the convergence, while for divergent series they estimate the generalized sum defined by Eq. (II.19).

Johnson shows how the contraction mapping takes place on several examples where the classical theory of Padé approximants is known - geometric series, Gauss' hypergeometric function, Euler's function $\sum n! z^n$, etc. As we shall argue in Section IV, this method may have great interest for proving convergence in physical applications where, precisely, the functional g is known and can be expressed. Although all mathematical details are not yet settled, this approach is very promising.

5. - The Padé conjecture

In most mathematical examples, although the convergence of Padé approximants has not been proved, numerical computations indicate that the method does converge. A conjecture was set forth by Baker and Gammel²¹⁾ which can be expressed in several ways, for instance : there exist subsequences of diagonal Padé approximants which converge uniformly on any compact in the convergence circle of the Taylor series. At present there is no counter example, for more details and numerical examples see Ref. 8). Using this conjecture, one can derive a variety of "quasi theorems"⁸⁾, in particular enlarge the convergence region.

6. - Padé approximants of type II

The Taylor series is an expansion around one point. One can actually build Padé approximants which contain information about the function and its derivatives at two and more points. This generalization of Padé approximants has been studied by Baker⁸⁾.

A particularly useful case is when we give only the value of the function at a certain number of points. We then obtain Padé approximants of type II¹⁴⁾. Let $z_1, z_2 \dots z_P$ be P complex numbers, and $f(z)$ an analytic function which takes the value $f(z_i)$ at these points. We define the $[N, M]$ Padé approximant of type II to $f(z)$ as the ratio of two polynomials in z , $P_N(z)$ and $Q_M(z)$ with $N+M = P-1$, which takes the values $f(z_i)$ at $z = z_i$

$$f^{[N, M]}(z) \equiv \frac{P_N(z)}{Q_M(z)} \quad (\text{II.21})$$

$$f^{[N, M]}(z_i) = f(z_i) \quad i = 1, P \quad (\text{II.22})$$

The $[N, M]$ approximant is unique, one can prove essentially the same properties as for ordinary approximants¹¹⁾ - orthogonality of denominators, convergence in the Stieltjes case, etc.

Padé approximants of type II have many useful practical applications as we shall see in the next section : approximation to functions, analytic extrapolation, etc.

III. - APPLICATIONS : GENERAL EXAMPLES

1. - Analytic continuation

The Padé approximation provides a practical method for performing numerically the analytic continuation of a function.

a) Divergent series

The applications to continuation of a Taylor series outside its circle of convergence are of course very important. Most physical applications use this property as we shall see in subsequent sections. It is interesting that, in many instances, the approximation may be used even if the radius of convergence is strictly zero, and gives a meaningful answer. A classical example is Euler's function

$$F(z) = \int_0^{\infty} \frac{e^{-xz} dx}{1 - z^2 x} \quad (\text{III.1})$$

whose formal Taylor series expansion is

$$F(z) = \sum_{n=0}^{\infty} n! z^n \quad (\text{III.2})$$

$F(z)$ has an essential singularity at $z = 0$. Although (III.2) diverges whatever the value of $z \neq 0$, the sequence of diagonal Padé approximants built from it converges to the value (III.1) outside the cut $[0, \infty]$. As in most similar cases, this can be proved because (III.1) defines a Stieltjes function. The continued fraction expansion of (III.2) from which diagonal Padé approximants are easily obtained is

$$F(z) = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{z}{1 - \cfrac{2z}{1 - \cfrac{2z}{1 - \cfrac{3z}{1 - \cfrac{3z}{\ddots}}}}}}} \quad (\text{III.3})$$

By truncating this expression one can check that the poles and zeros alternate on the interval $[0, \omega]$ and tend to accumulate near $z = 0$. A physical situation of this kind occurs for the energy levels of the anharmonic oscillator ²²⁾.

b) Analytic extrapolation

In many occasions, Padé approximants of type II are an easier way to approximate functions than those of type I. Given the values F_i of a function $F(z)$ at some points z_i , Padé approximants of type II give a very smooth interpolation of $F(z)$ between the z_i , often much more efficient than the Lagrange polynomial interpolation, provided $F(z)$ has some analyticity properties. More important is that the method allows the extrapolation outside the range of the z_i since it reconstructs the singularities of $F(z)$. Padé approximants absorb poles and enable us to perform numerically the analytic continuation of a function known at some points. For instance, one can build the type II Padé approximants for $\operatorname{tg} z$ starting with values on $-\pi/2 < z < \pi/2$; lowest orders give the nearby poles and zeros, and by increasing the order (number of points) one gradually builds more and more poles. It is interesting that the same features remain if one fits on the smaller interval $-\pi/4 < z < \pi/4$ where the function is actually very smooth; for z pure imaginary one obtains excellent approximations to $\operatorname{th}(\operatorname{Im} z)$. This can be very useful in practice, for instance in obtaining simple but accurate approximations to complicated functions ¹⁴⁾, etc.

In recent years, much interest has been devoted to the problem of continuing scattering amplitudes in unphysical regions, i.e., outside the convergence domain of the partial wave expansion in a given channel. Fleischer ²³⁾ has studied the box diagram (Fig. 1) and has shown the usefulness of Padé approximants in continuing from the s channel to the t channel. Major progress has been made in this direction recently by Yndurain ²⁴⁾ who proved, using unitarity, that given the first L πN partial waves and the πN total cross-sections, one can extrapolate to find the $\pi\pi \rightarrow N\bar{N}$ partial waves, and that Padé approximants converge on the cuts (in the average sense) within absolute error bands. This result is of great importance and should have many applications.

c) Carlson interpolation

This application is due to Zinn-Justin ¹¹⁾⁻¹³⁾. The problem is to reconstruct an analytic function $a(\ell)$ which

- (i) is holomorphic for $\text{Re } \ell > 0$,
- (ii) satisfies $|a(\ell)| < M \exp(\alpha |\text{Im } \ell| + \beta |\text{Re } \ell|)$, $\alpha < \pi$, and
- (iii) is known for integer values of ℓ : $a(\ell) = a_\ell$, $\ell = 1, 2, 3, \dots$

Formally, the solution to this problem is to construct the generating function of the a_ℓ

$$\varphi(z) = \sum_{n=1}^{\infty} a_n z^n \quad (\text{III.4})$$

the function $a(\ell)$ is then given by

$$a(\ell) = \frac{1}{2i\pi} \oint_C \frac{\varphi(t) dt}{t^{\ell+1}}, \quad \text{Re } \ell > 0 \quad (\text{III.5})$$

where the contour C surrounds the negative real axis. In practice, if we know a finite number of values a_ℓ ($\ell \leq \ell_{\max}$) we cannot use the corresponding polynomial (III.4) in (III.5) since the latter integral will only converge for $\text{Re } \ell > \ell_{\max}$, and give the trivial result $a(\ell) \equiv 0$ $\ell_{\text{integer}} > \ell_{\max}$. If, however, we build $\varphi_N(z)$, the $[N, N]$ Padé approximant to $\varphi(z)$, and insert it in Eq. (III.5), we obtain a quantity $a_N(\ell)$ which has the property of reproducing identically the values a_ℓ for $1 \leq \ell \leq \ell_{\max}$, as is easy to check, and of giving some approximation for the continuation elsewhere. Furthermore, Zinn-Justin has proved that $a_N(\ell)$ converges to $a(\ell)$ as $N \rightarrow \infty$ in the case where $\varphi(z)$ is a Stieltjes function. This has been applied by Common ²⁵⁾.

2. - Convergence acceleration

a) Approximants of type I

Even if a series converges, it may converge slowly, and Padé approximants can accelerate its convergence. Given a sequence of numbers, for instance the successive results of an iterative procedure to solve an equation, $u_1, u_2, \dots, u_n$ one can build the generating function

$$F(z) = u_1 + (u_2 - u_1)z + (u_3 - u_2)z^2 + \dots \quad (\text{III.6})$$

and, by definition, the exact answer is $u_{\infty} = F(1)$. Quite generally, provided $F(z)$ is regular at $z = 1$, Padé approximants to $F(z)$ at $z = 1$ will give a better estimation of the final answer than the last computed term u_n . Among the many applications, we note the study of numerical quadrature made recently by Chisholm, Genz and Rowlands ²⁶⁾. A very simple example is the following : we can integrate a function by the trapezoidal rule ; starting with one mesh point, we successively double the number of points at each step and obtain a set of approximations to the integral. Upon treating this sequence by Padé approximants, the result is roughly as good as that given by Gaussian integration ²⁷⁾ (the convergence can be calculated analytically) with as many over-all mesh points, which is quite remarkable.

b) Approximants of type II

Padé approximants of type II can be used when $F(z)$ in Eq. (III.6) is not regular at $z = 1$. Consider for instance the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (\text{III.7})$$

and let us denote by S_p its partial sums

$$S_p = \sum_{n=1}^p \frac{1}{n^2} \quad (\text{III.8})$$

The above series converges very slowly, however, we can consider the numbers S_p as functions of the variable $1/p$, i.e., $S_p = F(1/p)$ and the exact sum is $F(0)$. From the $N+M+1$ first partial sums, we can build the $[N, M]$ type II approximant to $F(z)$ and get an estimate for $F(0)$. Consider, for instance, the first three terms

$$S_1 = 1 \quad S_2 = 1.25 \quad S_3 = 1.361 \quad (\text{III.9})$$

the approximants will give us

$$F \begin{smallmatrix} [0,0] \\ (0) \end{smallmatrix} = 1 \quad F \begin{smallmatrix} [0,1] \\ (0) \end{smallmatrix} = 1.5 \quad F \begin{smallmatrix} [1,1] \\ (0) \end{smallmatrix} = 1.65... \quad (\text{III.10})$$

while the exact answer is $\pi^2/6 = 1.645...$ Convergence has been accelerated quite remarkably. Notice that to obtain $F \begin{smallmatrix} [1,1] \\ (0) \end{smallmatrix}$ we have used no more information than the values S_1, S_2 and S_3 .

c) Solving equations

Suppose $F(z)$ is an analytic function and we want to solve the equation

$$F(z) = 0$$

We can proceed as follows : start from three points z_1, z_2, z_3 where the values of F are F_1, F_2, F_3 . Next, we build the homographic function of z which has the values F_i for $z = z_i$ (type II Padé approximant). It is trivial to find the zero z_4 of this homographic function, and, using the value $F_4 = F(z_4)$, we continue the operation with z_2, z_3, z_4 and so on. Zinn-Justin has shown ^{11),13)} that the convergence is exponential : if at one step we are within an error ε of the zero, at the next step we will be within ε^3 (i.e., in two steps we go from 10% to 10^{-9}). Convergence towards the zero from a distance is quite fast.

We may remark that a) the method is very simple, all operations are linear, and b) it has no more difficulty in finding zeros in the complex plane than on the real axis.

This method is very powerful and useful.

3. - Padé approximants in the presence of noise (background)

a) Analytic information

When a Padé approximant of type II reconstructs a function, it does exactly the same thing as when one draws a curve "by hand", but it does it in a much more refined and powerful way. Padé understands the behaviour of as many derivatives as there are points and it is able to reconstruct the essential features of the analytic structure of the function.

When Padé approximants are used to accelerate the convergence of a sequence of numbers, we use the information that this sequence eventually converges. The approximants then evaluate how the sequence converges, find the "law" that relates successive terms and give an estimate for the final answer. This is possible only if there is some analyticity in the way the sequence converges.

It is worth being kept in mind that in any iterative method, the numbers obtained in successive iterations contain as a whole much more information than the last one individually (although the last one is usually considered as the best approximation). Padé approximants understand analyticity and are therefore able to extract information when some analyticity properties are present.

b) Perturbed information

Quite naturally, the following question arises : how do Padé approximants react when one feeds them with an analytic signal perturbed by some non-analytic background or noise ? For instance, there are always round-off errors in computers, also if we try to fit experimental data with Padé approximants of type II, we have to take into account experimental errors.

Again, from the theoretical point of view, major progress has been made in this direction by Yndurain²⁴⁾ who showed in the work mentioned above on πN that the Padé extrapolation was stable under perturbations. However, in general, we must rely on numerical experiments to investigate this problem. Froissart^{5),27)} has performed such experiments and the results appear as follows.

Consider the power series

$$S(z) = \sum_{n=0}^{\infty} (1 + \varepsilon r_n) z^n \quad (\text{III.11})$$

where ε is a small number (for instance 10^{-2} or 10^{-3}) and the r_n are random (complex) numbers such that

$$|r_n| \leq 1 \quad (\text{III.12})$$

Mathematically, $S(z)$ has a natural boundary on the circle $|z| = 1$ with probability one, i.e., it cannot be continued analytically outside this circle. However, we feel that the essential part of the function is $1/(1-z)$ which is analytic and has a pole at $z = 1$, while the non-analytic part is small and of order ε . We can build the $[N, N]$ Padé approximants to Eq. (III.11) and compute their poles and zeros. The pattern is as follows :

- (i) there is a pole near $z = 1$, which is very stable as the order changes ;
- (ii) associated to this pole, we find a far away and unstable (ghost) zero ; its distance is of the order $1/\varepsilon$, this simulates the fact that as $z \rightarrow \infty$, $1/(1-z)$ goes to zero while the diagonal Padé tends to a constant ;
- (iii) all the other poles and zeros group themselves in doublets (one pole + one zero) which gather in the vicinity of the unit circle, and therefore represent the natural boundary of the noise ; the distance of the pole and zero (and therefore the residue of the pole) in a doublet is of order ε if the doublet is close to $|z| = 1$, it decreases if the doublet enters the circle, and increases if it is further away ; the doublets are extremely unstable against the order N .

On Fig, 2, we give an example with a noise $\varepsilon = 5.10^{-2}$ and a $[4,4]$ Padé, the positions of the poles and zeros are

$$\text{stable pole} \quad z = 1.003 - i 8.10^{-4}$$

$$\text{ghost zero} \quad z = -55 + 5 i$$

doublets :

$$\text{poles} \quad (-0.97 - 0.19 i) \quad (-0.572 + 0.791 i) \quad (0.457 - 0.749 i)$$

$$\text{zeros} \quad (-0.95 - 0.21 i) \quad (-0.575 + 0.805 i) \quad (0.453 - 0.754 i)$$

The error made on the position of the true pole, which is of order of ε for the $[1,1]$ Padé, decreases as N increases and reaches a minimum value much smaller than ε . On the above example with $\varepsilon = 5.10^{-2}$ the error is $|z_{\text{pole}} - 1| \sim 3.10^{-3}$, with a noise $\varepsilon = 10^{-3}$ we obtain typically $|z_{\text{pole}} - 1| \sim 10^{-5}$. Padé approximants have somehow smoothed out the noise for what concerns the position of the pole.

On this very simple example we conclude that the approximation seems very stable against random perturbations. If we compute the approximants we will in general get an answer close to $1/(1-z)$ within ε , even if z is large and outside the unit circle, except if by bad luck a doublet is nearby. However, by changing the order, the doublets will most probably be displaced since they are very unstable, and the stability of the answer indicates its reliability. Padé approximants have rather damped the noise than amplified it.

The following more complicated example is very instructive, however 5),27). Consider the power series

$$S(z) = \sum_{n=1}^{\infty} \left(-\frac{1}{n} + \frac{\varepsilon r_n}{2^n} \right) z^n \quad (\text{III.13})$$

where ε and r_n are chosen as previously. The "regular" part of Eq. (III.13) is $\log(1-z)$ and converges for $|z| < 1$ while the "noise circle" (natural boundary) is now at $|z| = 2$. The pattern of poles and zeros of the approximants is shown on Fig. 3, there are three categories :

- (i) one set of poles and zeros gathers near the real axis in $[1, \infty]$ (with a stable zero of course near $z = 0$) in order to represent the cut of $\log(1-z)$;
- (ii) we have a set of doublets in the vicinity of $|z| = 2$, which represents the noise circle of (III.13),
- (iii) however, we observe another noise circle near $|z| = 1$.

This is interesting because the noise series itself $\sum \varepsilon r_n (z/2)^n$ is perfectly convergent in this region. However, there seems to be some reflection of the fact that the analytic part of the series diverges at $|z| = 1$. This is an illustration also of the fact that uniform convergence of Padé approximants is very difficult to prove in general since, here, poles of the approximants arise in the holomorphy domain of the function. Away from the doublets, the approximants give a very good approximation to $\log(1-z)$ in a region determined essentially by the order N .

Finally, it has been observed 5),27) that doublets created by the round-off errors of computers appear at high enough orders, they are located around the origin $z = 0$ at distances of the order of the accuracy.

c) Fits by rational functions

The same calculations as above can be made using Padé approximants of type II, they give essentially similar results. One is led naturally to try to fit experimental data with rational functions. The main ambition, there, is of course to use the fit in order to continue to unphysical regions. For instance, one can fit a total cross-section or an invariant mass distribution, and determine resonances by seeing if poles tend to appear and whether they are stable in going from one order to another. More ambitious is to try to determine t channel singularities by fitting angular distributions. There are, however, many difficulties. First is that since rational functions are not linear in

the parameters, least χ^2 fits are more complicated : several minima can occur. Secondly, the experimental errors are greater, and the functions more complicated, than what was considered in the above examples, the propagation of errors may become important. Thirdly, experimental quantities are usually moduli squared ; for instance pion exchange in NN scattering appears as a double pole in the transfer variable, and low order approximants may have trouble in representing such situations. The work of Yndurain ²⁴⁾ does give hope, however.

We mention, as examples, some investigations. The low energy nucleon-nucleon angular distribution was fitted in order to study pion exchange ²⁸⁾. The method does not really succeed in exhibiting the pion pole in a convincing manner basically for two reasons : (a) that electromagnetic contributions to the forward peak are important ; (b) that the one-pion exchange contribution is damped by the pseudoscalar interaction ($\sqrt{t}$ and $\sqrt{u}$ factors). However, when the pion pole is imposed, one gets a reasonable order of magnitude for the π N coupling constant, although not an accurate value. In reactions where pion exchange is a dominant contribution such as $\pi N \rightarrow \pi\pi N$, the method succeeds in expliciting the pion pole ²⁸⁾. In fact, in order to attain a singularity properly, one must in fact "see" the effect of the singularity on the data.

IV. - PHYSICAL APPLICATIONS

1. - Generalities

Very often, the equations describing a physical process are so complicated that the simplest way to obtain their solution, if not the only way, is to perform a power series expansion in some parameters. Furthermore, the physical values of the parameters may be such that this perturbation expansion does not converge, and therefore it is only a formal solution to the problem, i.e., it cannot be used quantitatively as such. However, the information is present in the coefficients of the perturbation series, and one may look for mathematical techniques that would be capable of treating this information in a convergent way. The Padé approximation is very well suited for this as we shall see.

The first physical applications of the Padé approximation have been made in statistical mechanics. We refer the reader to the excellent reviews of Baker ^{7),8)} and Fisher ⁹⁾ for complete details. For example, one can study phase transitions, starting from density expansions of thermodynamic functions. As a general rule, in this field one is interested in eliciting the singularities of the functions in the expansion parameter (e.g., the density), consequently one must use high order expansions, and approximants, in order to represent these singularities accurately ; this is clear from the above discussions.

As an example of how the approximation can be used, it is of interest to study briefly what can be considered as the most commonly used Padé approximant in physics : the Van der Waals equation ²⁹⁾. Consider a gas of N identical particles in a volume V , which interact through a two-body potential $\Phi(r)$. The potential consists of a hard core $[\Phi(r) = \infty, 0 < r < r_0]$ and a (small) attractive tail (see Fig. 4). The partition function Z of this gas can be written as

$$Z = Z_0 Q \quad (\text{IV.1})$$

where Z_0 is the partition function of the corresponding perfect gas, $Z_0 \sim [V/\beta^{3/2}]^N$ where $\beta^{-1} = kT$, and Q is the interaction correction

$$Q = \frac{1}{V^N} \int e^{-\beta \sum_{i,j} \Phi_{ij}} d^3r_1 \dots d^3r_N \quad (\text{IV.2})$$

From the partition function, we obtain the free energy and the pressure by the usual formulae

$$F = -\frac{1}{\beta} \log Z \quad p = -\frac{\partial F}{\partial V} \quad (\text{IV.3})$$

The expansion of Q in the density N/V is performed as follows

$$\begin{aligned} Q &= \frac{1}{V^N} \int \prod_{ij} (e^{-\beta \Phi_{ij}} - 1 + 1) d^3 r_1 \dots d^3 r_N \\ &= 1 + \frac{N(N-1)}{V^2} \int [e^{-\beta \Phi(r_{12})} - 1] d^3 r_1 d^3 r_2 + \dots \quad (\text{IV.4}) \end{aligned}$$

where we recognize in the second term the two-particle correlation function. We therefore have to first approximation

$$\log Q \simeq \frac{N^2}{V} \int (e^{-\beta \Phi(r)} - 1) d^3 r \quad (\text{IV.5})$$

The hard core contribution to the above integral (i.e., $r < r_0$, $\Phi = \infty$) is $-b = -\frac{4}{3} \pi r_0^3$, while for $r > r_0$ we have $e^{-\beta \Phi} - 1 \sim -\beta \Phi$ which contributes some positive quantity βa , therefore

$$\log Q \simeq \frac{N^2}{V} [-b + \beta a] \quad (\text{IV.6})$$

and, from Eq. (IV.3), we obtain the density expansion of the pressure

$$\begin{aligned} p &= \frac{1}{\beta} \frac{\partial}{\partial V} [\log Z_0 + \log Q] \\ &= \frac{N k T}{V} + \frac{N^2}{V^2} [b k T - a] \quad (\text{IV.7}) \end{aligned}$$

This, we can rewrite as

$$p + a \frac{N^2}{V^2} = k T \left[\frac{N}{V} + b \frac{N^2}{V^2} \right] \quad (\text{IV.8})$$

and, upon taking the $[1,1]$ Padé of the right-hand side in the density N/V , we obtain the Van der Waals equation

$$\left(p + a \frac{N^2}{V^2}\right)(V - V_0) = NkT \quad (\text{IV.9})$$

where we have set $V_0 = bN$.

This deserves several comments, first of which : why did we separate out the term aN^2/V^2 in going from Eq. (IV.7) to Eq. (IV.8), i.e., why did we not take the Padé approximant to the former ? The answer comes from physics : we cannot treat the potential of Fig. 4 as a whole, it is the singular part, the hard core, which leads to a singularity (the particles cannot come closer to each other than a distance r_0), it is that part only that we treat with Padé approximants, the attractive tail is small and smooth, we treat it as a "normal" perturbation and put it on the left-hand side (we expand it near $g = 0$, if g is the coupling strength, whereas the hard core can be considered as the limit of a square well as $g \rightarrow \infty$). This point is of course of crucial importance in practice, from this we learn that the understanding of the physical nature of the phenomena we want to study is a guide to how we use the Padé approximation.

Originally, the Van der Waals equation was obtained phenomenologically, however, we see how we can get higher order Van der Waals type equations by taking more terms into account in the virial expansion (IV.4)³⁰⁾. We see how we recover the equation of state for perfect gases in the limit of very low densities. We do not discuss the interpretation of the phase transition, which is well known, we note how at high densities (i.e., in the liquid phase) Eq. (IV.9) is an infinitely better description than the series (IV.8).

To summarize, the Padé approximation gives a good representation of the physical process on both sides of the singularity (phase transition), and even the singularity itself has a physical meaning if interpreted correctly.

The application of the Padé approximation to strong interaction physics was initiated by Gammel³¹⁾, and was revived a few years later by the work of Bessis and Pusterla³²⁾. It is necessary to point out that the spirit in which the approximation is used here is somewhat different from the above examples. In fact one starts from an expansion in some coupling constant g of, say, scattering amplitudes, but one is not interested so much in the

behaviour of the amplitude in the expansion parameter (as above) as in the variation in other parameters, for instance the energy, at a fixed value of g , where the amplitude is regular in g (except perhaps for poles : bound states and resonances). For this reason, one may expect that low order approximants are quite reliable, in contrast with the case of statistical mechanics. As we shall see, the investigations have been twofold : first, models for strong interaction amplitudes have been built starting from various Lagrangian theories, secondly, the properties of the approximation itself have been studied, e.g., the convergence problem.

Before going on, it is worth while mentioning the case of quantum electrodynamics. In usual perturbation calculations, e.g., the Lamb shift, the expansion parameter $\alpha = 1/137$ is so small that Padé approximants do not differ significantly from the perturbation series. If we now consider the bound state problem, the simplest idea would be to expand, for instance, the Coulomb amplitude $\Gamma(\ell + 1 + i\alpha/k)/\Gamma(\ell + 1 - i\alpha/k)$ in powers of α . However, since this is actually an expansion in α/k , the Padé approximants must reconstruct not only the bound state poles but also the essential singularity at $k = 0$. For this reason, although one can prove that the approximation converges, very high orders must be used in order to reproduce the first bound states correctly. A very elegant alternative has been proposed by Crater ³³⁾ who uses an expansion of the forward amplitude in powers of t/μ^2 where μ^2 is the (infinitesimal) photon mass and t the momentum transfer. This preserves the $O(4)$ symmetry in the expansion, and Crater showed the great similarity between the resulting spectrum and that obtained in the eikonal approximation. We also mention that the work of Garibotti, Pellicoro and Villani ³⁴⁾ on application of the Padé approximation to singular potentials (as examples of non-renormalizable theories) led to various applications to radiative processes ³⁵⁾, and that Graffi ³⁶⁾ proved the convergence of Padé approximants in the case of vacuum polarization by a constant external electromagnetic field.

2. - Application to the scattering problem

a) Statement of the problem

In general, one will start with a renormalizable Lagrangian field theory, and build the perturbation expansion of dynamical functions such as the S matrix, in the coupling constant g

$$S(g) = S_0 + g S_1 + g^2 S_2 + \dots \quad (\text{IV.10})$$

where the coefficients S_i depend on various kinematical variables, energy, momentum transfer, spin variables, etc. In Eq. (IV.10) $S(g)$ may be a scalar (e.g., a partial wave amplitude), a matrix of finite rank (e.g., coupled-channel calculation), an operator (the full S matrix), etc. The question asked is whether the Padé approximation to Eq. (IV.10) is a useful tool for studying quantitatively field theory in the strong coupling case, since the series apparently diverges. We note that given the Born term S_1 , and the renormalization prescription, the successive terms of the Feynman series (IV.10) are built from each other by using unitarity and crossing ¹⁴⁾.

There are essentially two different reasons why the perturbation series diverges. The first one is associated with the bound state problem : it is well known from potential theory that whenever a bound state (or resonance) occurs, the perturbation expansion diverges. In non-relativistic scattering, this is not a very serious problem since many methods are available for handling it - like solving the Lippmann-Schwinger integral equation, or using variational techniques - and, as we shall see, the Padé approximation provides a way to compute bound states directly in terms of the (divergent) perturbation expansion. In low energy strong interactions, where one is mainly interested in the spectrum, we only have the perturbation series at our disposal. The second reason is that in the relativistic case it is quite probable that an essential singularity is present in the coupling constant g at $g = 0$, i.e., the series has zero radius of convergence. This feature occurs in one dimensional relativistic models, in the Peres model ³⁷⁾, in the $g\varphi^3$ theory in four dimensions, in the Bethe-Salpeter equation ³⁸⁾. Dyson ³⁹⁾ has given an argument for a singularity at $\alpha = 0$ in quantum electrodynamics.

The convergence problem is twofold : first, we would like to know whether the method converges to the exact solution, but, almost as important is the question of how fast the approximation converges at a given energy. In fact, since the technical difficulties involved in the computation of Feynman graphs increase tremendously with the order of the perturbation expansion, we can only hope to compute the first few terms of the series in practice. We must know how reliable the low order approximants are.

In many instances we will refer to non-relativistic potential scattering. First, we believe that the structure of the bound state problem is essentially similar to the relativistic case, and this is of course our basic point of interest in practice. Secondly, it is a time-honoured tradition in particle physics that any approximation scheme for particle phenomena should be first tested for potential scattering where problems are well understood and exact solutions are known.

b) Proofs of convergence

α) - Potential scattering and the bound state problem

Considering a linear integral equation where the kernel is of finite rank N , one can easily verify that the $\overline{[M,N]}$ Padé approximants with $M \geq N$ give the exact solution. Chisholm⁴⁰⁾ used this to prove that Padé approximants give a convergent set of eigenvalues in the Lippmann-Schwinger equation with well-behaved potentials, e.g., Yukawa potentials, since the kernel is compact. The proof of convergence for the amplitude was then completed by Garibotti and Villani⁴¹⁾, using the minimal iteration method. The convergence proof below threshold can also be obtained by considering the analyticity properties of the scattering amplitude in the coupling constant (it is a Stieltjes function).

Another important property of the approximants is that they satisfy variational principles^{32),42)}. In particular, Bessis and Pusterla³²⁾ showed that diagonal approximants are identical to the approximations derived by Cini and Fubini⁴³⁾ from the Lippmann-Schwinger variational principle⁴⁴⁾.

For these reasons, we believe that the approximation handles the bound state problem correctly. This is not surprising since at fixed energy, bound states are fixed poles in the coupling constant, and we know that poles are not to be considered as singularities for the approximants (see Sections II and III).

β) - Essential singularities at $g = 0$

To investigate this problem, one studies so-called "models" of field theories, i.e., models with a small number of degrees of freedom which have a relativistic structure and exhibit an essential singularity at $g = 0$. For instance, in the Peres model³⁷⁾, where exact solutions are known, Baker and Chisholm⁴⁵⁾ have shown that Padé approximants to the S matrix elements

converge uniformly to the exact solution. Another celebrated example is the anharmonic oscillator, $V = \lambda x^2 + gx^4$ where the perturbation expansion in g of the energy levels has zero radius of convergence. Uniform convergence of Padé approximants for the energy levels was proved by Loeffel, Martin, Simon and Wightman^{22),46)} by making use of the analyticity properties in g .

Such proofs consist in finding analyticity properties in the coupling constant. Another study of this problem has been made by Ruijgrok⁴⁷⁾. From these analyses, we feel that it is quite possible that the Padé approximants can deal with essential singularities at $g = 0$.

8) - Further aspects

A general proof of convergence does not exist at present for realistic field theories. One possibility would consist in proving some analyticity properties in the coupling constants. However, it would be very interesting to know if the approach proposed by Johnson²⁰⁾ can be used. In fact, as mentioned above, the coefficients of the Feynman series are determined from one another by a functional equation consisting in unitarity and crossing. Therefore, it may be possible to use some techniques similar to those of Atkinson⁴⁸⁾ in order to prove that the contraction mapping suggested by Johnson does take place.

In some particular cases, it is sometimes possible to find more efficient summation procedures than the Padé algorithm⁴⁹⁾, but no example has ever been produced showing that it may fail in usual problems. Furthermore, it has very beautiful properties, as we shall see, like the unitarity properties, and it is systematic and well defined, so that its use in practice is particularly convenient. Even if it does not handle completely the essential singularity at $g = 0$, we feel that it is a much better asymptotic expansion of the amplitude than the perturbation series since most probably it deals with the bound state problem correctly.

3. - Properties

a) Unitarity and crossing

Consider the perturbation series for a partial wave amplitude

$$t^{\ell}(s) = g t_1^{\ell}(s) + g^2 t_2^{\ell}(s) + \dots \quad (\text{IV.11})$$

For this partial wave amplitude the unitarity relation is algebraic, we have

$$\text{Im } t^{\ell}(s) = [t^{\ell}(s)]^* \rho(s) t^{\ell}(s) \quad (\text{IV.12})$$

[the relation (IV.12) may be a matrix equation for a coupled channel problem ; the important criterion is that it is algebraic and not an integral equation].

It is easy to show that the Padé approximants to Eq. (IV.11)

$$t^{\ell [N,M]}(s) \equiv \frac{P_N(g)}{Q_M(g)} = t^{\ell}(s) + g^{N+M+1} R_1 \quad (\text{IV.13})$$

are unitary provided $M \geq N$ (31), (32), (50), (51), i.e., they satisfy Eq. (IV.12) exactly. This is in particular true for the diagonal ones $N = M$. This feature is due to the fact that there exist groups of transformations on the function and its Padé approximants, and that homographic properties such as $S S^* = 1$ are preserved under the mapping from the function to its Padé approximant (see Section II). These unitarity properties are of course very useful in the computation of strong interaction amplitudes ; we note that in many instances the non-unitary $[\overline{N}+1, \overline{N}]$ approximants do not violate unitarity by any considerable amount (14), (51), (52).

In the relativistic case, the total amplitude $A(s, t, u)$ satisfies crossing properties under s, t, u exchange, and furthermore, each term of the perturbation series individually satisfies crossing. Since partial wave approximants are unitary, the crossing symmetry present in the perturbation series is lost, that is to say that the amplitude $\tilde{A}(s, t, u)$ defined by

$$\tilde{A}(s, t, u) = \sum_{\ell} (2\ell+1) t^{\ell [N,M]}(s) P_{\ell}(z_s); \quad z_s = 1 + t/2q^2 \quad (\text{IV.14})$$

does not enjoy the crossing symmetry that $A(s,t,u)$ does ; moreover, the reconstructed amplitude $\tilde{A}$ does not have in principle the correct analytic structure in s at fixed t : it always has a left-hand cut starting at $s = 0$ and not at $s = -t$ (in the equal mass case). As an alternative to partial wave approximants, one may consider Padé approximants for the total amplitude

$$A(s,t,u) = g A_1(s,t,u) + g^2 A_2(s,t,u) + \dots \quad (\text{IV.15})$$

Consider the case of identical particles with no internal quantum numbers, the approximants $A^{[N,M]}(s,t,u)$ will identically satisfy crossing (the extension to more complicated cases is only an algebraic problem). In such approximations, however, exact unitarity is lost. These amplitudes are called "poloid" amplitudes ¹⁴⁾ for the following reason : consider the $[1,1]$ approximant to Eq. (IV.15)

$$A^{[1,1]}(s,t,u) = \frac{g A_1(s,t,u)}{1 - g \frac{A_2(s,t,u)}{A_1(s,t,u)}} \quad (\text{IV.16})$$

A pole of such an amplitude in energy s is given by the vanishing of the denominator

$$A_1(s,t,u) - g A_2(s,t,u) = 0 \quad (\text{IV.17})$$

However, it is quite obvious that the position of this "pole" will depend in general on the transfer variables t and u , whereas in the exact solution it should be a straight line in the Mandelstam plane. In turn, if we partial wave project Eq. (IV.16) this singularity will contribute small cuts in all partial waves, instead of a pole in only one (of course the residue of the small cut will in general be much larger in the relevant partial wave). We notice, however, that if elementary particles are present in the Lagrangian, the corresponding poloids will actually be flat as is easy to see. This may have great interest, for instance, in "unitarizing" the Veneziano model ⁵³⁾. In fact, if a "second order" Veneziano amplitude could be constructed, upon taking the Padé approximant of the total amplitude, such as (IV.16), one

would obtain an amplitude which would have the nice properties of being
 (i) exactly crossing symmetric with poles of definite angular momenta,
 (ii) Regge-behaved, and (iii) unitary in the vicinity of each pole.

b) Analytic structure

α) - Cuts

The numerators and denominators of partial wave Padé approximants are polynomials in the coupling constant g whose coefficients can be arranged to be themselves polynomials in the coefficients $t_n^\ell(s)$ of Eq. (IV.11) ; we have for instance

$$t^{\ell [1,1]}(s) = \frac{g [t_1^\ell(s)]^2}{t_1^\ell(s) - g t_2^\ell(s)} \quad (\text{IV.18})$$

Therefore, the Padé approximants have the same analytic structure in energy as the coefficients of the perturbation series, except for possible poles. For instance, in the equal mass case they have a unitarity cut in $[4m^2, \infty]$ and a left-hand cut in $[-\infty, 0]$.

For unitary approximants, the discontinuity across the right-hand cut is given by the unitarity relation, however, we may remark that in some instances the practical usefulness of the unitarity property may be obscured by the following effect. Consider a coupled channel problem, e.g., $\pi\pi$ and $K\bar{K}$, the partial wave amplitude being treated as a 2×2 matrix, we write its n th order perturbative term as

$$T_m^\ell(s) = \begin{pmatrix} t_m^\ell(\pi\pi \rightarrow \pi\pi) & t_m^\ell(\pi\pi \rightarrow K\bar{K}) \\ t_m^\ell(K\bar{K} \rightarrow \pi\pi) & t_m^\ell(K\bar{K} \rightarrow K\bar{K}) \end{pmatrix} \quad (\text{IV.19})$$

We can build matrix Padé approximants, for instance the $[1,1]$

$$T^{\ell [1,1]}(s) = g T_1^\ell(s) [T_1^\ell(s) - g T_2^\ell(s)]^{-1} T_1^\ell(s) \quad (\text{IV.20})$$

whose discontinuity across the right-hand cuts will satisfy identically coupled channel unitarity. However, the $K\bar{K} \rightarrow K\bar{K}$ amplitude has a left-hand cut starting at $s = 4m_K^2 - 4m_\pi^2$ which therefore extends above the $\pi\pi$ threshold $s = 4m_\pi^2$; in taking the first diagonal term of Eq. (IV.20), i.e., the $\pi\pi \rightarrow \pi\pi$ amplitude, this term will contain a (spurious) left-hand cut on top of the unitarity cut, reflecting the presence of the $K\bar{K}$ channel. Therefore, in such cases, it may be more convenient, for $s < 4m_K^2 - 4m_\pi^2$, to take Padé approximants of the $\pi\pi$ amplitude itself, than of the matrix; in doing this, we lose exact coupled-channel unitarity (but not elastic unitarity), and we have better analyticity properties.

β) - Poles

From the approximate amplitudes such as (IV.13) or (IV.20) one can extract various physical quantities, e.g., phase shifts from unitary approximants, resonance positions, and so on. Bound states and resonances appear as poles of the approximants. The positions of these poles in the complex energy plane are obtained, at a fixed value of the coupling constant g , by solving the equation

$$Q_M = 0 \quad (IV.21)$$

where Q_M is the Padé denominator, for instance, we have in the $[1,1]$ approximation

$$t_1^{\ell}(s) - g t_2^{\ell}(s) = 0 \quad (IV.22)$$

Poles which occur near the physical region are identified as bound states or resonances [Eqs. (IV.21) and (IV.22) must be understood as holding in the second Riemann sheet in the latter case] provided they are stable as the order of approximation changes. We notice that first-sheet poles may occur which do not have a physical meaning, as a general rule they are very unstable against the order of approximation and located far away in the complex plane, they can be considered as approximating left-hand cut effects.

As mentioned above, the Padé denominators $Q_M(g)$ have both a right-hand cut and a left-hand cut in the energy plane. In this respect, they do not have the same analyticity properties as the Jost function, although it has been proved by Garibotti and Villani ⁴¹⁾ in potential scattering that

they converge to the Jost function. It is very interesting to understand how this convergence is achieved in practice, i.e., how the left-hand singularities are gradually suppressed in the denominators [normalized as $Q_M(0) = 1$]. The case of the exponential potential is particularly interesting for this because the S matrix is known and can be expressed in closed form, and, in that case, the left-hand singularities consist of series of poles with equal spacing, and can be considered as a particularly simple example of the general structure of left-hand (interaction) singularities. Basdevant and Lee⁵²⁾ have proved that in this case, in the $[N, N]$ Padé approximant, the N interaction poles nearest to the physical threshold disappear from the Padé denominator, while the N next ones (i.e., from $N+1$ to $2N$) are present. In other words, as N increases, more interaction poles are removed near threshold and more are introduced far away. The more realistic case of the Yukawa potential has been treated numerically by Caser, Piquet and Vermeulen⁵¹⁾, and shows that the discontinuities across the left-hand cut of the Padé denominators get weaker from the nearby ones on, as the order of the approximation increases. The presence of left-hand singularities at finite order is of great importance in computing bound states. In general, bound state poles will be located near or even on top of left-hand singularities - e.g., the deuteron pole in NN scattering sits half-way between threshold and the left-hand cut - and at low orders, computation of binding energies may be inaccurate owing to the influence of left-hand singularities in Eq. (IV.21) [for further details on this point, see Refs. 51), 52), 54)]. A possible way to circumvent this problem is to consider off-shell amplitudes, e.g., off-energy-shell amplitudes in potential scattering, which have only the right-hand cut in the energy variable. The problem is then that we have more variables; the position of the pole will depend on the off-shell variables as in Eq. (IV.17) for the poloid. However, this method can actually be used, for instance, by requiring some minimal dependence on the extra variables and leads to quite satisfactory results^{51), 55), 56)}.

8) - Regge poles

Since the coefficients of the perturbation series satisfy the Mandelstam representation (both in potential scattering and in the relativistic case), one can extend the partial wave amplitudes $t_n^\ell(s)$ in Eq. (IV.11) to complex values of the angular momentum ℓ by using the Froissart-Gribov integral^{14), 32)}. The continuation will be defined for $\text{Re } \ell > n$, n being the number of subtractions (typically $\text{Re } \ell > -1$ in potential scattering, $\text{Re } \ell > 0$ or 1 in renormalizable field theories). In turn, for such values

of ℓ , the Padé approximants $t^{\ell}[\overline{N}, \overline{M}](s)$ can be extended in the complex angular momentum plane ^{14), 32)}. Regge poles will be defined as the continuation to complex values of ℓ of the solutions of Eqs. (IV.21), (IV.22).

We remark that (a) the resulting Regge trajectories have as much physical meaning, in principle, as the partial wave poles that they interpolate, (b) however, since crossing is not exact, they do not correspond to any asymptotic behaviour of the total amplitude such as (IV.16) at finite order, and (c) in computing Regge trajectories, we are faced with exactly the same problem of (spurious) left-hand cuts as above - the resulting trajectories $\alpha(s)$ will possess left-hand cuts in s at finite order, which they should not have in the exact solution - and therefore their behaviour near and below $s = 0$ may be unreliable.

8) - Unstable particles in Feynman graphs

In some field-theoretical models, one may want to introduce (elementary) particles in the Lagrangian which eventually appear as resonances in the final solution, for instance the σ resonance or the ρ resonance in $\pi\pi$ scattering (see next section). In such cases one is faced with the well-known problem of how to handle unstable particles in Feynman graphs. Although the Padé approximation does not provide a general answer to this problem, its unitarity properties give a solution as far as direct channel resonances are concerned. For example, consider the σ model for $\pi\pi$ scattering, where in first order the σ mass m_{σ} is real and we have $m_{\sigma}^2 > 4m_{\pi}^2$. In this model, the $I = 0$ s wave is unique in the sense that it contains a pole corresponding to the elementary σ particle. The first and second Born terms t_1 and t_2 in this channel, behave in the vicinity of $s = m_{\sigma}^2$ as

$$t_1 \sim a + b/(s - m_{\sigma}^2) \quad (\text{IV.23})$$

$$t_2 \sim A(s)/(s - m_{\sigma}^2)^2 + B(s)/(s - m_{\sigma}^2) + C(s) \quad (\text{IV.24})$$

where A , B and C have elastic cuts along $s \geq 4m_{\pi}^2$ - the relevant Feynman graphs are shown in Fig. 5 - (notice that t_2 contains a double pole, t_3 will contain a triple pole, etc.)

The $[1,1]$ Padé approximant, however, behaves as

$$t^{[1,1]} \sim \frac{g s^2}{[b - g B(s)](s - m_\sigma^2) - g A(s)} \quad (\text{IV.25})$$

in the vicinity of $s = m_\sigma^2$, so that its pole will appear in the second sheet as it should. This correct renormalization of the σ mass is a consequence of the unitarity properties, and will remain for all unitary approximants. We have here the example of a series whose sum does not have the same analytic structure as its coefficients. Notice, however, that the box diagram such as Fig. 6 has its t channel discontinuity starting at $t = 4m_\sigma^2$; in the final solution, the "two σ " exchange contribution should be located in the second sheet of the 4π cut, whereas in any approximant it will remain on the real axis at $4m_\sigma^2$. It is in this latter respect that the unstable particle problem is not completely handled.

4. - Practical convergence

a) Soluble cases

The question asked is : what is the order of approximation which gives a satisfactory answer compared to the exact solution ? This has been studied in potential scattering in Refs. 51), 52), and one can summarize the conclusions as follows. If there is at most one bound state or resonance, (which is always the case in particle physics at low energies), the $[2,2]$ approximation essentially gives a perfect answer for amplitudes in the physical region in energy - the determination of binding energies may in some cases be a little less accurate owing to the "left-hand cut" problems mentioned above. This is exemplified on Fig. 7 for the case of Yukawa-type potentials, one can notice that the $[1,1]$, $[2,1]$ and $[1,2]$ already give very useful information.

The Bethe-Salpeter equation can be solved numerically, it is interesting to test the Padé approximation in that case which bears some relativistic aspects. Nuttall ⁵⁷⁾ has shown that for a $g\phi^2\psi$ interaction with scalar particles the $[1,1]$ gives very good results, while Nieland and

Tjon ⁵⁸⁾ in a π nucleon model found that the $[1,1]$ was unreliable (see, however, remarks below) but the $[2,2]$ was very good (they do not seem to have examined the $[2,1]$ and $[1,2]$ approximants). Similar results were obtained by Gammel, Mentzel and Wortman ⁵⁹⁾.

Such results are very encouraging in that they show that in many cases, computations up to the second order of perturbation theory, which are quite easy, will give the gross features of the amplitudes, while third and fourth order ones will presumably give a very reliable answer at low energies.

b) Realistic field theories

At present, in such cases, the only true way to test convergences is by seeing how stable the answers are as the order of approximation increases. The only case which has been studied this way is the $g\varphi^4$ theory for mesons, where Bessis and Pusterla ³²⁾ have computed Feynman graphs up to the fourth order ; some of these graphs are represented on Fig. 8. The gross result is that the basic low energy features, e.g., lowest lying resonances, are extremely stable in going from third to fourth order (i.e., a posteriori the fourth order calculation was somewhat unnecessary). This is quite remarkable, further details and explanations have been given by Zinn-Justin ¹¹⁾.

There is also an indirect way to test convergence, and this is by studying the crossing properties of Padé amplitude. In fact, if we project the poloid amplitudes of Eq. (IV.16) (which are identically crossing symmetric) onto partial waves we obtain

$$\tilde{t}^{\ell}(s) = \frac{1}{2} \int_{-1}^1 A^{[N,M]}(s, t, u) P_{\ell}(\omega\theta) d(\omega\theta) = t^{\ell}(s) + g^{N+M+1} R_3 \quad (\text{IV.26})$$

On the other hand, partial wave approximants which are unitary, satisfy

$$t^{[N,M]}(s) = t^{\ell}(s) + g^{N+M+1} R_1 \quad (\text{IV.27})$$

Since the exact solution is both unitary and crossing symmetric, if we test numerically that in a certain energy region we have $t^{[N,M]}(s) \sim \tilde{t}^{\ell}(s)$, and that $\tilde{t}$ is approximately unitary while $t^{[N,M]}$ is approximately crossing symmetric, we get an idea of the orders of magnitude of the remainders R_1 and R_3 , and therefore of practical convergence.

There are many ways to perform such tests. The first method used consisted in computing the trajectory of the poloid amplitude ¹⁴⁾ in the Mandelstam plane, as defined by, e.g., Eq. (IV.17) ; if crossing is good it must be reasonably flat, and it must approximate the position of the pole as found in partial wave approximants. An example is given in Fig. 9 where we see the poloid of the ρ meson in $\pi\pi$ scattering in the $g\varphi^4$ theory. We notice that the agreement is good and that the poloid flattens in going from third to fourth order. In order to test crossing properties of partial wave approximants, it is particularly convenient to use the crossing constraints derived by Balachandran and Nuyts ⁶⁰⁾ and by Roskies ⁶¹⁾, and the inequalities of Martin ⁶²⁾. This was done first by Basdevant, Cohen-Tannoudji and Morel ⁶³⁾ on the $g\varphi^4$ theory, and used systematically afterwards ^{64),65)}. One can also check forward dispersion relations ^{64),65)}.

The conclusion is that all the crossing tests as a whole are successful and indicate that (a) the approximation appears to converge quickly, (b) the direct channel resonances produced are also effectively present in the left-hand cuts, and (c) the Padé amplitudes have a consistent physical structure.

c) Some problems with lowest orders

In some instances, the Born term of the perturbation series has a behaviour different from that of all the other terms in some domain, and in that domain the $[1,1]$ approximant [see Eq. (IV.18)] may be unreliable. One example, mentioned above, is the complication due to left-hand cuts in computing binding energies [see Ref. 54) for the discussion of the deuteron pole in NN scattering]. The problem may be of physical origin : in the ρ exchange model of $\pi\pi$ scattering for instance, the Born term is a short range force (ρ exchange) while all other terms contain long range forces (two-pion exchange) (see Fig. 10). Consequently, in higher waves $\ell \geq 2$, the $[1,1]$ approximant is unreliable near threshold ⁶⁵⁾. Other examples can be found, e.g., in πN scattering ^{66),67)} ; in the φ^4 meson theory, the Born term is purely s wave, therefore the $[1,1]$ approximant does not even exist for $\ell \geq 1$ ^{14),32)}.

There can be a kinematic problem : in the pseudoscalar theory for NN scattering with pion exchange, the Born term contains only the pseudoscalar amplitude in the crossed channel whereas all other perturbative terms contain the five invariants SVTAP, this leads to anomalies such as : s waves having the threshold behaviour of a p wave, etc. ⁵⁴⁾.

In such cases, the perturbation series may be a better approximation than the $[1,1]$ in some domains. Anyhow, such problems do not affect the approximation scheme itself, but only the lowest order approximants.

d) Final remarks

In its application to strong interaction physics, the Padé approximation is basically a low energy approximation since it is based on exact unitarity. It allows the computation of Regge trajectories which can be used in turn for part of the high energy behaviour, but we may remark that in Padé calculations there is never any sign of a Pomeron singularity. This is quite normal since the Pomeron is due to inelastic effects, which are not taken into account. From the various tests that have been performed, we feel quite confident that the approximation is capable of reproducing the basic low energy effects both as direct and crossed channel contributions. It also seems that it is practical in the sense that quantitative results can be obtained with the few Feynman graphs which can be actually computed in practice.

It is interesting that this approximation may be regarded as an S matrix spirit as realizing the "nearest singularity" concept in a way compatible with unitarity¹⁴⁾. In fact, the Born term of the Lagrangian can be thought of as defining a property of the S matrix in some domain (e.g., a pole singularity, or the value at some point) and the successive iterations of this Born term, through the analyticity-crossing-unitarity mechanism, introduces the various unitarity cuts. In this respect, one understands that the successive approximations will be valid in larger and larger energy regions around the input (Born term) starting points as the order is increased.

V. - STRONG INTERACTION AMPLITUDES

In applying the Padé approximation to strong interaction processes, e.g., $\pi\pi$, πN , NN scattering, etc., there have been essentially two steps : first, in order to understand the approximation itself, and to gain some intuition, one has studied the simplest possible Lagrangians in each case. Secondly, one has constructed models with more physical content, essentially by incorporating the physical concepts coming from current algebra. We shall now review the various achievements in each case. As a general rule, renormalizable Lagrangians have been considered, however, we shall see that non-renormalizable ones have also been used in an "effective Lagrangian" spirit.

1. - $\pi\pi$ scattering

a) φ^4 theory

This is the simplest Lagrangian for $\pi\pi$ scattering

$$\mathcal{L} = \frac{1}{2} \left[(\partial_\mu \vec{\pi})^2 - \mu_\pi^2 (\vec{\pi})^2 \right] - g/4 (\vec{\pi} \cdot \vec{\pi})^2 \quad (V.1)$$

It was first studied by Alexanian and Wellner⁶⁸⁾ who, however, used Padé approximants to the phase shifts thereby generating essential singularities in the amplitude. A few years later, Copley and Masson⁶⁹⁾, and Bessis and Pusterla³²⁾ constructed Padé approximants to the partial wave amplitudes. The latter authors performed a fourth order calculation - the Feynman graphs have been shown on Fig. 8 - their work was continued by Basdevant, Bessis and Zinn-Justin¹⁴⁾. Renormalization is performed by requiring that g be the value of the $\pi\pi$ amplitude at the centre of the Mandelstam triangle $s = t = u = 4/3 m_\pi^2$, and the value of g is fixed so that the ρ resonance occurs at its physical mass ($g \sim 6$).

The s wave phase shifts are then both negative, as shown in Fig. 11, the scattering lengths are

$$a_0 \sim -0.6 m_\pi^{-1} \quad a_2 \sim -0.3 m_\pi^{-1} \quad (V.2)$$

their ratio is close to the ratio $5/2$ of the total $I = 0$ and $I = 2$ amplitudes at the symmetry point $s = t = u = 4/3 m_\pi^2$.

In the d wave, an $I = 0$ resonance is obtained and identified with the f_0 . It has a mass $m_{f_0} \sim 1600$ MeV, however, together with this " f_0 " an $I = 2$ exotic resonance occurs practically at the same mass.

The widths of the ρ and f_0 are too small, typically $\Gamma_\rho \sim 30$ MeV $\Gamma_{f_0} \sim 50$ MeV.

The Regge trajectories have reasonable intercepts :

$$\alpha_\rho(0) \sim 0.5 \qquad \alpha_{f_0}(0) \sim 0.6 \qquad (V.3)$$

they rise with a parabolic law : $\alpha(s) \sim \sqrt{s}$ (see Fig. 12). It is interesting that as $g \rightarrow \infty$, although the Froissart bound is violated, the intercepts stay finite, and no pole appears for negative s , as an indication that crossing is not badly violated. One has as $g \rightarrow \infty$, $\alpha_\rho(0) = 0.9$, $\alpha_{f_0}(0) = 1.5$, $\alpha_{I=2}(0) = 1.1$.

For what concerns the s wave, the results are quite similar to those of the Chew-Mandelstam theory ⁷⁰⁾. This is not surprising since the physical input is the same, but it is encouraging as to the reliability of the Padé approximation. The two theories differ, however, in the p , d , etc., waves where the Chew-Mandelstam approach cannot yield resonances ⁷¹⁾. Since crossing tests are successful, one sees the advantage of this method for computing higher partial wave resonances.

The inclusion of $K\bar{K}$ forces splits the f_0 from its $I = 2$ unwanted partner ¹⁴⁾, however, since experimentally the s wave phase shifts have opposite signs ^{72),73)}, and since no exotic resonance has been detected at low energies, it seems clear that the physical content of the φ ⁴ Lagrangian does not correspond to the actual situation.

b) The σ model

The extra physical information which seems to be lacking in the previous analysis can be taken from the current algebra considerations, as suggested by Lee ⁷⁴⁾. The calculation of $\pi\pi$ scattering in the σ model of Gell-Mann and Levy ⁷⁵⁾ has been performed by Basdevant and Lee ⁶⁴⁾. The Lagrangian of the σ model is (neglecting nucleon fields)

$$\mathcal{L} = \frac{1}{2} [(\partial_\mu \vec{\pi})^2 + (\partial_\mu \sigma)^2] - \frac{\mu^2}{2} (\vec{\pi}^2 + \sigma^2) - \frac{g}{4} (\sigma^2 + \vec{\pi}^2)^2 + c \sigma \quad (\text{V.4})$$

where σ and $\vec{\pi}$ are the isoscalar, scalar and isovector, pseudoscalar meson fields. Except for the term $c\sigma$, the Lagrangian (V.4) is invariant under $SU(2) \times SU(2)$ transformations; the PCAC condition is satisfied

$$\partial_\mu \vec{A}_\mu(x) = c \vec{\pi}(x) \quad (\text{V.5})$$

where $\vec{A}_\mu(x)$ is the axial vector current, and the tree diagrams of the Lagrangian (V.4) give precisely the result of current algebra in the soft pion limit⁷⁶⁾. Furthermore, it is possible to renormalize the theory in such a way as to preserve the current algebra constraints and PCAC in each order of the perturbation expansion^{74),77)}.

The Feynman rules for the σ model can be derived most succinctly if we translate the σ field by its vacuum expectation value v in Eq. (V.4)

$$\sigma = s + v \quad (\text{V.6})$$

The resulting Lagrangian is

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} [(\partial_\mu s)^2 - \mu_\sigma^2 s^2] + \frac{1}{2} [(\partial_\mu \vec{\pi})^2 - \mu_\pi^2 \vec{\pi}^2] \\ & - \frac{g}{4} (s^2 + \vec{\pi}^2)^2 - (gr) s (s^2 + \vec{\pi}^2) \end{aligned} \quad (\text{V.7})$$

where

$$\mu_\pi^2 = \mu^2 + gr^2, \quad \mu_\sigma^2 = \mu^2 + 3gr^2, \quad c = v(\mu^2 + gr^2) \quad (\text{V.8})$$

The expansion is performed according to the number of loops ⁷⁴⁾, and the corresponding first and second order Feynman diagrams are shown in Fig. 13.

To lowest order, if we compare Eq. (V.5) with the physical PCAC condition

$$\partial_\mu \vec{A}_\mu(x) = f_\pi m_\pi^2 \vec{\pi}(x) \quad (V.9)$$

where f_π is the pion decay constant ($f_\pi \sim 95$ MeV from $\pi \rightarrow \mu \nu$), we have

$$C = f_\pi m_\pi^2 \quad (V.10)$$

Therefore, the theory (V.4) involves three parameters, μ^2 , C and g , which can be eliminated in favour of the pion mass m_π^2 , the known pion decay constant f_π , and a dimensionless coupling constant g , as yet undetermined.

At present, the calculation of $\pi\pi$ amplitudes in this model has been done in the one-loop approximation, i.e., only the $[1,1]$ approximant has been computed (a two-loop calculation is underway ⁷⁸⁾). The $I = 0$ s wave is used to fix the value of the coupling constant g , for instance by requiring that $\delta_0^0(m_\rho^2) \sim 90^\circ$; one uses typically $g \sim 6$.

The $I = 0$ and $I = 2$ s wave phase shifts agree well with experiment as can be seen on Fig. 14. The physical (renormalized) resonance obtained is very broad, typically

$$M_\sigma \sim 500 \text{ MeV} \quad \Gamma_\sigma \sim 300 \text{ MeV}$$

With this value of g , the ρ and f_0 resonances are generated dynamically, and one obtains

$$M_\rho = 780 \text{ MeV} \quad \Gamma_\rho = 35 \text{ MeV}$$

$$M_{f_0} = 1115 \text{ MeV} \quad \Gamma_{f_0} = 180 \text{ MeV}$$

in fair agreement with experiment (remember this is the lowest order approximation). An exotic $I = 2$ resonance is still present, but it is clearly split from the f_0 and there are indications that it may disappear at higher orders.

Various crossing tests are well satisfied⁶⁴⁾. In all respects the Padé approximation is much more satisfactory than previous "unitarizations" of current algebra amplitudes⁷⁹⁾. Furthermore, at low energies, the Padé amplitudes do not differ much from the linear Weinberg amplitudes⁸⁰⁾; the unitarization maintains the smoothness assumption usually made in conjunction with PCAC.

From a phenomenological point of view, all gross features of low energy $\pi\pi$ scattering are present in this model. Of course, higher order calculations are certainly necessary in order to be absolutely conclusive quantitatively.

c) ρ -exchange model

Another approach, by Basdevant and Zinn-Justin⁶⁵⁾, considers ρ exchange as the dominant force between pions, treating the ρ as a massive Yang-Mills field⁸¹⁾ universally coupled to the conserved isospin current. This corresponds to the idea of Sakurai⁸²⁾ that strong interactions may be mediated by vector mesons. It is appealing since we know that such forces forbid exotic resonances, and furthermore the celebrated KSFR⁸³⁾ relation indicates that in such a model, the orders of magnitude should essentially be the same as predicted by current algebra.

The Lagrangian is

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \left(\partial_\mu \vec{\pi} - g \vec{\rho}_\mu \times \vec{\pi} \right)^2 - \frac{m_\pi^2}{2} (\vec{\pi} \cdot \vec{\pi}) \\ & - \frac{1}{4} \left(\partial_\mu \vec{\rho}_\nu - \partial_\nu \vec{\rho}_\mu - g \vec{\rho}_\mu \times \vec{\rho}_\nu \right)^2 - \frac{m_\rho^2}{2} \vec{\rho}_\mu \cdot \vec{\rho}^\mu \end{aligned} \quad (V.11)$$

where $\vec{\rho}_\mu$ is the isovector vector field. Owing to the partial gauge invariance (it is only the ρ mass term which breaks it), and to the ensuing Ward identities, the most divergent parts which come from the $k_\mu k_\nu / m_\rho^2$ term of the ρ propagator, cancel at a given order, and the theory behaves as a renormalizable one⁸⁴⁾. In the theory (V.11) the (renormalized) parameters m_ρ and g are directly related to the physical ρ mass and width. Therefore, the only arbitrary constant in the theory is the subtraction constant of the second order term. This is fixed in order to satisfy Adler's self-consistency condition⁸⁵⁾ at each order of perturbation theory. Thereby, the consequences of PCAC are satisfied.

Here again, the $[1,1]$ approximant has been computed. The s wave phase shifts, shown on Fig. 15, agree well with experiment, with a strong positive δ_0^0 and a small negative δ_0^2 . Furthermore, an $I = 0$ s wave σ resonance is produced, it is very broad

$$M_\sigma \sim 450 \text{ MeV} \quad \Gamma_\sigma \sim 500 \text{ MeV}$$

The scattering lengths agree with current algebra predictions

$$a_0^0 \sim 0.17 \text{ } m_\pi^{-1} \quad a_0^2 \sim -0.06 \text{ } m_\pi^{-1}$$

The f_0 resonance is also generated dynamically, in very good agreement with experiment

$$M_{f_0} \sim 1280 \text{ MeV} \quad \Gamma_{f_0} \sim 250 \text{ MeV}$$

and no exotic resonance appears.

Of course, since the ρ is elementary (input) in this theory, the parameters of the $I = 1$ p wave agree very well with experiment.

The σ model and the ρ model give very similar results although the starting points are different. This has the same reason as the success of the KSFR relation⁸³⁾ between current algebra and ρ exchange. It is interesting that in the σ model the ρ is generated dynamically while in the ρ model the σ is generated dynamically. In some sense this is a reciprocal bootstrap situation between ρ and σ . In both models the f_0 is created, we believe that this is the only theoretical approach that gives a consistent and well defined explanation for this particle. Many Lagrangians may be equivalent if one calculates high enough orders, the main difference being the efficiency in reproducing such or such a feature of the dynamics at lower orders. Higher order calculations are certainly necessary both in the σ model and in the ρ model in order to be really conclusive.

We notice, finally, that the present results agree with other theoretical approaches to $\pi\pi$ scattering, this provides a better understanding of the underlying physics⁸⁶⁾.

2. - Other meson-meson channels

In this sector, the investigations are rather scarce up to now. The only systematic calculation of $\pi\pi$, πK , $K\bar{K}$, KK , etc., amplitudes at present, has been done by Basdevant, Bessis and Zinn-Justin¹⁴⁾, it is based on the φ^4 theory. Owing to the particular structure of the φ^4 Lagrangian, and the dominance of s wave forces, this leads to exchange degeneracies between exotic and non-exotic channels - e.g., in such a theory $I = \frac{1}{2} \pi K$ amplitudes are exactly degenerate with $I = \frac{3}{2}$ ones - and this is not a realistic situation physically. However, it is interesting to see that the known two-meson resonances - the $J^P = 1^-$ octet and the 2^+ nonet - are very well generated, although they are accompanied by unwanted objects, as shown in the Table. If we remember that in $\pi\pi$ scattering, in going from the φ^4 theory to the σ and ρ models, one essentially preserves the positive features and corrects the bad ones, this gives great hope that a consistent picture of all meson-meson amplitudes could also be obtained by incorporating some $SU(3) \times SU(3)$ structure in the Lagrangian. The calculation of meson-meson amplitudes with vector meson models (ρ, K^*, φ) is underway⁸⁷⁾.

3. - πN scattering

For some time, the situation in this sector was very unsatisfactory ⁶⁶⁾⁻⁶⁸⁾. The basic reason lies in the deficiency of the $[1,1]$ approximant for the Lagrangians considered, as mentioned in Section IV.4c.

The pseudoscalar theory, with the interaction Lagrangian

$$\mathcal{L}_{int} = -ig \bar{N} \gamma_5 \vec{\epsilon} N \vec{\pi} - \frac{\lambda}{4} (\vec{\pi} \cdot \vec{\pi})^2 \quad (V.12)$$

was first computed in the one loop approximation by Gammel, Mentzel and Kubis ⁸⁸⁾ and by Mignaco, Pusterla and Remiddi ⁶⁶⁾. From this latter work, one can see clearly the inadequacy of the $[1,1]$ approximation; one can play with the $\pi\pi$ coupling λ in order, for instance, to bring the N_{33}^* to its physical mass, however, the shape of the resonance is then disastrous, ghosts occur, etc. Besides that, the theory (V.12) does not satisfy current algebra constraints and the physics is incorrect.

In order to satisfy the current algebra requirements, Mignaco and Remiddi ⁶⁷⁾ studied the σ model

$$\begin{aligned} \mathcal{L} = & \bar{N} [i \not{\partial} - g(\sigma + i \vec{\pi} \cdot \vec{\epsilon} \gamma_5)] N + \frac{1}{2} [(\partial_\mu \sigma)^2 + (\partial_\mu \vec{\pi})^2] \\ & - \frac{1}{2} \mu^2 [\sigma^2 + \vec{\pi}^2] - \frac{\lambda}{4} (\sigma^2 + \vec{\pi}^2)^2 + c \sigma \end{aligned} \quad (V.13)$$

Again, the calculation was done in the $[1,1]$ approximation, using the renormalization procedure of Lee and Gervais ⁸⁹⁾.

Here, it is even easier to see why the $[1,1]$ is inadequate. In fact, although the Lagrangian (V.13) gives the results of current algebra in the tree approximation, the on-shell values of the scattering lengths (which agree so well with experiment) are obtained from the Born term only if one has $m_\sigma > m_N > m_\pi$, as a rapid computation can show. Since the experimental situation is rather $m_N > m_\sigma > m_\pi$, the Born term by itself does not reproduce these scattering lengths - it even predicts that they are of the same sign instead of the opposite. This is of course due to the present particular realization of chiral symmetry. Hence, since the Born term does

not have the correct physical structure, the $[1,1]$ approximant is unreliable (in fact the truncated perturbation series gives much better values for the scattering lengths), although one can certainly play with the various parameters in order to adjust some waves. It is quite sure that in a third order calculation such troubles would disappear.

As an alternative to computing higher orders, where the technical difficulties may, in a first step, mask the underlying physics in the absence of a good first order approximation, one may think of using the $[1,1]$ for a Lagrangian whose Born term has the correct physical structure. "Smooth" Born terms are provided by the (non-renormalizable) effective Lagrangians⁹⁰⁾ of current algebra. If one works in the one-loop approximation, the number of subtraction parameters is usually very small, and the theory behaves in a way similar to a renormalizable one. An approach of this kind has been attempted by Filkov and Palyushev⁹¹⁾, and the results are quite remarkable as we shall now see.

In the presentation of their work, Filkov and Palyushev do not follow actually a Lagrangian approach, they operate in a very simple on-shell way. Following a method suggested in Ref. 14), they iterate (once) π N dispersion relations starting from a very simple first order (Born term) which contains, besides the nucleon pole, a constant in the isospin-even crossed channel amplitude :

$$\text{First order : } A_1^{(+)} = a \quad (\text{V.14a})$$

$$A_1^{(-)} = 0 \quad (\text{V.14b})$$

$$B_1^{(\pm)} = g^2 \left[\frac{1}{m_N^2 - s} \mp \frac{1}{m_N^2 - u} \right] \quad (\text{V.14c})$$

This could be interpreted in a Lagrangian spirit as the following interaction

$$\mathcal{L}_{int} \sim \lambda (\bar{N} N) (\vec{\pi} \cdot \vec{\pi}) - i g \bar{N} \gamma_5 \vec{\tau} N \cdot \vec{\pi} \quad (\text{V.15})$$

with a (non-renormalizable) point interaction, and the corresponding first order diagrams would be as shown in Fig. 16. In second order, it is easy to see that the divergences are logarithmic (and not quadratic) so that they can be absorbed in the two parameters a and g . The "renormalized" g is the πN coupling constant, and the subtraction points are such that a is simply related to a combination of (known) scattering lengths,

$$a = 4\pi \left\{ \frac{2m+\mu}{6m} (a_1 + 2a_3) - \frac{2m\mu}{3} [(a_{11} + 2a_{31}) - (a_{13} + 2a_{33})] \right\} \quad (V.16)$$

Therefore, the parameters of the model are fixed, and Filkov and Palyushev have computed the unitary $[1,1]$ approximant in the various partial wave amplitudes.

We start the discussion of their results with the p waves since the N_{33}^* is the prominent feature of πN physics at low energy, they are shown on Fig. 17. It is remarkable that the N_{33}^* is obtained very well both in position and shape. We also notice, on the figures, the great improvement over previous calculations, which is a feature of all these results. We note the remarkable agreement of P_{11} and P_{31} , the only exception being P_{13} which agrees only up to ~ 1350 MeV. In the s waves, Fig. 18, and in the d waves, Fig. 19, the same comments apply : good agreement with the data and drastic improvement over other models. Even more, it seems that basically whenever differences occur with experiment they can be attributed to inelastic effects - see, e.g., S_{11} , S_{31} , D_{15} , D_{13} near 1600 MeV. In the present calculation such effects have not been taken into account, this could be done easily.

This calculation is very impressive and certainly deserves further analysis and better understanding. For instance, crossing properties should be analyzed, in particular to understand how the Padé algorithm has generated strong $I_t = 1$ (e.g., ρ exchange) contributions which are necessary in πN dispersion relations, but are absent in the Born terms (V.14). The second point is : how would the results be modified if one used a true chiral Lagrangian⁹⁰⁾, where the parameters would be fixed in terms of weak interaction parameters (g_A/g_V or f_π , etc.).

The only serious defect that one could think of in the present calculation is that since it is based on a non-renormalizable theory, higher orders would introduce more arbitrary parameters and therefore one would lose

one of the attractive features of the scheme, i.e., that it is systematic and well defined. However, we do not consider this as a basic drawback for the following reasons. The present $[1,1]$ calculation with a non-renormalizable interaction must be considered as an easy representation of what a higher order calculation, e.g., $[2,2]$, would give in a renormalizable theory such as the σ model. The physics is the same, what changes is its practical realization ; it is only technical problems which render the $[1,1]$ useless in the σ model. In any case what is important is the predictive power of the theory, i.e., the fact that it contains as few parameters as possible.

We finally mention the work of Redei ⁹²⁾ who attempted to compute the P_{33} resonance in an effective Hamiltonian approach, and the very interesting analysis of Jacobs and Lambert ⁹³⁾ who apply the Padé approximation to the full nucleon propagator in order to compute the P_{11} Roper resonance.

4. - NN scattering

In the nucleon-nucleon sector, the only complete investigation has been made in the pseudoscalar theory with pion exchange

$$\mathcal{L}_{int} = -ig \bar{N} \gamma_5 \vec{\tau} N \vec{\pi} \quad (V.17)$$

this has been done by Wortman ⁹⁴⁾ and by Bessis, Graffi, Grecchi and Turchetti ⁵⁴⁾. The calculations have been performed to second order, i.e., in the $[1,1]$ approximation, the only parameter being the πN coupling constant g .

We take the usual notation $^{2S+1}L_J$, the isospin being related to L and S by $(-1)^{I+L+S+1} = 1$. As mentioned in Section IV.4c, the $[1,1]$ approximant causes some trouble in certain channels owing to the particular behaviour of the Born term in this theory.

It is in the singlet partial waves $S = 0$ that the deficiency of the $[1,1]$ is felt most. In fact, in the 1S_0 wave, the Born term t_1 behaves at threshold as a p wave ($\sim k^3$) ; since the second order term t_2 has a normal s wave behaviour, it is easy to see that actually the $[1,1]$ approximant will have the behaviour of a d wave ⁵⁴⁾ !

$$t^{[1,1]} = \frac{g t_2^2}{t_2 - g t_2} \sim k^5 \quad k \rightarrow 0 \quad (V.18)$$

As a consequence, the 1S_0 phase shift is essentially meaningless, the 1D_2 and 1G_4 are also very bad : they give rise to sharp resonances at low energies. The odd sequence $^1P_1, ^1F_3$, etc., is better (see Fig. 20).

In the uncoupled triplet waves $S = 1, L = J$, there is no technical problem with the $[1,1]$, and the agreement with experiment ⁹⁵⁾ is good up to 300 MeV as can be seen on Fig. 21.

In the coupled triplet waves, to which the deuteron channel belongs, the possible defects of the $[1,1]$ are partially cured by the fact that one uses matrix Padé approximants which restore a correct threshold behaviour ⁵⁴⁾. The deuteron pole is found with a binding energy of 4.8 MeV, a factor of two away from the experimental 2.2 MeV. However, this must be considered as very satisfactory in view of the difficulties created by left-hand singularities in determining this binding energy at lowest order, as explained in Section IV.3b. In Fig. 22, we see that the corresponding 3S_1 phase compares acceptably with experiment, but the 3D_1 which is coupled to it and the mixing angle ϵ_1 , are poor. In Fig. 23, we show other members of this sequence, we note that 3P_0 is the only one which does not reproduce qualitatively experimental points, however, it is the only wave of this series which is actually not coupled and for which the matrix approach does not hold.

To summarize, we think that provided we are aware of the difficulties caused by the $[1,1]$, a great deal of the pion-exchange contributions to NN dynamics is well understood in this framework. We see that the simple Lagrangian (V.17) is useful for NN scattering [as opposed to φ^4 in $\pi\pi$ and (V.12) in πN] since π exchange is a dominant contribution here in certain regions. It is very nice in particular that an explanation of the deuteron comes out of the calculation.

In order to complement the physical structure of the Lagrangian, Bessis and Turchetti ⁹⁶⁾ have started calculating NN scattering in the σ model, using the renormalization technique of Symanzik ^{77),97)}. Unfortunately their calculation is limited to the s wave scattering lengths. They show that it is possible to obtain the 1S_0 and 3S_1 scattering lengths in very acceptable agreement with experiment. A further calculation, in the non-linear σ model, by Turchetti ⁹⁸⁾ does not give really new information.

In this field, as in others, one feels that the only conclusive way to work with renormalizable Lagrangians must go through a higher order (i.e., g^6) calculation. However, in view of the π N results of Filkov and Palyushev, it would certainly be very interesting to perform a $[1,1]$ calculation in NN, with a true effective Lagrangian, involving the main contributions known phenomenologically ⁹⁹⁾, e.g., ρ , ω , etc., exchange. This would allow a better understanding of nucleon-nucleon physics, and is quite promising in view of the excellent results obtained here with only pion exchange. If the results would be good, this might provide immediately a scheme for theoretical calculations of other baryon-baryon amplitudes, which would be of great practical interest.

VI. - CONCLUSION

There are still many interesting open questions concerning the mathematical properties of Padé approximants. But, if we accept, say, the Padé conjecture it is obvious that they are of great practical usefulness. The important features are that they give larger convergence domains, they allow to perform analytic continuations numerically, and they accelerate the convergence of various processes. They provide an excellent approximation method. It is perhaps not the best method in all cases, but the crucial point is that rational functions are no more difficult to use as polynomials, whereas they are infinitely more powerful.

The usefulness of Padé approximants in physical applications is beyond any doubt ; in many instances they have been proved to give the correct answer. In the particular domain of strong interaction physics there are several different aspects.

The question of proving convergence of Padé approximants in quantum field theory is quite hopeful in the sense that convergence has been proved in many models which bear realistic features - bound state poles, essential singularities. It is worth noting that up to now, no realistic counter-example has been given to what one might call a "physical" Padé conjecture ; no basic inconsistency has been found. Progress towards a convergence proof can certainly be made.

The Padé approximation seems capable of eliciting the complex spectrum that a Hamiltonian possesses. For this reason, it seems to be particularly suited as an approximation scheme for constructing a theory of strong interactions based on Lagrangian field theory. The great virtue of the approximation is that it is systematic and well defined. Even though in practice the theory will be limited to the prediction of low energy amplitudes, owing to the technical complexity of Feynman graphs, it can nevertheless be carried through up to any order in principle. It can be used to test various dynamical assumptions, such as the current algebra predictions. In these respects, the Padé approximation appears as a new powerful tool for treating old (but good) physical ideas.

In practical calculations, the results which have been obtained up to now in $\pi\pi$, πN , and in some respects in NN scattering are quite impressive. Many things, however, remain to be done. First, there are other

meson-meson, meson-baryon, and baryon-baryon amplitudes to be computed, other Lagrangians to be investigated. Secondly it is certainly necessary to perform higher order calculations, which are worth while in view of the present results, in order to obtain more reliable information. Finally, from the point of view of generality, one would like to obtain a consistent set of, e.g., $\pi\pi$, πN and NN amplitudes from a single unique Lagrangian, with the same values for the parameters.

It is certain that from the point of view of efficiency - i.e., man-hours of work compared to practical results - the Padé approximation compares quite favourably with other approximation schemes for strong interaction physics, although one may of course share the opinion of d'Alembert ¹⁰⁰⁾ : "Pour moi j'avoue que tous les raisonnements et les calculs fondés sur des séries qui ne sont pas convergentes ... me paraîtront toujours très suspects, même quand les résultats de ces raisonnements s'accorderaient avec des vérités connues d'ailleurs".

J ^P	I	S	Channel	Degeneracy	Name	Experimental mass (MeV)	Calculated (1)	Calculated (2)
1 ⁻	1	0	$\pi\pi$	-	ρ	776	781	790
1 ⁻	$\frac{1}{2}$	± 1	πK	$I = \frac{3}{2}$	$K^*(890)$	892	819	833
1 ⁻	0	0	$K\bar{K}$	$KK\bar{K} \quad I = 1 \quad J = 1$ $KK \quad I = 0 \quad J = 1$	ϕ	1019	1018	1039
2 ⁺	0	0	$\pi\pi$	-	f_0	1254	1242	1192
2 ⁺	0	0	$K\bar{K}$	-	f'_0	1514	1508	1490
2 ⁺	1	0	$K\bar{K}$	$KK \quad I = 1 \quad J = 2$	A_2	1310	1389	1440
2 ⁺	$\frac{1}{2}$	± 1	πK	$I = \frac{3}{2}$	$K^*(1415)$	1417	1455	1483
2 ⁺	2	0	$\pi\pi$	-	-	-	1460	1479

TABLE : Masses (in MeV) of the resonances computed in the meson ϕ^4 theory of Ref. 14).
I is the isospin and S the strangeness, we indicate when a degenerate channel is present. We show two possible solutions (1) and (2) from Ref. 14).

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"Les séries divergentes sont, en général, quelque chose de bien fatal, et c'est une honte qu'on ose y fonder aucune démonstration ... la partie la plus essentielle des mathématiques est sans fondement. Pour la plus grande partie les résultats sont justes, il est vrai, mais c'est là une chose bien étrange. Je m'occupe à en chercher la raison, problème très intéressant" [Quoted in Ref. 3].

FIGURE CAPTIONS

- Figure__1 A box diagram of field theory, studied in Ref. 23).
- Figure__2 Pattern of poles and zeros of a $[4,4]$ approximant to Eq. (III.11).
- Figure__3 Pattern of poles and zeros for a Padé approximant to Eq. (III.13).
- Figure__4 Form of the two-body potential $\Phi(r)$ which is used to derive the Van der Waals equation.
- Figure__5 Pole terms in first and second order, in the σ model.
- Figure__6 Box diagram with two σ exchange in the σ model.
- Figure__7 Phase shifts as computed by Padé approximants in potential scattering.
(a) Exponential potential in the s wave, with no bound state, Ref. 52).
(b) Yukawa potential with one bound state, Ref. 51).
- Figure__8 Feynman graphs of the $g\varphi^4$ theory computed in Ref. 32). Each graph must be understood as contributing to all crossed channels.
- Figure__9 Poloid of the ρ meson, from Ref. 14) ;
—— : fourth order ; ---- : third order. The straight lines are the positions of the corresponding partial wave poles. The value of g is constant.
- Figure__10 Range of forces in, e.g., the ρ exchange model. T_1 is short range (ρ exchange), T_2, T_3 , etc. are long range (two-pion exchange).
- Figure__11 $\pi\pi$ s wave phase shifts in the φ^4 theory of Refs. 14) and 32).
- Figure__12 $\pi\pi$ Regge trajectories in the φ^4 theory, from Ref. 32).

- Figure 13 Feynman graphs in the σ model ; tree diagrams : first order ; one-loop diagrams : second order ⁶⁴⁾.
- Figure 14 s wave $\pi\pi$ phase shifts in the σ model ⁶⁴⁾ data are from Baton et al. ⁷²⁾ and Malamud and Schlein ⁷³⁾. Curve A corresponds to $\delta_0^0(m_\rho^2) \sim 90^\circ$ ($g = 5.72$, $f_\pi = 125$ MeV), curve B to $\delta_0^0(m_\rho^2) \sim 75^\circ$ ($g = 6.22$, $f_\pi = 120$ MeV).
- Figure 15 s wave $\pi\pi$ phase shifts in the ρ model of Ref. 65). Data are from Ref. 72).
- Figure 16 Diagrammatic representation of the Born term, Eq. (V.14a-c).
- Figure 17 πN p wave phase shifts. The data are from the CERN phase shift analysis.
 — calculation of Filkov and Palyushev ⁹¹⁾,
 ----- χ_5 theory by Mignaco et al. ⁶⁶⁾,
 -.-.- σ model by Mignaco et al. ⁶⁷⁾.
- Figure 18 Same as Fig. 17 for the πN s waves.
- Figure 19 Same as Fig. 17 for the πN d waves.
- Figure 20 NN singlet odd waves. Theoretical curves are from Bessis et al. ⁵⁴⁾. Experimental data from the Livermore phase shift analysis ⁵⁹⁾.
- Figure 21 Same as Fig. 20 for the NN uncoupled triplet waves.
- Figure 22 Same as Fig. 20 for the deuteron waves, ϵ_1 , is the mixing angle between 3S_1 and 3D_1 .
- Figure 23 Same as Fig. 20 for the coupled triplet waves.

FIG.1

FIG.2

FIG.3

FIG. 4

FIG. 5

FIG.6

FIG.7

FIG.8

FIG.9

FIG.10

FIG.11

FIG.12

FIG. 13

FIG.14

FIG.15

FIG.16

FIG. 17_a

FIG 17 b

FIG. 18

FIG 19a

FIG.19 b

FIG. 20

FIG.21

FIG. 22

FIG.23